

THE COMPLEXITY OF THE CLASSIFICATION OF RIEMANN SURFACES AND COMPLEX MANIFOLDS

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ABSTRACT. In answer to a question by Becker, Rubel, and Henson, we show that countable subsets of \mathbb{C} can be used as complete invariants for Riemann surfaces considered up to conformal equivalence, and that this equivalence relation is itself Borel in a natural Borel structure on the space of all such surfaces. We further proceed to precisely calculate the classification difficulty of this equivalence relation in terms of the modern theory of Borel equivalence relations.

On the other hand we show that the analog of Becker, Rubel, and Henson's question has a negative solution in (complex) dimension $n \geq 2$.

1. Introduction

In this paper we consider the problem of classifying various classes of complex manifolds. The investigation is completely abstract, since we are not so much concerned with specific schemes of classification and their success or failure, as with considering what *kinds* of complete invariants could in principle be produced. Here we have in mind that there is a hierarchy of levels of difficulty of classifying various mathematical objects and this paper joins Dougherty-Jackson-Kechris [94], Harrington-Kechris-Louveau [90], Hjorth [9?a], Hjorth-Kechris [95], Kechris [92], Kechris [98], and others, as one more piece in a general project to compare the classification problems across a variety of mathematical disciplines and obtain a language that can contrast their various forms.

At perhaps the simplest level are schemes of classification which provide a single point in some highly concrete space as a complete invariant. In ergodic theory the real number corresponding to the entropy of a Bernoulli shift is a complete invariant for this class of measure preserving transformations. One similarly finds in the theory of Riemann surfaces that *compact* complex surfaces considered up to conformal equivalence may be cataloged as points in highly concrete spaces.

The work below came about after we saw the following theorem:

THEOREM 1.1 (Becker-Rubel-Henson [80]). *There is no "reasonably concrete space" X and "reasonably intrinsic" or "reasonably definable" assignment*

$$f: \mathcal{D} \rightarrow X$$

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from \mathcal{D} , the collection of complex domains, such that for any two $R_1, R_2 \in \mathcal{D}$,

$$R_1 \cong R_2 \Leftrightarrow f(R_1) = f(R_2).$$

In other words, there is no reasonable way to assign points in X as complete invariants for complex domains considered up to conformal equivalence.

Of course the phrases “reasonably concrete space” and “reasonably intrinsic assignment” are deliberately vague. For the purposes of this introduction we simply ask that the reader accept that this can and will be made precise, that the various ways in which one might do so are not subject to serious controversy, and that all the competing explications of these phrases give rise to similar outcomes. Indeed while one may have doubts about how a definition of such concepts should be crafted, it should certainly be clear that for instance the space of *all* subsets of \mathbb{C} is not sufficiently concrete and that a function obtained by invoking the axiom of choice to well order the complex domains and then thus armed producing an injection from \mathcal{D} into the ordinals or into \mathbb{C} or into $\mathcal{P}_{\aleph_0}(\mathbb{C})$ (the set of all countable subsets of \mathbb{C}) can not be considered “reasonably intrinsic” or “reasonably definable”.

Perhaps in passing we can mention that for us “reasonably concrete space” means something like a Polish space or a standard Borel space, and that for us a “reasonably definable function” means something like a function that is Borel measurable from some standard Borel space of parametrizations. More generously one may consider, as in the Ulm invariants from abelian group theory (see Kaplansky [69]), spaces such as the set of all countable subsets of the first uncountable ordinal, or equivalently, countable transfinite sequences from \mathbb{C} , and functions that are universally Baire measurable, or projective, or even ordinal definable from reals. As we discuss in Section 5.C, Theorem 1.1 survives in some form even in these contexts.

Our first concern is the extent to which classification may be obtained if we consider invariants more general than a *single* point in some space. In direct response to a question by Becker, Rubel, and Henson, we show that *countable unordered subsets* of \mathbb{C} can provide complete invariants for \mathcal{R} , the class of all Riemann surfaces:

THEOREM 1.2. *There is a “reasonably definable” function*

$$f: \mathcal{R} \rightarrow \mathcal{P}_{\aleph_0}(\mathbb{C}),$$

from the Riemann surfaces to countable subsets of \mathbb{C} , such that for any two R_1, R_2 ,

$$R_1 \cong R_2 \Leftrightarrow f(R_1) = f(R_2).$$

In other words, there is a “reasonably definable” way to assign countable subsets of a concrete space as complete invariants for Riemann surfaces considered up to conformal equivalence.

One might draw an analogy between this result and the Halmos-von Neumann [42] invariants for *discrete spectrum* ergodic measure preserving transformations. There,

as here, there is no reasonable method to assign points in say \mathbb{C} as complete invariants, but the countable subset of \mathbb{C} corresponding to the eigenvalues completely classifies an ergodic discrete spectrum measure preserving transformation up to isomorphism. The proof of 1.2 in 4.A below is based on an abstract method and does not seem to provide “geometrically meaningful” invariants $f(R)$. One can wonder if it is possible to sharpen this theorem by providing such invariants.

The method of proof in 4.A below actually gives a much better upper bound on the complexity of conformal equivalence for Riemann surfaces and in 4.B we show that this is precise.

THEOREM 1.3. *The classification problem for Riemann surfaces considered up to conformal equivalence is “equal in difficulty” to that of the universal countable Borel equivalence relation, E_∞ .*

Here “equal in difficulty” indicates that each equivalence relation can be embedded in the other using a function that is Borel measurable in some suitable space of parameters. E_∞ is known to have a number of instantiations. For instance, it can be realized as the orbit equivalence relation produced by the shift action of F_2 (the free group on 2 generators) on $2^{F_2} = \{f \mid f: F_2 \rightarrow \{0, 1\}\}$, or as isomorphism on finitely branching trees (Jackson-Kechris-Louveau [9?]), or (very recently, Thomas-Velickovic [99]) isomorphism on finitely generated groups.

Put another way, this means that the “moduli space” of all Riemann surfaces is “Borel equivalent” to the very complicated quotient space $2^{F_2}/E_\infty$. Actually, as it follows from the proof of 1.3 given in Section 4 below, this holds as well for Riemann surfaces homeomorphic to the infinitely punctured plane, i.e., $\mathbb{C} \setminus S$, where $S \subseteq \mathbb{C}$ is infinite discrete. Thus this gives a precise measure of the set theoretic complexity of the moduli space of these Riemann surfaces. It is much more complex than the moduli spaces of finitely punctured compact Riemann surfaces, which are fairly “concrete” and admit a rich geometrical structure.

Finally, the higher dimensional case is discussed in Section 6. Here we provide a new lower bound on the complexity of biholomorphism, and indicate a sense—which when translated into the theory of Borel equivalence relations can be made totally precise—in which the passage from complex dimension 1 to complex dimension 2 brings an increase in classification difficulty. The following is a simple corollary of 6.1 below.

THEOREM 1.4. *Let \mathcal{M}^2 be the class of two dimensional complex manifolds. Then there is no “reasonably definable” assignment*

$$f: \mathcal{M}^2 \rightarrow \mathcal{P}_{\aleph_0}(\mathbb{C})$$

such that for any $M_1, M_2 \in \mathcal{M}^2$,

$$M_1 \cong M_2 \Leftrightarrow f(M_1) = f(M_2).$$

2. Borel equivalence relations

Definition 2.1. A topological space is said to be *Polish* if it is separable and admits a complete metric. We then define the *Borel* sets to be those appearing in the smallest σ -algebra containing the open sets. A function between two Polish spaces is said to be *Borel* if the inverse image of any open set under f is Borel.

If X is a Polish space and E is an equivalence relation, then E is said to be *Borel* if it is Borel as a subset of $X \times X$ (in the product topological structure). For $x \in X$, we let $[x]_E = \{y \in X: xEy\}$. An equivalence relation is *countable* if every $[x]_E$ is countable.

Examples of Polish spaces.

- (i) \mathbb{R}, \mathbb{C} , in their usual topologies; $2^{\mathbb{N}} =_{df} \{f \mid f: \mathbb{N} \rightarrow \{0, 1\}\}$ equipped with the metric

$$d(x, y) = 2^{-\text{least } n \text{ such that } x^{(n)} \neq y^{(n)}}$$

is a compact metric space, and so certainly the underlying topological space is Polish.

- (ii) Polish spaces are closed under countable products, and thus $\mathbb{R}^{\mathbb{N}}, \mathbb{C}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}}, \mathbb{C}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}$ are all Polish.
- (iii) Any G_δ subset of a Polish space is Polish (see Kechris [95, 3C]).
- (iv) The space of all countable models of a given countable language is a Polish space; this is an especially important example from the point of view of logic. Let \mathcal{L} be a countable (relational) language and let $\text{Mod}(\mathcal{L})$ be the set of all \mathcal{L} -structures with underlying set \mathbb{N} and equipped with the topology induced by taking as subbasic open sets those of the form

$$\{\mathcal{M} \in \text{Mod}(\mathcal{L}): \mathcal{M} \models R(n_1, \dots, n_k)\},$$

$$\{\mathcal{M} \in \text{Mod}(\mathcal{L}): \mathcal{M} \models \neg R(n_1, \dots, n_k)\},$$

for $n_1, \dots, n_k \in \mathbb{N}$ and $R \in \mathcal{L}$ a relation of arity k . This space is Polish since it is homeomorphic to

$$\prod_{R \in \mathcal{L}} 2^{\mathbb{N}^{a(R)}},$$

where $a(R)$ is the arity of the relation symbol R (see Becker-Kechris [96]). (A similar definition works even for languages with function symbols, since we can show that the space $\text{Mod}(\mathcal{L})$ is homeomorphic to a G_δ subset of $\text{Mod}(\mathcal{L}')$ for a suitably chosen relational \mathcal{L}' .)

Many mathematical objects can be thought of as points in some appropriately chosen Polish space considered up to some notion of isomorphism or equivalence.

For instance, we may think of countable groups as elements of some appropriate $\text{Mod}(\mathcal{L})$ considered up to isomorphism. A fundamental notion used in comparing equivalence relations is that of Borel reducibility.

Definition 2.2. For E and F equivalence relations on Polish spaces X and Y , we write $E \leq_B F$, and say that E is *Borel reducible to F* , to indicate that there is a Borel function $f: X \rightarrow Y$ such that for all $x_1, x_2 \in X$,

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

We use $E \sim_B F$ for $E \leq_B F \leq_B E$; $E <_B F$ indicates that $E \leq_B F$ but it is not the case that $F \leq_B E$.

Examples of equivalence relations.

- (i) $\text{id}(\mathbb{C})$, the equality relation on \mathbb{C} ; more generally for any Polish space X we can consider the identity relation $\text{id}(X)$ on X . Since any two uncountable Polish spaces are Borel isomorphic, it follows that for any two uncountable Polish spaces X and Y , $\text{id}(X) \sim_B \text{id}(Y)$.
- (ii) We let E_0 be the equivalence relation of eventual agreement on $2^{\mathbb{N}}$. This means that $x_1 E_0 x_2$ if and only if there is some $N \in \mathbb{N}$ such that for all $m > N$, $x_1(m) = x_2(m)$. Here it is well known that $\text{id}(\mathbb{C}) <_B E_0$ (see Hjorth [9?a, Chapter 3], or Hjorth-Kechris [95], or Kechris [00]).
- (iii) Given a countable group G , we can let it act on the Polish space $2^G (=_{df} \{f \mid f: G \rightarrow \{0, 1\}\})$ in the product topology) by *shift*: for $g \in G$ and $f \in 2^G$ we define $g \cdot f$ by

$$(g \cdot f)(h) = f(g^{-1}h)$$

for $h \in G$. We denote the orbit equivalence by $E(G, 2)$.

- (iv) In specific cases the complexity under \leq_B of $E(G, 2)$ is exactly understood. For instance it is known that

$$E_0 \sim_B E(\mathbb{Z}, 2)$$

(see Dougherty-Jackson-Kechris [94]) and that for any Borel equivalence relation E one has either

$$E \leq_B \text{id}(\mathbb{C})$$

or

$$E_0 \leq_B E,$$

but not both (this was proved in Harrington-Kechris-Louveau [90]).

- (v) Let F_2 be the free group on 2 generators. Here it is known that for any countable Borel equivalence relation E , i.e., having countable equivalence classes,

$$E \leq_B E(F_2, 2).$$

In this sense, we say that $E(F_2, 2)$ is a *universal countable Borel equivalence relation*, and we will denote it by E_∞ .

- (vi) More generally, we consider orbit equivalence relations induced by the continuous actions of *Polish groups* (that is, topological groups whose underlying space is Polish). If G is a Polish group acting continuously on a Polish space X , we will use E_G^X for the orbit equivalence relation given by

$$x_1 E_G^X x_2 \Leftrightarrow \exists g \in G (g \cdot x_1 = x_2).$$

- (vii) Finally, for \mathcal{L} a countable language, we let $\cong_{\text{Mod}(\mathcal{L})}$ be the equivalence relation of isomorphism on elements of $\text{Mod}(\mathcal{L})$. Thus $\mathcal{M} \cong_{\text{Mod}(\mathcal{L})} \mathcal{N}$ iff there is some permutation π of the natural numbers such that for all relations R in the language,

$$\mathcal{M} \models R(n_1, \dots, n_k) \Leftrightarrow \mathcal{N} \models R(\pi(n_1), \dots, \pi(n_k)).$$

Definition 2.3. We say that an equivalence relation E is *smooth* if $E \leq_B \text{id}(X)$ for some Polish space X ; note that here we may assume without loss of generality that $X = \mathbb{C}$. We say that E *admits classification by countable structures* if there is some countable language \mathcal{L} with

$$E \leq_B (\cong_{\text{Mod}(\mathcal{L})}).$$

Let us further say that E *admits classification by countable subsets of \mathbb{C}* if there is a countable sequence of Borel functions $(f_n)_{n \in \mathbb{N}}$ into \mathbb{C} such that for all x_1, x_2 ,

$$x_1 E x_2 \Leftrightarrow \{f_n(x_1): n \in \mathbb{N}\} = \{f_n(x_2): n \in \mathbb{N}\}.$$

Note that in the definition of classification by countable subsets we ask that $\{f_n(x_1): n \in \mathbb{N}\}$ and $\{f_n(x_2): n \in \mathbb{N}\}$ are equal as unordered sets; if we were to alternatively require something along the lines of the equality of the *sequences* $(f_n(x_1))_{n \in \mathbb{N}}$ and $(f_n(x_2))_{n \in \mathbb{N}}$, then this would be nothing other than the notion of smoothness, since $\mathbb{C}^{\mathbb{N}}$ is itself a Polish space.

The next couple of lemmas clarify these definitions.

LEMMA 2.4. *If E is a countable Borel equivalence relation, then E admits classification by countable subsets of \mathbb{C} .*

Proof. Let X be the Polish space on which E is defined. Following the uniformization theorem for Borel sets in the plane with countable sections (see KeCHRIS [95, 18.C]) we may find a countable sequence of Borel functions

$$h_n: X \rightarrow X$$

such that for all $x \in X$,

$$[x]_E = \{h_n(x) : n \in \mathbb{N}\}.$$

Then if $h: X \hookrightarrow \mathbb{C}$ is a Borel injection we may let $f_n = h \circ h_n$ witness the definition at 2.3. \square

This implication does *not* reverse (see, for instance, Hjorth [97a, §2.3]).

LEMMA 2.5. *If E admits classification by countable subsets of \mathbb{C} , then E admits classification by countable structures.*

Proof. Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of unary predicates, $\{U_n : n \in \mathbb{N}\}$ a countable basis for the topology of \mathbb{C} , and let

$$\langle \cdot, \cdot \rangle : \mathbb{N}^2 \rightarrow \mathbb{N}$$

be a bijection. Let $(f_n)_{n \in \mathbb{N}}$ be the sequence of Borel functions witnessing the classifiability by countable subsets of \mathbb{C} . Let \mathcal{L} be the countable language determined by $(P_n)_{n \in \mathbb{N}}$.

Then for any point x in the space on which E is defined, we let \mathcal{M}_x be the \mathcal{L} -model with underlying set \mathbb{N} defined by

$$\mathcal{M}_x \models P_n(\langle k, l \rangle)$$

if and only if $f_k(x) \in U_n$. (Here each $f_k(x)$ is quite deliberately repeated infinitely often; allowance needs to be made for the fact that the sets $\{f_n(x_1) : n \in \mathbb{N}\}$ and $\{f_n(x_2) : n \in \mathbb{N}\}$ may be equal, but the sequences $(f_n(x_1))_{n \in \mathbb{N}}$ and $(f_n(x_2))_{n \in \mathbb{N}}$ may list a given complex number with different frequency.) \square

The implication here also fails to reverse (see, for example, Hjorth-Kechris-Louveau [98]).

Although we confine ourselves to Borel functions, a similar reducibility theory arises for any general class of functions including the Borel, closed under composition, and satisfying reasonable regularity properties (such as Lebesgue measurability and the pullbacks of open sets having the Baire property). This is an important point. Key theorems are not held hostage to circumstantial and subjective choices used in our explication of *reasonably definable*.

For instance we could equally work with the C -measurable functions, i.e., those that pull back open sets to sets lying in the smallest σ -algebra containing the open sets and closed under Souslin's \mathcal{A} -operation (compare Hjorth-Kechris [95]). Alternatively we might use the *absolutely* Δ_2^1 functions.

For functions of higher complexity, it is necessary to work in systems stronger than ZFC to avoid the whole pursuit's dissolving into a quicksand of indistinguishable independence results. For instance, we can develop a similar theory for functions in

$L(\mathbb{R})$ (the smallest class of sets containing all the reals and closed, in some sense, under transfinite iterations of rudimentarily definable operations) assuming sufficient *determinacy* or the existence of *large cardinals*. The authors of Becker-Henson-Rubel [80] choose to use the collection of all “ordinal definable functions” and in this context one again obtains a similar theory by working in the very specific model of set theory known as the *Solovay model*.

3. Parametrizing complex manifolds

In order to precisely formulate the problem of classifying species of complex manifolds we need a method of parametrizing complex manifolds. Here the method must be completely general, and in some sense given in advance of any specific classification result. We will use points in a standard Borel space—indeed without loss of generality we can take the space \mathbb{R} —to describe or *parametrize* complex manifolds. The obvious properties will become Borel in the space of parametrizations and the equivalence relation of biholomorphism will be Σ_1^1 (analytic)—that is to say, defined by the Borel image of a Borel set. It should not be felt that there is any great mystery or presumption in the method of parametrization. It is an empirical fact that most classes of concrete mathematical objects can be represented by points in a standard Borel space and all such methods of representation tend to be “Borel equivalent”.

We recount some basic definitions, all detailed in Kodaira [86].

Definition 3.1. A complex manifold of dimension n is a connected Hausdorff second countable topological space M along with a chart $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \Lambda}$, where:

- (i) $\bigcup_{\alpha \in \Lambda} U_\alpha = M$;
- (ii) each $\varphi_\alpha: U_\alpha \rightarrow V_\alpha$ is a homeomorphism between $U_\alpha \subseteq M$, open in M , and $V_\alpha \subseteq \mathbb{C}^n$, open in \mathbb{C}^n ;
- (iii) the overlap maps

$$\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta), \text{ when } U_\alpha \cap U_\beta \neq \emptyset,$$

are holomorphic.

A complex manifold of dimension 1 is called a *Riemann surface*. For M and N complex manifolds, with charts $\{U_\alpha, \varphi_\alpha\}$, $\{W_\beta, \psi_\beta\}$, we say that $\tau: M \rightarrow N$ is *holomorphic* if each $\psi_\beta \tau \varphi_\alpha^{-1}$ is holomorphic. A *biholomorphic* map is a holomorphic map with holomorphic inverse. We shall write

$$M \cong N$$

if they are *biholomorphically equivalent*, that is to say, there is some biholomorphic

$$\pi: M \rightarrow N.$$

For $n = 1$, i.e., for Riemann surfaces, one usually says that M, N are *conformally equivalent* if $M \cong N$.

A manifold is a “separable” object, and thus we might hope to represent or parametrize a manifold by describing the arrangement of a countable dense subset. There is some judgment regarding the kind of objects to take as satisfactory parameterizations; plainly the class of all manifolds would be absurdly complicated and the set of all subsets of say $\mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^3 \dots$ only slightly less so. Our idea of a *reasonable* space is something like \mathbb{R} , or the set of all subsets of \mathbb{N} , or $C([0, 1])$, or infinite strings from a finite alphabet, or any of the classical Banach spaces. In each case we are less interested in the topological structure of these spaces than their *Borel structure*.

Definition 3.2. A set S equipped with a σ -algebra \mathcal{B} of subsets is said to be a *standard Borel space* if there is a completely metrizable separable topology τ on S which gives rise to \mathcal{B} as the σ -algebra generated by its open sets.

As a remark on terminology, we say that a completely metrizable separable topological space is *Polish*, and that the *Borel sets* for a Polish topology are those appearing in the smallest σ -algebra containing its open sets. Thus we may say that (S, \mathcal{B}) is a standard Borel space if there is a Polish topology τ on S that has \mathcal{B} as its Borel sets.

As well as the usual examples of Polish spaces, such as $\mathbb{R}^n, \mathbb{C}^n, 2^{\mathbb{N}}$, stripped down to their Borel structure, we should also mention that given any Polish space X the collection of all closed subsets, $\mathcal{F}(X)$, is a standard Borel space, when equipped with the σ -algebra generated by sets of the form $\{F \in \mathcal{F}(X): F \cap U \neq \emptyset\}$ for $U \subseteq X$ open; this more subtle example is known as the Effros standard Borel space and is discussed in §12.C of Kechris [95]. It follows that the space of all *open* subsets of X is a standard Borel space in the σ -algebra generated by sets of the form $\{O \subseteq X \text{ open}: O \cap U = \emptyset\}$.

The collection of standard Borel spaces is closed under countable disjoint unions and countable products. As a consequence of §13.A in Kechris [95], any Borel subset of a standard Borel space is again standard Borel in the induced subspace Borel structure.

We can then proceed to define for each $n \geq 1$, a standard Borel space \mathcal{M}^n and a map

$$p \in \mathcal{M}^n \mapsto M_p$$

assigning to each $p \in \mathcal{M}^n$ an n -dimensional complex manifold M_p such that for every n -dimensional complex manifold M there is at least one $p \in \mathcal{M}^n$ with $M \cong M_p$. We call \mathcal{M}^n the *parameter space of n -dimensional complex manifolds*. The construction of \mathcal{M}^n and the verification that it has a number of reasonable properties that we will need in various parts of this paper is technically cumbersome, although mathematically rather shallow. We will thus postpone the precise definition and verification of these facts until §7. In our proofs between now and then we will simply state as lemmas the various facts about this coding that we need and return to their proofs in §7.

To start off, the following fact gives an upper bound for the complexity of biholomorphic equivalence.

PROPOSITION 3.3. *For each $n \geq 1$, the relation*

$$p \cong_n q \Leftrightarrow p, q \in \mathcal{M}^n \text{ and } M_p \cong M_q$$

is Σ_1^1 .

For $n = 1$ we will let \mathcal{M}^1 be denoted by \mathcal{R} and \cong_1 by $\cong_{\mathcal{R}}$ (for Riemann surfaces) and we will see in §4 that actually $\cong_{\mathcal{R}}$ is Borel, i.e., the relation of conformal equivalence of Riemann surfaces is Borel (in the parameters). However we doubt if this is true in higher dimensions.

CONJECTURE 3.4. *For $n \geq 2$, the biholomorphic equivalence relation \cong_n of n -dimensional complex manifolds (in the parameters) is not Borel.*

In this paper we will also discuss a particular class of Riemann surfaces, namely *domains* (open connected sets) in \mathbb{C} . Again we will need to parametrize these by elements of a standard Borel space. We will thus define in §7 a standard Borel space \mathcal{D} and a map $d \in \mathcal{D} \mapsto D_d$ assigning to each $d \in \mathcal{D}$ a domain $D_d \subseteq \mathbb{C}$ such that for each domain $D \subseteq \mathbb{C}$ there is at least one $d \in \mathcal{D}$ with $D = D_d$. We call \mathcal{D} the *parameter space of domains in \mathbb{C}* . Of course for every domain D there is also a $p \in \mathcal{R}$ such that $M_p \cong D$. The next fact asserts that such a p can be computed in a Borel way from any $d \in \mathcal{D}$ such that $D = D_d$.

PROPOSITION 3.5. *There is a Borel function $f: \mathcal{D} \rightarrow \mathcal{R}$ such that $D_d \cong M_{f(d)}$.*

In particular if we denote by \cong_D the relation of conformal equivalence (in the parameters), i.e.,

$$d \cong_D e \Leftrightarrow d, e \in \mathcal{D} \text{ \& } D_d \cong D_e,$$

then we have

$$(\cong_D) \leq_B (\cong_{\mathcal{R}}).$$

4. Riemann surfaces

Our main goal in this section is to compute the precise complexity of conformal equivalence of Riemann surfaces and planar domains in the hierarchy of Borel equivalence relations.

THEOREM 4.1. *The conformal equivalence relations $\cong_{\mathcal{R}}, \cong_D$ (in the parameters) of Riemann surfaces and planar domains are Borel and*

$$(\cong_{\mathcal{R}}) \sim_B (\cong_D) \sim_B E_{\infty}.$$

We will split the proof of this theorem in two parts:

The *upper bound*, i.e., showing that \cong_R is Borel and $(\cong_R) \leq_B E_\infty$.

The *lower bound*, i.e., showing that $E_\infty \leq_B (\cong_D)$.

Since, as we already pointed out in §3, $(\cong_D) \leq_B (\cong_R)$ this will complete the proof of 4.1.

4.A. *The upper bound* We will need some basic facts from the Uniformization Theory of Riemann Surfaces (see, e.g., Forster [81]).

Given a Riemann surface M , we will denote by \widehat{M} its *universal covering* Riemann surface and by $\pi: \widehat{M} \rightarrow M$ the *covering map*. Thus (\widehat{M}, M, π) has the following properties:

- (i) \widehat{M} is simply connected and π is a holomorphic surjection of \widehat{M} onto M .
- (ii) π *evenly covers* M , i.e., for each $x \in M$ there is open U containing x , so that $\pi^{-1}[U]$ is the disjoint union of open sets $\{V_i\}$, where each $\pi|_{V_i}: V_i \rightarrow U$ is a biholomorphism.
- (iii) \widehat{M} is uniquely, up to biholomorphism, determined by (i), (ii) above, i.e., if $f: M \rightarrow N$ is a biholomorphism and (\widehat{N}, N, ρ) satisfy (i), (ii) above, then there is a biholomorphism $\widehat{f}: \widehat{M} \rightarrow \widehat{N}$ such that $\rho \circ \widehat{f} = f \circ \pi$. Moreover, for each $\widehat{x}_0 \in \widehat{M}$, $\widehat{y}_0 \in \widehat{N}$ with $f(\pi(\widehat{x}_0)) = \rho(\widehat{y}_0)$, there is a unique \widehat{f} as above with $\widehat{f}(\widehat{x}_0) = \widehat{y}_0$.

The following fundamental result is variously called the *Uniformization Theorem* or the *Riemann Mapping Theorem for Riemann Surfaces*.

THEOREM 4.2. *Every simply connected Riemann surface is conformally equivalent to exactly one of the following:*

- (i) $\mathbb{C}_\infty =$ *the Riemann Sphere* ($= \mathbb{C} \cup \{\infty\}$);
- (ii) $\mathbb{C} =$ *the complex plane*;
- (iii) $\mathbb{H} = \{x + iy : y > 0\} =$ *the upper half plane*.

The group of automorphisms (i.e., biholomorphic correspondences) of a Riemann surface M will be denoted by $\text{Aut}(M)$. It turns out that the automorphism groups of the simply connected Riemann surfaces can be explicitly described as follows (see, e.g., Beardon [84], Bedford et al. [91]).

- (i) $\text{Aut}(\mathbb{C}_\infty) = PSL_2(\mathbb{C})$.

Here $PSL_2(\mathbb{C})$ is the quotient of the group $SL_2(\mathbb{C})$ of all complex matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$ by its center $\{+I, -I\}$, where I is the identity matrix. This acts on \mathbb{C}_∞ as a *Möbius transformation*, i.e., for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{C})$, $z \in \widehat{\mathbb{C}}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}.$$

(ii) $\text{Aut}(\mathbb{C}) =$ the “ $az + b$ ” group.

This is the group of all pairs $(a, b) \in \mathbb{C}^* \times \mathbb{C}$, when $\mathbb{C}^* = \mathbb{C} - \{0\}$, with multiplication $(a, b) \cdot (a', b') = (aa', ab' + b)$. It acts on \mathbb{C} by $(a, b)(z) = az + b$.

(iii) $\text{Aut}(\mathbb{H}) = PSL_2(\mathbb{R})$.

Here $PSL_2(\mathbb{R})$ is the quotient of the group $SL_2(\mathbb{R})$ of all real matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$ by its center $\{+I, -I\}$. This again acts on \mathbb{H} by Möbius transformations.

In particular, these automorphism groups are Lie groups, therefore Polish locally compact, and their actions on the underlying spaces are continuous.

Now consider a Riemann surface M and a universal covering $\pi: \widehat{M} \rightarrow M$, where we now take M to be one of $\mathbb{C}_\infty, \mathbb{C}, \mathbb{H}$. Let

$$F_\pi = \{(x, y) \in \widehat{M} : \pi(x) = \pi(y)\}.$$

Then F_π is a closed subset of \widehat{M}^2 , i.e., $F_\pi \in \mathcal{F}(\widehat{M}^2)$. Now let $f: M \rightarrow N$ be a biholomorphic map and $\rho: \widehat{N} \rightarrow N$ a universal covering. Then $\widehat{N} = \widehat{M}$ and there is $\sigma \in \text{Aut}(\widehat{M})$ such that $\pi = \rho \circ \sigma$. Thus

$$\begin{aligned} (x, y) \in F_\rho &\Leftrightarrow \rho(x) = \rho(y) \\ &\Leftrightarrow \pi(\sigma^{-1}(x)) = \pi(\sigma^{-1}(y)) \\ &\Leftrightarrow (\sigma^{-1}(x), \sigma^{-1}(y)) \in F_\pi \\ &\Leftrightarrow (x, y) \in \sigma \cdot F_\pi, \end{aligned}$$

where $\text{Aut}(\widehat{M})$ acts on $\mathcal{F}(\widehat{M}^2)$ by

$$\sigma \cdot F = \{(x, y) : (\sigma^{-1}(x), \sigma^{-1}(y)) \in F\}$$

Thus F_ρ, F_π belong to the same orbit of this action.

Conversely, suppose $\pi: \widehat{M} \rightarrow M, \rho: \widehat{M} \rightarrow N$ are two universal coverings and for some $\sigma \in \text{Aut}(\widehat{M}), F_\rho = \sigma \cdot F_\pi$. Then we can define $f: M \rightarrow N$ by

$$f(x) = \rho(\sigma(y)), \text{ where } \pi(y) = x.$$

Notice that this is well defined, since if $\pi(y) = \pi(y') = x$, i.e., $(y, y') \in F_\pi$, then $(\sigma(y), \sigma(y')) \in F_\rho$, so $\rho(\sigma(y)) = \rho(\sigma(y'))$. It is now clear that f is a biholomorphism between M, N .

Thus we have seen that fixing $\widehat{M} \in \{\mathbb{C}_\infty, \mathbb{C}, \mathbb{H}\}$ there is an injection from the set of conformal equivalence classes of Riemann surfaces with universal covering surface \widehat{M} into the orbits of the action of $\text{Aut}(\widehat{M})$ on $\mathcal{F}(\widehat{M}^2)$. We have thus essentially reduced our problem to the study of this action. But then we can use the following general result:

THEOREM 4.3 (Kechris [92]). *Let G be a Polish locally compact group, X a standard Borel space, $(g, x) \mapsto g \cdot x$ a Borel action of G on X . Then there is a Borel set $S \subseteq X$ which meets every orbit in a countable nonempty set.*

This implies the following, concerning the orbit equivalence relation E_G^X :

$$x E_G^X y \Leftrightarrow \exists g \in G (g \cdot x = y).$$

First, it is well known that E_G^X is Borel (see Kechris [95, 35.49]). Thus, $E_G^X|_S$ is countable Borel and satisfies $(E_G^X|_S) \leq_B E_\infty$. Also there is a Borel function $f: X \rightarrow S$ with $f(x) E_G^X x$, for all $x \in X$ (see Kechris [95, 18C]). Thus $E_G^X \leq_B E_G^X|_S$, so

$$E_G^X \leq_B E_\infty$$

In particular, this applies to the action of $\text{Aut}(\widehat{M})$ on $\mathcal{F}(\widehat{M}^2)$, where $\widehat{M} \in \{\mathbb{C}_\infty, \mathbb{C}, \mathbb{H}\}$, since this is clearly a Borel action.

One way to proceed then is to show the following:

- (i) For each $\widehat{M} \in \{\mathbb{C}_\infty, \mathbb{C}, \mathbb{H}\}$, the set $U_{\widehat{M}} = \{z \in \mathcal{R}: M_z \text{ has universal covering surface } \widehat{M}\}$ is Borel.
- (ii) There is a Borel function $F: U_{\widehat{M}} \rightarrow \mathcal{F}(\widehat{M}^2)$ so that $F(z)$ is of the form F_π , for some covering $\pi: \widehat{M} \rightarrow M$.

Then letting $E_{\widehat{M}}$ be the orbit equivalence relation induced by the action of $\text{Aut}(\widehat{M})$ on $\mathcal{F}(\widehat{M}^2)$, for $z, w \in U_{\widehat{M}}$ we have

$$\begin{aligned} z \cong_R w &\Leftrightarrow M_z \cong M_w \\ &\Leftrightarrow F(z) E_{\widehat{M}} F(w), \end{aligned}$$

i.e., $(\cong_R) \leq_B E_{\widehat{M}}$ and since $E_{\widehat{M}} \leq_B E_\infty$ we have that $(\cong_R) \leq_B E_\infty$.

The drawback of this approach is that the construction of the function F in (ii) above, although intuitively rather clear, involves some messy computations, which we don't want to commit to print. So we will follow an alternative, somewhat indirect, approach that will minimize the technicalities.

We will use the following criterion (a proof of which can be found in Hjorth [9?a, 5.2]).

PROPOSITION 4.4 (Kechris). *For a Borel equivalence relation E on a standard Borel space X , the following are equivalent:*

- (i) $E \leq_B E_\infty$.
- (ii) *There is a Borel function $f: X \rightarrow Y$, with Y a standard Borel space, such that for any $x, y \in X$,*

- (a) $f([x]_E)$ is countable, and
- (b) $\neg x E y \Rightarrow f([x]_E) \cap f([y]_E) = \emptyset$.

Put $\widehat{M}_1 = \mathbb{C}_\infty$, $\widehat{M}_2 = \mathbb{C}$, $\widehat{M}_3 = \mathbb{H}$. Consider the action of $\text{Aut}(\widehat{M}_i)$ on $\mathcal{F}(\widehat{M}_i^2)$ and let E_i be the corresponding orbit equivalence relation. Also let S_i be Borel sets that meet every E_i -class in a countable nonempty set. Then there are Borel functions $f_{i,j}$, $i = 1, 2, 3$, $j = 1, 2, \dots$, $f_{i,j}: \mathcal{F}(\widehat{M}_i^2) \rightarrow \mathcal{F}(\widehat{M}_i^2)$ such that $\{f_{i,1}(F): j = 1, 2, \dots\} = S_i \cap [F]_{E_i}$ (see Kechris [95, 18.C]). We can of course assume that $f_{i,1}(F) = F$ for $F \in S_i$. Thus if we put $f_i = f_{i,1}$, f_i satisfies (ii) of 4.4 for E_i . Now define the following relation $P \subseteq \mathcal{R} \times Y$, where $Y = S_1 \cup S_2 \cup S_3$, and we of course consider S_1, S_2, S_3 as pairwise disjoint:

$$P(z, y) \Leftrightarrow \exists i \in \{1, 2, 3\} \exists \pi: \widehat{M}_i \rightarrow M_z \ (\pi \text{ is a covering map and } y = f_i(F_\pi)).$$

Denote by P_z the section of P determined by $z \in \mathcal{R}$. Then if $z_1 \cong_R z_2$ and $y_1 \in P_{z_1}$, $y_2 \in P_{z_2}$, say $y_1 = f_i(F_{\pi_1})$, $y_2 = f_i(F_{\pi_2})$ (for some $i \in \{1, 2, 3\}$), then clearly $F_{\pi_1} E_i F_{\pi_2}$, so $y_1 \in f_i([F_{\pi_1}]_{E_i}) = f_i([F_{\pi_2}]_{E_i}) \ni y_2$. Thus $P_{z_1}, P_{z_2} \subseteq f_i([F_{\pi_1}]_{E_i})$. It follows that for each $z \in \mathcal{R}$, $\bigcup_{w \cong_R z} P_w = \bar{P}_z$ is countable.

Now, if $\bar{P}_{z_1} \cap \bar{P}_{z_2} \neq \emptyset$ and $y \in \bar{P}_{z_1} \cap \bar{P}_{z_2}$, then $y = f_i(F_{\pi_1}) = f_i(F_{\pi_2})$, where $i \in \{1, 2, 3\}$ and $\pi_1: \widehat{M}_i \rightarrow M_{z'_1}$, $\pi_2: \widehat{M}_i \rightarrow M_{z'_2}$ are covering maps with $z'_1 \cong_R z_1$, $z'_2 \cong_R z_2$. Then $F_{\pi_1} E_i F_{\pi_2}$, so $z'_1 \cong_R z'_2$, and thus $z_1 \cong_R z_2$.

So we have:

- (i) $z \in \mathcal{R} \Rightarrow \bar{P}_z$ is countable;
- (ii) $z_1 \not\cong_R z_2 \Rightarrow \bar{P}_{z_1} \cap \bar{P}_{z_2} = \emptyset$.

We can now complete the proof by showing the following two facts:

- (a) \cong_R is Borel;
- (b) P is Borel.

Indeed, from (b) and the fact that every section of P is countable, we can find a Borel function $f: \mathcal{R} \rightarrow Y$ such that f uniformizes P , i.e.,

$$P(z, f(z))$$

for any $z \in \mathcal{R}$. Then clearly f satisfies condition (ii) of 4.4 and so by (a) above we can apply 4.4 to conclude that $(\cong_R) \leq_B E_\infty$.

Proof of (a). We need the following lemma concerning our parametrization, which will be proved in §7.

LEMMA 4.5. Let $T_i \subseteq \mathcal{R} \times \mathcal{F}(\widehat{M}_i^2)$ be defined by

$$(z, F) \in T_i \Leftrightarrow \exists \pi: \widehat{M}_i \rightarrow M_z \ (\pi \text{ is a covering map and } F = F_\pi).$$

Then T_i is Σ_1^1 .

It follows that if $C_i \subseteq \mathcal{R}$ is defined by

$$z \in C_i \Leftrightarrow \text{the universal cover of } M_z \text{ is } \widehat{M}_i,$$

then $C_i = \text{proj}_{\mathcal{R}}(T_i)$, so $C_i \in \Sigma_1^1$ as well. Since $\{C_1, C_2, C_3\}$ is a partition of \mathcal{R} , it follows that actually each C_i is Borel.

We have already seen in §2 that \cong_R is Σ_1^1 , so to show that it is Borel it is enough to show that it is Π_1^1 . Recall that the equivalence relations E_i induced by the action of $\text{Aut}(\widehat{M}_i)$ on $\mathcal{F}(\widehat{M}_i^2)$ are Borel. Then we have

$$z_1 \cong_R z_2$$

$$\Leftrightarrow \exists i \in \{1, 2, 3\} [z_1, z_2 \in C_i \ \& \ \forall F_1 \forall F_2 [(z_1, F_1) \in T_i \ \& \ (z_2, F_2) \in T_i \Rightarrow F_1 E_i F_2]]$$

which shows that \cong_R is Π_1^1 .

Proof of (b). We have

$$P(z, y) \Leftrightarrow \exists i \in \{1, 2, 3\} [(z, F) \in T_i \ \& \ f_i(F) = y],$$

so clearly P is Σ_1^1 . To see that it is also Π_1^1 , notice that

$$P(z, y) \Leftrightarrow \exists i \in \{1, 2, 3\} [z \in C_i \ \& \ \forall F ((z, F) \in T_i \Rightarrow \exists j (f_i(f_{i,j}(F)) = y))].$$

4.B. The lower bound We will use the following realization of E_∞ :

Recall that we let F_2 be the free group with 2 generators and let $p(F_2)$ be the set of all subsets of F_2 with the topology given by its identification with 2^{F_2} , the latter having the product topology. Consider the left-translation action of F_2 on $p(F_2)$,

$$g \cdot A = gA,$$

which is clearly continuous. Remember that we denote by $E(F_2, 2)$ the associated orbit equivalence relation and we identify $E(F_2, 2)$ with E_∞ . It is therefore enough to show that

$$E(F_2, 2) \leq_B (\cong_D).$$

To do this we will associate to each $A \subseteq F_2$ a discrete subset $S_A \subseteq \mathbb{H}$ and consider the domain $D_A = \mathbb{H} \setminus S_A$. We will show that

$$A E(F_2, 2) B \Leftrightarrow D_A \cong D_B$$

(where $D_A \cong D_B$ means of course that D_A, D_B are conformally equivalent). Moreover the construction is very explicit, so that there is a Borel function $f: p(F_2) \rightarrow \mathbb{H}^{\mathbb{N}}$ such that for any $A \subseteq F_2$, $f(A)$ enumerates S_A . We will prove in §7 the following easy fact about our parametrization of domains.

PROPOSITION 4.6. *There is a Borel function $g: \mathbb{H}^{\mathbb{N}} \rightarrow \mathcal{D}$ so that for any $x \in \mathbb{H}^{\mathbb{N}}$ with $\{x_n: n \in \mathbb{N}\}$ discrete we have $D_{g(x)} = \mathbb{H} \setminus \{x_n: n \in \mathbb{N}\}$.*

Then if $h = g \circ f$ we have

$$A E(F_2, 2)B \Leftrightarrow h(A) \cong_D h(B),$$

i.e., $E(F_2, 2) \leq_B (\cong_D)$.

We will now proceed to the construction of $A \mapsto S_A$.

Consider the group $PSL(2, \mathbb{Z})$ of all integer matrices in $PSL(2, \mathbb{R}) = \text{Aut}(\mathbb{H})$. This acts properly discontinuously on \mathbb{H} , thus each orbit of this action is a discrete subset of \mathbb{H} (see, e.g., Katok [92]). Moreover, there are subgroups of $PSL(2, \mathbb{Z})$ isomorphic to F_2 which have no fixed points under this action. An example is the group generated by the two generators

$$\sigma = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

(see Wagon [93, p. 61, 7.1]). From now on we will identify F_2 with the subgroup $\langle \sigma, \tau \rangle$ of $PSL(2, \mathbb{Z})$ generated by σ, τ . Then for any fixed $x \in \mathbb{H}$, $\{g(x): g \in F_2\}$ is discrete and $g(x) = x$ iff $g = 1$ (the identity of F_2).

So from now on fix $x_1 \in \mathbb{H}$ and put $x_g = g(x)$. Then $g(x_h) = x_{gh}$.

Next we consider the hyperbolic metric ρ on \mathbb{H} given by

$$\rho(z, w) = \ln \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}.$$

It of course also induces the usual topology on \mathbb{H} . The main fact that we will use is that each element of $\text{Aut}(\mathbb{H}) = PSL(2, \mathbb{R})$ acts by isometries on (\mathbb{H}, ρ) (again, see Katok [92]). Since $\{x_g: g \in F_2\}$ is discrete, there is an $\epsilon > 0$ such that $\{y \in \mathbb{H}: \rho(x_1, y) < \epsilon\} \cap \{x_g: g \in F_2\} = \{x_1\}$. It follows that for any $g \in F_2$, $\{y \in \mathbb{H}: \rho(x_g, y) < \epsilon\} \cap \{x_g: g \in F_2\} = \{x_g\}$.

Next put $x_1^{(0)} = x_1$ and choose three points $x_1^{(1)}, x_1^{(2)}, x_1^{(3)}$ in \mathbb{H} , distinct from each other and from $x_1^{(0)}$, such that $\rho(x_1^{(0)}, x_1^{(i)}) < \frac{\epsilon}{5}$ and the hyperbolic distances $\epsilon_{ij} = \rho(x_1^{(i)}, x_1^{(j)})$, $0 \leq i \neq j \leq 3$, are all different from each other, except for $\epsilon_{ij} = \epsilon_{ji}$. Then define $x_g^{(i)}$, for $i = 0, 1, 2, 3$, by

$$x_g^{(i)} = g(x_1^{(i)}),$$

(so that $x_g^{(0)} = x_g$). Then $\rho(x_g^{(i)}, x_g^{(j)}) = \rho(x_1^{(i)}, x_1^{(j)})$. It follows that $\{y \in \mathbb{H}: \rho(y, x_g^{(0)}) < \frac{\epsilon}{2}\} \cap \bigcup_{h \in F_2} \{x_h^{(i)}: 0 \leq i \leq 3\} = \{x_g^{(i)}: 0 \leq i \leq 3\}$ and that the set $\bigcup_{g \in F_2} \{x_g^{(i)}: 0 \leq i \leq 3\}$ is also discrete. Notice also that $g(x_h^{(i)}) = x_{gh}^{(i)}$.

For any $A \in F_2$ put

$$S_A = \{x_g^{(i)} : g \in F_2, 0 \leq i \leq 2\} \cup \{x_h^{(3)} : h \in A\}.$$

We will show that this works.

First assume that $A \in E(F_2, 2)B$ and let $g \in F_2$ be such that $gA = B$. Then clearly $g(S_A) = S_B$ and so $g(D_A) = D_B$, thus $D_A \cong D_B$.

Conversely, assume that $D_A \cong D_B$ via the biholomorphism π . If Ω is a domain, \mathbb{D} the unit disk, $a \in \Omega$ and a function $f: \Omega \rightarrow \mathbb{D}$ is holomorphic in $\Omega \setminus \{a\}$, then by a classical theorem of Riemann (see, e.g., Rudin [66, 10.20]) a is a removable singularity, so f can be defined at a so that it remains holomorphic on Ω . Since \mathbb{H}, \mathbb{D} are conformally equivalent it follows that there are holomorphic extensions $\pi^+: \mathbb{H} \rightarrow \mathbb{H}, \pi^-: \mathbb{H} \rightarrow \mathbb{H}$ of π, π^{-1} , respectively. Then $\pi^- \circ \pi^+$ is the identity on D_A , and vice versa, so $(\pi^+)^{-1} = \pi^-$. Thus there is a biholomorphism $\pi^+ \in \text{Aut}(\mathbb{H})$ extending π . For convenience we will just write π instead of π^+ from now on. Clearly $\pi(S_A) = \pi(S_B)$. Next we will find $g \in F_2$ such that $\pi = g$. To find this g consider $\pi(x_1^{(0)})$. For some $g, \pi(x_1^{(0)}) = x_g^{(i)}, i \in \{0, 1, 2, 3\}$. We want to argue that $i = 0$, i.e., $\pi(x_1^{(0)}) = x_g^{(0)}$. So assume $i \neq 0$, towards a contradiction. Let us look at $\pi(x_1^{(i)})$. We have $\rho(\pi(x_1^{(i)}), x_g^{(i)}) = \rho(\pi(x_1^{(i)}), \pi(x_1^{(0)})) = \rho(x_1^{(i)}, x_1^{(0)}) < \frac{\epsilon}{5}$, so as $\rho(x_g^{(i)}, x_g^{(0)}) < \frac{\epsilon}{5}$, clearly $\rho(\pi(x_1^{(i)}), x_g^{(0)}) < \frac{\epsilon}{2}$, so that $\pi(x_1^{(i)})$ must be one of $x_g^{(j)}, j \in \{0, 1, 2, 3\}, j \neq i$. Thus, $\rho(x_g^{(i)}, x_g^{(j)}) = \epsilon_{ij} = \rho(x_1^{(0)}, x_1^{(i)}) = \epsilon_{0i}$, so, by the choice of the ϵ_{ij} , we must have $j = 0$, i.e., $\pi(x_1^{(i)}) = x_g^{(0)}, \pi(x_1^{(1)}) = x_g^{(0)}$. Then let $j \in \{1, 2\}, j \neq i$, and consider $\pi(x_1^{(j)})$. Again it must be one of the $x_g^{(k)}, k \neq i, 0$. But then $\epsilon_{k0} = \rho(x_g^{(k)}, x_g^{(0)}) = \rho(\pi(x_1^{(j)}), \pi(x_1^{(i)})) = \rho(x_1^{(j)}, x_1^{(i)}) = \epsilon_{ji}$, a contradiction.

So $\pi(x_1^{(0)}) = x_g^{(0)}$. Then since $\rho(\pi(x_1^{(1)}), x_g^{(0)}) = \rho(x_1^{(1)}, x_1^{(0)}) = \epsilon_{10}$ we must have $\pi(x_1^{(1)}) = x_g^{(1)}$ and similarly $\pi(x_1^{(2)}) = x_g^{(2)}$. Thus $\pi(x_1^{(i)}) = g(x_1^{(i)}), i \in \{0, 1, 2\}$. Since both π, g are Möbius transformations this implies that $\pi = g$. Finally we need to show that $gA = B$, and by symmetry it is enough to see that $gA \subseteq B$. If $h \in A$, then $x_h^{(3)} \in S_A$. So $g(x_h^{(3)}) = \pi(x_h^{(3)}) = x_{gh}^{(3)} \subseteq S_B$, so $gh \in B$.

It is obvious from our construction that there is a Borel function $f: p(F_2) \rightarrow \mathbb{H}^{\mathbb{N}}$ with $f(A)$ enumerating S_A , so the proof is complete. \square

5. Some consequences

5.A. We first notice that the result in §4 answers the following question in Becker-Henson-Rubel [80]:

Q10. Is it consistent with the ZFC axioms of set theory to assume that there is no complete system of invariants Φ , definable by a formula in set theory, where $\Phi(G)$ is a countable (unordered) subset of \mathbb{C} for each domain G ?

From 4.1 and the fact that any two uncountable standard Borel spaces are Borel isomorphic, it follows that there is a Borel function $f: \mathcal{D} \rightarrow \mathbb{C}$ and a countable Borel

equivalence relation E on \mathbb{C} such that

$$d \cong_R e \Leftrightarrow f(d)Ef(e).$$

In particular, if to any planar domain D we assign the invariant $\Phi(D) = f([d]_{\cong_R})$, for any $d \in \mathcal{D}$ with $D_d = D$, we obtain complete conformal invariants for planar domains which have the form of countable subsets of \mathbb{C} . Moreover, since the work in §4 is quite effective, both f and E are actually effectively Borel, or Δ_1^1 , and thus the invariant $\Phi(D)$ can be defined by an explicit simple formula in set theory. So Q10 is answered negatively in a strong form.

5.B. By loosening-up somewhat a definition given in Hjorth-Kechris [95], let us say that an equivalence relation E on a standard Borel space X is *Ulm-classifiable* if there is a map $U: X \rightarrow Y^{<\omega_1}$ where Y is a standard Borel space and $Y^{<\omega_1} = \{f: \alpha \rightarrow Y: \alpha < \omega_1\}$, with $xEy \Leftrightarrow U(x) = U(y)$, and such that U satisfies the following technical condition expressing the “niceness” of U :

- (i) If $\alpha(x) = \alpha^U(x) = \text{domain of } U(x) < \omega_1$, then for each $\alpha < \omega_1$,

$$A_\alpha = A_\alpha^U = \{x \in X: \alpha(x) = \alpha\}$$

has the universal Baire property and the map $U|_{A_\alpha}: X \rightarrow Y^\alpha$ is universally Baire measurable.

- (ii) The set

$$R_U = \{(x, y) \in X^2: \alpha(x) \leq \alpha(y)\}$$

has the universal Baire property in X^2 .

Here a set has the *universal Baire property* if its preimage by any Borel function (on any Polish space) has the Baire property and a function is *universally Baire measurable* if its pre-composition with any such Borel function is Baire measurable.

Recall that E_0 is the equivalence relation on $2^{\mathbb{N}}$ given by $x E_0 y \Leftrightarrow \exists n \forall m \geq n (x(m) = y(m))$. Then it is a folklore fact that E_0 is not Ulm-classifiable. To see this, first recall that E_0 is generically ergodic in the sense that any E_0 -invariant set with the property of Baire is either meager or comeager. Now assume that $U: 2^{\mathbb{N}} \rightarrow 2^{<\omega_1}$ verifies that E_0 is Ulm-classifiable. By the Kuratowski-Ulam Theorem (see Kechris [95]) applied to R_U , there is some $\alpha_0 < \omega_1$, with A_{α_0} non-meager. Thus, A_{α_0} is comeager. Since there is a Borel embedding from Y into $2^{\mathbb{N}}$ and so a Borel embedding of Y^{α_0} into $(2^{\mathbb{N}})^{\alpha_0}$ we can assume, without loss of generality, that $Y = 2$. Then for $\alpha < \alpha_0$, there is a unique $i_\alpha \in \{0, 1\}$ so that $\{x \in 2^{\mathbb{N}}: U(x)(\alpha) = i_\alpha\} = B_\alpha$ is comeager, so $B = \bigcap_{\alpha < \alpha_0} B_\alpha$ is comeager, which is absurd, since B is a single E_0 -equivalence class, thus countable.

Since $E_0 \leq_B E(F_2, 2) \sim_B (\cong_D) \sim_B (\cong_R)$, it follows that \cong_R, \cong_D are not Ulm-classifiable either, i.e., one cannot find complete conformal invariants for planar

domains which take the form of countable transfinite sequences of reals, complex numbers, or members of any standard Borel space, and which can be defined in a “reasonable” way.

In Becker-Henson-Rubel [80], the authors show that in the generic model $V[G]$, obtained by adding a Cohen real G to the universe V , there is no $U: \{\text{domains}\} \rightarrow \mathbb{C}^{<\omega_1}$ which is a complete system of invariants and can be defined in $V[G]$ by a formula in set theory with parameters in V . Their proof can be recast in our language as follows: If such a U existed, since $E_0 \leq_B (\cong_D)$ by a Δ_1^1 and thus explicitly defined function, we would have (in $V[G]$) a $\Psi: 2^{\mathbb{N}} \rightarrow 2^{<\omega_1}$ definable by a formula of set theory with parameters in V , such that $x E_0 y \Leftrightarrow \Psi(x) = \Psi(y)$. By standard homogeneity properties of the Cohen poset it follows that $\Psi(H_1) = \Psi(H_2)$ for any two Cohen generics $H_1, H_2 \in V[G]$, which is absurd if we take H_1, H_2 to be the odd, even parts of G .

Also since Shelah [84] showed that it is consistent relative to ZF+DC that every set in a Polish space has the property of Baire, it follows that it is consistent with ZF that there is no map (definable or not) from planar domains into $Y^{<\omega_1}$, Y a standard Borel space, which gives complete invariants for conformal equivalence.

5.C. We have seen in §4 that the conformal equivalence relation on domains of the form $\mathbb{H} \setminus S$, S discrete, has exactly the complexity of E_∞ . Of course the same holds for domains of the form $\mathbb{D} \setminus S$, S discrete in \mathbb{D} (\mathbb{D} the disc). In some sense these are the simplest Riemann surfaces for which conformal equivalence can be so complex.

First if we restrict our attention to the compact case, a concrete classification of \cong_R for compact Riemann surfaces can be achieved, and we refer the reader to the classical theory of moduli spaces (see, e.g., Imayoshi-Taniguchi [92]). A similar result holds for the Riemann surfaces obtained by subtracting finitely many points from a compact Riemann surface (in particular all domains of the form $\mathbb{C} \setminus \{x_1, \dots, x_n\}$) and domains of the form $\mathbb{H} \setminus \{x_1, \dots, x_n\}$. Next we consider domains of the form $\mathbb{C} \setminus S$, with S a discrete subset of \mathbb{C} . From the Picard Great Theorem, it follows that if $\mathbb{C} \setminus S, \mathbb{C} \setminus S'$ are conformally equivalent, then there is $\pi \in \text{Aut}(\mathbb{C})$ with $\pi(S) = S'$ and of course the converse is true as well. Thus, up to \sim_B , conformal equivalence on domains of the form $\mathbb{C} \setminus S$, S a discrete subset of \mathbb{C} , is the same as that of the orbit equivalence relation induced by the action

$$(g, S) \mapsto g \cdot S = g(S)$$

of $\text{Aut}(\mathbb{C})$ on the standard Borel space of discrete subsets of \mathbb{C} (a Borel subset of $\mathcal{F}(\mathbb{C})$). We will use this fact to show that conformal equivalence on the domains of the form $\mathbb{C} \setminus S$ is actually $<_B E_\infty$.

To see this, we will use the theory of amenability of countable Borel equivalence relations; see, e.g., Kechris [91].

As in that paper, we call a countable Borel equivalence relation E *amenable* if there is a map $C \mapsto \Phi_C$ associating to each E -equivalence class $C = [x]_E$ a mean Φ_C

on C (i.e., a continuous functional Φ_C on $l^\infty(C)$, with $\inf(f) \leq \Phi_C(f) \leq \sup(f)$) such that $C \mapsto \Phi_C$ is universally measurable in the following sense: if $F: X^2 \rightarrow [-1, 1]$ is Borel, the function $G: X \rightarrow [-1, 1]$ given by $G(x) = \Phi_{[x]_E}(F_x)$ (where $F_x: [x]_E \rightarrow \mathbb{R}$ is defined by $F_x(y) = F(x, y)$) is universally measurable. (A function is *universally measurable* if it is μ -measurable for any probability Borel measure μ .)

Since $E(F_2, 2)$ is not amenable (see Kechris [91, 2.3]) and amenability is preserved downwards under \leq_B , it follows that if E is amenable, then $E <_B E_\infty$.

Denote the space of discrete subsets of \mathbb{C} by $\mathcal{F}_d(\mathbb{C})$ and the orbit equivalence relation induced by the action of $\text{Aut}(\mathbb{C})$ on $\mathcal{F}_d(\mathbb{C})$ by E_d . Then, by 4.3, there is a Borel set $T \subseteq \mathcal{F}_d(\mathbb{C})$ meeting every E_d -equivalence class in a countable nonempty set. Then $E_d \sim_B E_d|T$ so in order to show that $E_d <_B E_\infty$, it is enough to show that $E_d|T = E$ is amenable. Since the statement “ $E_d <_B E_\infty$ ” is clearly Π_2^1 , so absolute under generic extensions, we can assume that the *Continuum Hypothesis*, CH, holds.

By a theorem of Mokobodzki (see Dellacherie-Meyer [83, pp. 102–108]), CH implies that there is a universally measurable shift-invariant mean $\Phi_{\mathbb{N}}$ on \mathbb{N} . This means that $\Phi_{\mathbb{N}}|[-1, 1]^{\mathbb{N}}: [-1, 1]^{\mathbb{N}} \rightarrow [-1, 1]$ is universally measurable and $\Phi_{\mathbb{N}}((x_0, x_1, \dots)) = \Phi_{\mathbb{N}}(x_1, x_2, \dots)$. Next we will use the fact that $\text{Aut}(\mathbb{C})$, the “ $az + b$ ” group, is solvable, thus amenable, and so it has a *Følner sequence* $\{K_n\}$. Thus $\{K_n\}$ is a sequence of compact subsets of $\text{Aut}(\mathbb{C})$ which have the following properties, where λ is the left-invariant Haar measure on $\text{Aut}(\mathbb{C})$:

- (i) $\lambda(K_n) > 0$;
- (ii) $\frac{\lambda(hK_n \Delta K_n)}{\lambda(K_n)} \rightarrow 0$ for each $h \in \text{Aut}(\mathbb{C})$ (see Patterson [88, 4.16]).

We now define a mean on $L^\infty(\text{Aut}(\mathbb{C}), \lambda)$, i.e., a continuous linear functional Λ on this Banach space with $\text{essinf}(f) \leq \Lambda(f) \leq \text{esssup}(f)$, as follows:

$$\Lambda(f) = \Phi_{\mathbb{N}} \left(n \mapsto \frac{\int_{K_n} f(g) d\lambda(g)}{\lambda(K_n)} \right).$$

Then property (ii) of the Følner sequence and the shift-invariance of $\Phi_{\mathbb{N}}$ imply that Λ is left-invariant, i.e., for any f as above, if $f_h(g) = f(hg)$, then

$$\Lambda(f) = \Lambda(f_h), \quad \forall h \in \text{Aut}(\mathbb{C}).$$

Now let $\pi: \mathcal{F}_d(\mathbb{C}) \rightarrow T$ be Borel such that $\pi(x)E_d x, \forall x \in \mathcal{F}_d(\mathbb{C})$. Then for each $E (= E_d|T)$ -equivalence class $C = [x]_{E_d} \cap T$ define a mean Φ_C on $l^\infty(C)$ by the formula

$$\Phi_C(p) = \Lambda(g \mapsto p(\pi(g \cdot x)))$$

for any $p \in l^\infty(C)$, where $g \cdot x$ is the action of $\text{Aut}(\mathbb{C})$ on $\mathcal{F}_d(\mathbb{C})$.

The left-invariance of Λ shows that this is well-defined independently of the choice of x in the same orbit of the action of $\text{Aut}(\mathbb{C})$ on $\mathcal{F}_d(\mathbb{C})$. Next we check that $C \mapsto \Phi_C$

is universally measurable, and thus E is amenable. Fix Borel $F: T^2 \rightarrow [-1, 1]$. We have to verify that $G: T \rightarrow [-1, 1]$, given by $G(x) = \Phi_{[x]_E}(F_x) = \Lambda(g \mapsto F_x(\pi(g \cdot x))) = \Lambda(g \mapsto F(x, \pi(g \cdot x)))$, is universally measurable. From the definition of Λ we have

$$G(x) = \Phi_{\mathbb{N}} \left(n \mapsto \frac{\int_{K_n} F(x, \pi(g \cdot x)) d\lambda(g)}{\lambda(K_n)} \right).$$

Put

$$\varphi(n, x) = \frac{\int_{K_n} F(x, \pi(g \cdot x)) d\lambda(g)}{\lambda(K_n)}.$$

Then $\varphi: \mathbb{N} \times T \rightarrow [-1, 1]$ is Borel, so

$$\psi(x) = (n \mapsto B(n, x))$$

is a Borel function from T into $[-1, 1]^{\mathbb{N}}$. Since $G = \Phi_{\mathbb{N}} \circ \psi$, and universally measurable functions are closed under composition, G is universally measurable.

On the other hand, let $E(\mathbb{Z}, 2)$ be the equivalence relation induced by the shift on $p(\mathbb{Z})$ and consider the subset $X \subseteq p(\mathbb{Z})$ defined by

$$X = \{A \subseteq \mathbb{Z}: A \text{ contains two consecutive integers.}\}$$

Then for $A, B \in X$,

$$AE(\mathbb{Z}, 2)B \Leftrightarrow C \setminus A, C \setminus B \text{ are conformally equivalent.}$$

Now the map

$$A \in p(\mathbb{Z} \setminus \{\emptyset\}) \mapsto A' \in X$$

defined by

$$A' = A \cup \{n + 1: n \in A\}$$

clearly has the property that

$$AE(\mathbb{Z}, 2)B \Leftrightarrow A'E(\mathbb{Z}, 2)B,$$

thus $E(\mathbb{Z}, 2)|_{(p(\mathbb{Z}) \setminus \{\emptyset\})} \leq_B E(\mathbb{Z}, 2)|_X$. But it is known, see Dougherty-Jackson-Kechris [94], that

$$E_0 \sim_B E(\mathbb{Z}, 2)|_{(p(\mathbb{Z}) \setminus \{\emptyset\})},$$

from which it follows that E_0 is \leq_B the conformal equivalence of planar domains of the form $\mathbb{C} \setminus S$, S a discrete subset of \mathbb{C} . We in fact conjecture the following.

CONJECTURE 5.1. *The conformal equivalence relation on planar domains of the form $\mathbb{C} \setminus S$, S discrete in \mathbb{C} , is $\sim_B E_0$.*

Notice also that, as in 5.B above, the conformal equivalence relation on planar domains of the form $\mathbb{C} \setminus S$, S discrete, is not Ulm-classifiable.

5.D. As in Becker-Henson-Rubel [80, 6.8], we want to discuss some implications of our result to the conjugacy action on $PSL_2(\mathbb{R}) = \text{Aut}(\mathbb{H})$ on its discrete subgroups. Put

$$\mathcal{S}_d = \{F \subseteq PSL_2(\mathbb{R}) : F \text{ is a discrete subgroup}\}.$$

Then \mathcal{S}_d is Borel in $\mathcal{F}(PSL_2(\mathbb{R}))$, so it is a standard Borel space. Consider the conjugacy action of $PSL_2(\mathbb{R})$ on \mathcal{S}_d , which is clearly Borel, and denote by E_c the corresponding orbit equivalence relation. Thus,

$$FE_cG \Leftrightarrow F, G \text{ are conjugate discrete subgroups of } PSL_2(\mathbb{R}).$$

It is essentially shown in 6.8 of Becker-Henson-Rubel [80] that $E_0 \leq_B E_c$. Again we compute the exact complexity of E_c .

THEOREM 5.2. $E_c \sim_B E_\infty$.

Proof. By 4.3, $E_c \leq_B E_\infty$ and, by 4.B, $E_\infty \leq_B (\cong_R |C_3)$, where C_3 is the set of all $z \in \mathcal{R}$ with M_z having universal covering surface \mathbb{H} , so it is enough to show that $(\cong_R |C_3) \leq_B E_c$. (Recall here that the universal covering space of any domain of the form $\mathbb{H} \setminus S$, S discrete, must be \mathbb{H} .)

If $\pi: \mathbb{H} \rightarrow M$ is the universal covering of a Riemann surface M , let G_π be the covering group, i.e., the group of all $g \in \text{Aut}(\mathbb{H}) = PSL_2(\mathbb{R})$ with $\pi \circ g = \pi$. The group $G_\pi \subseteq PSL_2(\mathbb{R})$ is discrete and acts freely on \mathbb{H} . Moreover with the usual definition of \mathbb{H}/G_π , \mathbb{H}/G_π is conformally equivalent to M . Finally, if $G, F \subseteq PSL_2(\mathbb{R})$ are discrete subgroups acting freely on \mathbb{H} , then \mathbb{H}/G is conformally equivalent to \mathbb{H}/F iff G, F are conjugate in $PSL_2(\mathbb{R})$ (see Beardon [84]).

Next we will need the following technical lemma about our parametrization, that we will prove in §7.

LEMMA 5.3. *There is Borel map $\varphi: \mathcal{S}_d \rightarrow \mathcal{R}$ such that if $G \in \mathcal{S}_d$ acts freely on \mathbb{H} , then \mathbb{H}/G is conformally equivalent to $M_{\varphi(G)}$.*

Let now $T \subseteq X = \mathcal{S}_d \cap \{G : G \text{ acts freely on } \mathbb{H}\}$ be a Borel set meeting every E_c -equivalence class in X at a countable nonempty set. Define $Q \subseteq C_3 \times T$ by

$$(z, G) \in Q \Leftrightarrow \varphi(G) \cong_R z.$$

Thus Q is Borel and each section Q_z is countable, nonempty, so let $F: C_3 \rightarrow T$ be Borel with $(z, F(z)) \in Q$. Then

$$z \cong_R w \Leftrightarrow F(z)E_cF(w),$$

so $\cong_R |C_3 \leq_B E_c$, and we are done. \square

6. Complex manifolds

We will see here that for $n \geq 2$ the biholomorphic equivalence relation on n -dimensional complex manifolds is much more complicated than that of Riemann surfaces. We have the following result:

THEOREM 6.1. *For $n \geq 2$ the relation of biholomorphic equivalence \cong_n of n -dimensional complex manifolds (in the parameters) does not admit classification by countable structures.*

Proof. It is enough to consider the case $n = 2$. The proof will be an application of the theory of turbulence, a consequence of which is the following result.

PROPOSITION 6.2 (Hjorth [9?, 3.3.3]). *Consider the Polish group $G = (\mathbb{R}^{\mathbb{N}}, +)$ and let $H \subseteq G$ be a proper Polishable subgroup. If H is strongly dense (i.e., for every $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ there is $x_{n+1}, x_{n+2} \dots$ with $(x_0, x_1, \dots, x_n, x_{n+1}, \dots) \in H$), then the equivalence relation*

$$xE_Hy \Leftrightarrow x - y \in H$$

does not admit classification by countable structures.

Thus it will be enough to find such an H with

$$E_H \leq_B (\cong_2).$$

Consider the unit disk \mathbb{D} and the group $H_{\mathbb{D}}$ of all holomorphic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ with pointwise addition. It is easy (using, e.g., Rudin [66, 10.27]) to see that with the topology of uniform convergence on compacts this is a Polish group. Consider the closed subgroup $H_{\mathbb{D}}^*$ of all $f \in H_{\mathbb{D}}$ such that $f(1/n) \in \mathbb{R}$ for all $n \in \mathbb{N}$, $n \geq 2$. Define $p: H_{\mathbb{D}}^* \rightarrow \mathbb{R}^{\mathbb{N}}$ by $p(f) = (f(\frac{1}{n+2}))$. Then p is a continuous homomorphism from $H_{\mathbb{D}}^*$ into $\mathbb{R}^{\mathbb{N}}$ and so $p(H_{\mathbb{D}}^*) = H$ is a Polishable subgroup of $\mathbb{R}^{\mathbb{N}}$ (see Becker-Kechris [96, 1.6]) and is clearly proper (as the sequence $(1, 0, \dots) \notin H$). Next we verify that H is strongly dense in $\mathbb{R}^{\mathbb{N}}$: if $a_0, \dots, a_{m-1} \in \mathbb{R}$, then there is a polynomial f with real coefficients for which $f(\frac{1}{i+2}) = a_i$ for $i < m$, so

$$\left(f\left(\frac{1}{2}\right), \dots, f\left(\frac{1}{m+1}\right), f\left(\frac{1}{m+2}\right), \dots \right) = p(f) \in H.$$

So it is enough to show that

$$E_H \leq_B (\cong_2).$$

For $x = (x_n) \in \mathbb{R}^{\mathbb{N}}$, let $M(x)$ be the 2-dimensional complex manifold

$$M(x) = (\mathbb{D} \times \mathbb{C}) \setminus \left(\bigcup_{n \in \mathbb{N}} \left\{ \frac{1}{n+2} \right\} \times \{x_n + (n+2), x_n + (n+3), x_n + 2(n+2)\} \cup \{0\} \times \mathbb{R} \right)$$

with the obvious chart. The following easy fact about our parametrization will be checked in §7.

LEMMA 6.3. *There is a Borel function $F : \mathbb{R}^{\mathbb{N}} \rightarrow M^2$ such that*

$$M(x) = M_{F(x)}$$

Then it is enough to show that

$$x - y \in H \Leftrightarrow M(x) \cong M(y).$$

If $x - y \in H$, so that $y - x \in H$ let $f \in H_{\mathbb{D}}^*$ be such that $p(f) = y - x$, i.e., f is holomorphic on \mathbb{D} and $f(\frac{1}{n+2}) = y_n - x_n$. Then the map $\sigma : M(x) \rightarrow M(y)$ given by

$$\sigma(z, w) = (z, w + f(z))$$

is a biholomorphism between $M(x)$, $M(y)$, i.e., $M(x) \cong M(y)$.

Conversely assume that $M(x) \cong M(y)$ and $\sigma : M(x) \rightarrow M(y)$ is a biholomorphism. Fix $z \in \mathbb{D}$. The map $w \in \mathbb{C} \mapsto \pi_1(\sigma(z, w)) \in \mathbb{D}$, where $\pi_1 : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{D}$ is the first projection function, is holomorphic and bounded, so by Liouville's Theorem (see Rudin [66, 10.23]) it is constant, say $C(z)$. (Strictly speaking, when $z = \frac{1}{n+2}$ this map is only defined in the complement of a finite set, but it clearly has a holomorphic extension to all of \mathbb{C} .)

It is obvious that for each n , $C(\frac{1}{n+2}) = \frac{1}{m+2}$ for some m . Thus the map $w \in \mathbb{C} \setminus \{x_n + (n+2), x_n + (n+3), x_n + 2(n+2)\} \mapsto \pi_2(\sigma(w)) \in \mathbb{C} \setminus \{x_m + (m+2), x_m + (m+3), x_m + 2(m+2)\}$, where $\pi_2 : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{C}$ in the second projection function, is a biholomorphism, so clearly $\mathbb{C} \setminus \{n+2, n+3, 2(n+2)\}$, $\mathbb{C} \setminus \{m+2, m+3, 2(m+2)\}$ are conformally equivalent. Thus, as in 5.C, a transformation of the form $z \mapsto az + b$ of $\{n+2, n+3, 2(n+2)\}$ gives us $\{m+2, m+3, 2(m+2)\}$; by considering distances we see that $|a| = 1$ and therefore $a = 1$ and $m = n$, i.e., $C(\frac{1}{n+2}) = \frac{1}{n+2}$.

Since C is holomorphic (being given by $C(z) = \pi_1(\sigma(z, i))$), it follows that $C(z) = z$ for all $z \in \mathbb{D}$.

Thus σ has the form

$$\sigma(z, w) = (z, \rho_z(w))$$

where $\rho_z \in \text{Aut}(\mathbb{C})$, i.e., $\rho_z(w) = a(z)w + b(z)$. As $b(z) = \rho_z(0) = \pi_2(\sigma(z, 0))$ and $a(z) = -b(z) + \frac{\pi_2(\sigma(z, i))}{i}$, it follows that a, b are holomorphic. Since the map $a(\frac{1}{n+2})w + b(\frac{1}{n+2})$ maps $\{x_n + (n+2), x_n + (n+3), x_n + 2(n+2)\}$ onto $\{y_n + (n+2), y_n + (n+3), y_n + 2(n+2)\}$ it follows that $a(\frac{1}{n+2}) = 1$ and $x_n + (n+2) = b(\frac{1}{n+2}) + y_n + (n+2)$, or $x_n - y_n = b(\frac{1}{n+2})$, so $b \in H_{\mathbb{D}}^*$ and $x - y = p(b) \in H$, and the proof is complete. \square

It is clear that the complex manifold used above are not compact. So the following is open:

Problem 6.4. What is the complexity of the biholomorphic equivalence relation on 2-dimensional compact complex manifolds?

We do not even know if it is Borel or even concretely classifiable.

7. Technicalities of the parametrization

We start with some definitions.

Definition 7.1. Let Y_0^n be the set of all triples (O_0, O_1, φ) such that O_0 and O_1 are open sets in \mathbb{C}^n and $\varphi: O_0 \rightarrow O_1$ is holomorphic. We provide Y_0^n with the Borel structure generated by sets of the form

$$\{(O_0, O_1, \varphi): U \subseteq O_0\}$$

$$\{(O_0, O_1, \varphi): U \subseteq O_1\}$$

$$\{(O_0, O_1, \varphi): \varphi(V) \subseteq U\}$$

for $U, V \subseteq \mathbb{C}^n$ open.

For each n fix in advance some enumeration $(q_i^n)_{i \in \mathbb{N}}$ (or just $(q_i)_{i \in \mathbb{N}}$ when context makes n clear) of a dense subset of \mathbb{C}^n and let Y_1^n the collection of 4-tuples

$$(O_0, O_1, (c_i), (d_i)),$$

such that:

- (i) O_0, O_1 are open sets in \mathbb{C}^n , each c_i is in \mathbb{C}^n , each d_i is in $M_n(\mathbb{C})$ (the space of all $n \times n$ matrices over \mathbb{C}).
- (ii) For q_i not in O_0 the values of c_i and d_i are both $(0, 0, \dots)$ (or any other fixed default value in \mathbb{C}^n).
- (iii) There exists $\varphi: O_0 \rightarrow O_1$ analytic such that for each $q_i \in O_0$,

$$\varphi(q_i) = c_i,$$

$$\varphi'(q_i) = d_i.$$

Then letting $\mathcal{O}(\mathbb{C}^n)$ be the standard Borel space of open subsets of \mathbb{C}^n , we have that Y_1^n is a subspace of the standard Borel space $\mathcal{O}(\mathbb{C}^n) \times \mathcal{O}(\mathbb{C}^n) \times (\mathbb{C}^n)^{\mathbb{N}} \times M_n(\mathbb{C})^{\mathbb{N}}$. Here $\mathcal{O}(\mathbb{C}^n)$ is understood to have the Borel structure generated by sets of the form

$\{O \in \mathcal{O}(\mathbb{C}^n): U \subseteq O\}$ for U open in \mathbb{C}^n . To see that this is a standard Borel space define

$$\pi: \mathcal{F}(\mathbb{C}^n) \rightarrow \mathcal{O}(\mathbb{C}^n)$$

by

$$F \mapsto \mathbb{C}^n \setminus F;$$

this is obviously a Borel bijection with Borel inverse between $\mathcal{F}(\mathbb{C}^n)$ equipped with the Effros standard Borel structure and $\mathcal{O}(\mathbb{C}^n)$.

LEMMA 7.2. Y_1^n is a standard Borel space.

Proof. By Kechris [95, 13.A], it suffices to show that Y_1^n is a Borel subset of $\mathcal{O}(\mathbb{C}^n) \times \mathcal{O}(\mathbb{C}^n) \times (\mathbb{C}^n)^{\mathbb{N}} \times M_n(\mathbb{C})^{\mathbb{N}}$.

Let $(C_p)_{p \in \mathbb{N}}$ be the family of compact sets in \mathbb{C}^n consisting of all closures of basic open precompact sets. Then note that $(O_0, O_1, (c_i), (d_i)) \in Y_1^n$ if and only if for all C_p with $C_p \subseteq O_0$ we have the following:

(a) For all $\epsilon \in \mathbb{Q}^+$ there exists $\delta \in \mathbb{Q}^+$ such that

$$\forall q_i, q_j \in C_p (|q_i - q_j| < \delta \Rightarrow |d_i - d_j|, |c_i - c_j| < \epsilon).$$

(b) For all $\epsilon \in \mathbb{Q}^+$ there exists i_0, i_1, \dots, i_l , a finite sequence in \mathbb{N} , and $\delta_0, \delta_1, \dots, \delta_l$ in \mathbb{Q}^+ such that

$$C_p \subseteq \bigcup_{j \leq l} B_{\delta_j}(q_{i_j})$$

and for all $j \leq l$ and $q_m, q_{m'} \in B_{\delta_j}(q_{i_j})$ (the open ball of radius δ_j around q_{i_j}) if

$$(\zeta_1, \dots, \zeta_n) = (q_m - q_{m'}) \in \mathbb{C}^n,$$

$$(\zeta'_1, \dots, \zeta'_n) = (c_m - c_{m'}) \in \mathbb{C}^n,$$

$[\xi_{i,i'}]_{i,i' \leq n} = d_{i_j}$, then

$$\begin{aligned} & |(\zeta_1, \dots, \zeta_n)[\xi_{i,i'}] - (\zeta'_1, \dots, \zeta'_n)| / |q_m - q_{m'}| \\ &=_{df} \left| \left(\sum_{i \leq n} \zeta_i \xi_{i,1}, \sum_{i \leq n} \zeta_i \xi_{i,2}, \dots, \sum_{i \leq n} \zeta_i \xi_{i,n} \right) - (\zeta'_1, \dots, \zeta'_n) \right| / |q_m - q_{m'}| < \epsilon. \end{aligned}$$

Since all continuous functions on compact sets are uniformly continuous, (a) asserts that the assignments

$$q_i \mapsto c_i,$$

$$q_i \mapsto d_i$$

extend to continuous functions on C_p while (b) asserts that $q_i \mapsto d_i$ indicates the derivative of $q_i \mapsto c_i$. We leave to the reader the routine verifications that (a) and (b) describe Borel sets. \square

COROLLARY 7.3. *For each n , Y_0^n is a standard Borel space.*

Proof. The natural map from Y_1^n to Y_0^n that associates to each $(O_0, O_1, (c_i), (d_i))$ the unique holomorphic function φ with

$$\begin{aligned}\varphi(q_i) &= c_i \\ \varphi'(q_i) &= d_i\end{aligned}$$

for $q_i \in O_0$ is clearly one to one and an isomorphism with respect to the relevant Borel structures. \square

Definition 7.4. Let X'_0 be the space of all $(y_{i,i'})_{i,i' \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$ such that there exists a complete metric space with dense subset $\{a_i : i \in \mathbb{N}\}$, where for each i, i' ,

$$d(a_i, a_{i'}) = y_{i,i'}.$$

Note that X'_0 is a Borel subspace of $\mathbb{R}^{\mathbb{N} \times \mathbb{N}}$, since the only restrictions on the $(y_{i,i'})_{i,i' \in \mathbb{N}}$ is that they are ≥ 0 and satisfy the triangle inequality, and that $y_{i,i'} = 0$ exactly when $i = i'$. For each such $(y_{i,i'})$ we let $X(y_{i,i'})$ be a complete metric space with dense subset $\{a_i : i \in \mathbb{N}\}$ as indicated; the exact choice of $X(y_{i,i'})$ will be irrelevant in what follows.

We then let X_0 be the collection of all $(y_{i,i'}) \in X'_0$ such that for each $i \in \mathbb{N}$ the open ball

$$B_1(a_i)$$

of radius 1 around a_i in $X(y_{i,i'})$ is precompact.

Since precompactness can be phrased in terms of being ϵ -bounded for every rational $\epsilon > 0$, this is a Borel subset of X'_0 , and thus X_0 is a standard Borel space. It is also seen that $X(y_{i,i'})$ will be locally compact for each $(y_{i,i'})$ in X_0 . Conversely we have:

LEMMA 7.5. *If X is a locally compact separable metric space, then there is some $(y_{i,i'})$ in X_0 such that $X(y_{i,i'})$ is homeomorphic to X .*

Proof. Starting with X we may fix a complete d and an increasing union $(O_m)_{m \in \mathbb{N}}$ of precompact open sets so that for each m the closure of O_m , \bar{O}_m , is included in O_{m+1} and such that

$$X = \bigcup_{m \in \mathbb{N}} O_m.$$

Then we may choose continuous $f_m: X \rightarrow [m, m + 2]$ such that f_m is constantly m on \bar{O}_m and constantly $m + 2$ on $X \setminus O_{m+1}$. Then we may define d' on X by

$$d'(a, a') = d(a, a') + \sum_{m \in \mathbb{N}} |f_m(a) - f_m(a')|.$$

Letting $\{a_i: i \in \mathbb{N}\}$ be any dense subset of X , we obtain $(y_{i,i'})$ in X'_0 with

$$(y_{i,i'}) = d'(a_i, a_{i'}). \quad \square$$

The next technical definition is rather cumbersome. In any case, we ultimately parametrize the complex manifolds in a Borel manner, and this is the only real concern. By a *basic open set* in \mathbb{C}^n we mean a ball with rational center and radius.

Definition 7.6. Let \mathcal{M}^n be the space of sequences $((y_{i,i'}), (A_i), (\xi_{i,i'}))_{i,i' \in \mathbb{N}}$ (or just $((y_{i,i'}), (A_i), (\xi_{i,i'}))$ for short) such that:

- (i) $(y_{i,i'}) \in X_0$.
- (ii) $A_i \in 2^{\mathbb{N}}$ (which we equip with the product topology and identify with the set of all subsets of \mathbb{N}).
- (iii) $\xi_{i,i'} \in \mathbb{C}^n$.
- (iv) A_j corresponds to a regular open set in $X(y_{i,i'})$, in the sense that $i \in A_j$ if and only if $\exists \delta \in \mathbb{Q}^+ \forall i' \in \mathbb{N} \forall q \in \mathbb{Q}^+ (y_{i,i'} < \delta \Rightarrow \exists i'' \in A_j (y_{i',i''} < q))$.
- (v) $\{a_i: i \in A_j\}$ is guaranteed to be precompact by having diameter less than or equal to 1, in the sense that for all $i, i' \in A_j$,

$$y_{i,i'} < 1.$$

- (vi) If we define $U_j =_{df} \overline{\{a_i: i \in A_j\}}^\circ$ for each j to be the interior of the closure of $\{a_i: i \in A_j\}$ and $V_j =_{df} \overline{\{\xi_{j,i}: i \in A_j\}}^\circ$ for each j to be the interior of the closure of $\{\xi_{j,i}: i \in A_j\}$, then the assignment

$$a_i \mapsto \xi_{j,i}$$

extends to a homeomorphism

$$\varphi_j: U_j \rightarrow V_j$$

and the sets V_j are basic open and connected in \mathbb{C}^n .

- (vii) The overlap maps

$$\varphi_j \circ \varphi_i^{-1}: \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap V_j)$$

are biholomorphic.

- (viii) The topological space $X(Y_{i,i'})$ is connected.

- (ix) For each i the closed set $\overline{B_1(a_i)}$ is covered by finitely many open sets of the form U_j as described in (vi).

For $p \in \mathcal{M}^n$ we will use M_p to denote the complex manifold of dimension n , $X(y_{i,i'})$, with chart $\{U_i, \varphi_i\}$ as indicated above.

LEMMA 7.7. *\mathcal{M}^n is a standard Borel space.*

Proof. First let M_0^n be the collection of all $((y_{i,i'}), (A_i), (\xi_{i,i'}), (V_{i,j}), (\varphi_{i,j}))$ such that:

- (i) $((y_{i,i'}), (A_i), (\xi_{i,i'}))$ satisfy (i)-(vi) and (ix) of 7.6.
- (ii) For each i, j we have $(V_{i,j}, V_{j,i}, \varphi_{i,j})$ in Y_0^n .
- (iii) Each $V_{i,j}$ is the interior of the closure of $\{\xi_{i,l} : l \in A_j \cap A_i\}$.
- (iv) For each $l \in A_j \cap A_i$, we have

$$\varphi_{i,j}(\xi_{i,l}) = \xi_{j,l}.$$

- (v) For all i, i' there is a finite sequence $j_1 = i, j_2, \dots, j_k = i'$ such that each $A_{j_n} \cap A_{j_{n+1}} \neq \emptyset$.

Note that (v) says that M_p is connected. Thus $((y_{i,i'}), (A_i), (\xi_{i,i'}))$ is in \mathcal{M}^n if and only if there is some $(V_{i,j}), (\varphi_{i,j})$ with $((y_{i,i'}), (A_i), (\xi_{i,i'}), (V_{i,j}), (\varphi_{i,j}))$ in M_0^n and the functions

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$$

are equal to $\varphi_{i,j}$.

M_0^n is a subset of the standard Borel space $X_0 \times (2^{\mathbb{N}})^{\mathbb{N}} \times (\mathbb{C}^n)^{\mathbb{N} \times \mathbb{N}} \times (Y_0^n)^{\mathbb{N} \times \mathbb{N}}$ and so we endow it with the relative Borel structure. To see it is a standard Borel space we only need check that it is a Borel subset, and the first issue here is whether 7.6 (vi) corresponds to a Borel condition; this in turn follows using the precompactness of $\{a_l : l \in A_j\}$, as in the proof of 7.2. We also need to be concerned with showing 7.6 (ix) is Borel; but this amounts to the assertion that there are finite sequences $i_1, i_2, \dots, i_n, j_1, \dots, j_n$ in \mathbb{N} and $\delta_1, \dots, \delta_n \in \mathbb{Q}^+$, $\delta'_1, \dots, \delta'_n \in \mathbb{Q}^+$ such that:

- (a) Each $\delta_l < \delta'_l$.
- (b) For all $i' \in \mathbb{N}$ and $l \leq n$, if $y_{i,i'} < \delta'_l$ (i.e., $a_{i'} \in B_{\delta'_l}(a_{i_l})$), then $i' \in A_{j_l}$ (i.e., $a_{i'} \in U_{j_l}$); and so, by $\delta_l < \delta'_l$, we have $B_{\delta_l}(a_{i_l}) \subseteq U_{j_l}$.
- (c) For all i' with $y_{i',i} < 1$ (i.e., $a_{i'} \in B_1(a_i)$), we have some l with $y_{i',i_l} < \delta_l$ (i.e., $a_{i'} \in B_{\delta_l}(a_{i_l})$).

Granting all this, the projection function

$$((y_{i,i'}), (A_i), (\xi_{i,i'}), V_{i,j}, (\varphi_{i,j})) \mapsto ((y_{i,i'}), (A_i), (\xi_{i,i'}))$$

$$M_0^n \rightarrow X_0 \times (2^{\mathbb{N}})^{\mathbb{N}} \times (\mathbb{C}^n)^{\mathbb{N} \times \mathbb{N}}$$

is one-to-one, Borel, and has \mathcal{M}^n as its image. Thus, by Kechris [95, 15.1], \mathcal{M}^n is a Borel subset of $X_0 \times (2^{\mathbb{N}})^{\mathbb{N}} \times (\mathbb{C}^n)^{\mathbb{N} \times \mathbb{N}}$ and therefore standard Borel. \square

It is easily seen that for each M , a complex manifold of dimension n , there is some $p \in \mathcal{M}^n$ with M_p and M isomorphic. The only restriction on our charts is that the sets V_j from 7.6 (vi) are basic open and connected, but refining a chart as needed we may always make this assumption.

Recall that for $p, q \in \mathcal{M}^n$ we set $p \cong_n q$ if M_p and M_q are biholomorphic. We are finally in a position to verify the lemmas in Sections 3–6, whose proofs we postponed until now.

PROPOSITION 3.3. \cong_n is Σ_1^1 .

Proof. Fix $p = ((y_{i,i'}), (A_i), (\xi_{i,i'}))$ and $q = ((y'_{i,i'}), (A'_i), (\xi'_{i,i'}))$ in \mathcal{M}^n , let V_i, U_i and V'_i, U'_i be the respective sets from 7.6 (vi) for p and q , and let φ_i and φ'_i be the respective functions. Note that we may parametrize the holomorphic functions from M_p to M_q by sequences of 4-tuples $(V_{i,j}, W_{i,j}, \psi_{i,j}, k(i, j))$ with $(V_{ij}, W_{ij}, \psi_{ij})$ in Y_0^n and $k(i, j) \in \mathbb{N}$, such that:

- (i) V_i is the union of $\{V_{i,j} : j \in \mathbb{N}\}$.
- (ii) $W_{i,j} \subseteq V'_{k(i,j)}$.
- (iii) For $l \in A_i \cap A_{i^*}$, $\varphi_i(a_l) =_{df} \xi_{i,l}$, $\varphi_{i^*}(a_l) =_{df} \xi_{i^*,l}$, $\xi_{i,l} \in V_{i,j}$, $\xi_{i^*,l} \in V_{i^*,j^*}$, we have

$$(\varphi'_{k(i,j)})^{-1}(\psi_{i,j}(\xi_{i,l})) = (\varphi'_{k(i^*,j^*)})^{-1}(\psi_{i^*,j^*}(\xi_{i^*,l})).$$

(ii) and (iii) are clearly Borel; local compactness of the space gives the same conclusion for (i). The significance of (iii) is to ensure that the partial functions knit together in a well-defined fashion and yield a holomorphic function from M_p to M_q . So given $(V_{i,j}, W_{i,j}, \psi_{i,j}, k(i, j))_{i,j \in \mathbb{N} \times \mathbb{N}}$ as above we define

$$\psi: M_p \rightarrow M_q$$

extending the assignment

$$\psi(a_l) = (\varphi'_{k(i,j)})^{-1}(\psi_{i,j}(\xi_{i,l}))$$

for $l \in V_{i,j}$.

Thus we have a Borel subset, Z_0^n , of $\mathcal{M}^n \times \mathcal{M}^n \times (Y_0^n \times \mathbb{N})^{\mathbb{N} \times \mathbb{N}}$ that provides parameters for the collection of all pairs of complex n -dimensional manifolds and holomorphic functions between those spaces. For $(p, q, \vec{u}) \in Z_0^n$, let $\varphi_{\vec{u}}: M_p \rightarrow M_q$ be the resulting holomorphic function. Since it is easily seen that

$$Z_1^n = \{(p, q, \vec{u}, \vec{v}) : (p, q, \vec{u}), (q, p, \vec{v}) \in Z_0^n, \varphi_{\vec{u}} = (\varphi_{\vec{v}})^{-1}\}$$

is a Borel subset of $\mathcal{M}^n \times \mathcal{M}^n \times (Y_0^n \times \mathbb{N})^{\mathbb{N} \times \mathbb{N}} \times (Y_0^n \times \mathbb{N})^{\mathbb{N} \times \mathbb{N}}$, it follows that \cong_n , being the projection of Z_1^n (over \vec{u}, \vec{v}), is Σ_1^1 . \square

Definition 7.8. We put $\mathcal{D} = \{O \in \mathcal{O}(\mathbb{C}): O \text{ is connected}\}$. It is not hard to see that \mathcal{D} is a Borel subset of $\mathcal{O}(\mathbb{C})$. For $d \in \mathcal{D}$, we let $D_d = d$.

Recall that \mathcal{R} is just \mathcal{M}^1 , the space of parameters of Riemann surfaces.

PROPOSITION 3.5. *There is a Borel function $f: \mathcal{D} \rightarrow \mathcal{R}$ such that for each $d \in \mathcal{D}$, $D_d \cong M_{f(d)}$.*

Proof. For $d = U$ an open subset of \mathbb{C} , we can let (a_i) enumerate $U \cap (\mathbb{Q} + i\mathbb{Q})$ and let (V_i) enumerate the basic open sets of diameter < 1 included in U . We may then let $y_{i,i'}$ = $d(a_i, a_{i'})$, $A_j = \{i: a_i \in V_j\}$ and $\xi_{j,i} = a_i$ for $a_i \in V_j$. By local compactness of the spaces this can be done in a Borel in U fashion. Then (i)–(ix) of 7.6 follow easily. \square

LEMMA 4.5. *Let $\widehat{M}_1 = \mathbb{C}_\infty, \widehat{M}_2 = \mathbb{C}, \widehat{M}_3 = \mathbb{H}$. For each $i = 1, 2, 3$, let $T_i \subseteq \mathcal{R} \times \mathcal{F}(\widehat{\mathcal{M}}_i^2)$ be the set of (p, F) for which there exists*

$$\pi: \widehat{M}_i \rightarrow M_p$$

with π a covering map and $F = F_\pi = \{(x, y): \pi(x) = \pi(y)\}$. Then T is Σ_1^1 .

Proof. Following the proof of 3.1, we can obtain a standard Borel space $Z_{2,i}$ of pairs (\vec{u}, p) such that \vec{u} appropriately represents a holomorphic $\pi_{\vec{u}}: \widehat{M}_i \rightarrow M_p$. The construction of 3.1 shows that for each $\zeta, \zeta' \in D$, the fixed countable subset of \widehat{M}_i and $q, r \in \mathbb{Q}^+$ the set of pairs $(\vec{u}, p) \in Z_{2,i}$ with $d(\pi_{\vec{u}}(\zeta), \pi_{\vec{u}}(\zeta')) < q$ and $d(\pi_{\vec{u}}(\zeta), a_n) < r$ is Borel, where d is the metric for M_p and (a_n) the fixed dense subset of M_p .

Now the statement that $\pi_{\vec{u}}$ be a covering map amounts to the requirement that each unit ball around each a_n in M_p can be covered by finitely many basic open sets, $B_{\delta_1}(a_{n_1}), \dots, B_{\delta_k}(a_{n_k})$, so that for all $j \leq k$, letting $W_j = \pi_{\vec{u}}^{-1}(B_{\delta_j}(a_{n_j}))$ we have:

- (i) Any connected component of W_j is precompact and any two distinct connected components of W_j have disjoint closures.
- (ii) For any connected component U of $W_j, \pi_{\vec{u}}|U: U \rightarrow B_{\delta_j}(a_{n_j})$ is biholomorphic.

We fix j , and we verify that (i) and (ii) are Borel conditions. Put $D_j = D \cap W_j$. Then if we define the equivalence relation

$$\zeta \sim \zeta' \Leftrightarrow \zeta, \zeta' \text{ are in the same connected component of } W_j,$$

for $\zeta, \zeta' \in D_j$, we see that the connected components of W_j are exactly the sets of the form

$$U_C = \text{the interior of the closure of } C,$$

where C is a \sim -equivalence class.

Thus (i), (ii) above are equivalent to (i)', (ii)' below, where C varies over all the \sim -equivalence classes:

- (i)' Every U_C is precompact and for any two distinct $C_1, C_2, \bar{U}_{C_1} \cap \bar{U}_{C_2} = \emptyset$.
- (ii)' $\pi_{\bar{u}}|_{U_C}$ is biholomorphic.

By the methods of 7.2 and the proof of 3.1 we see that these conditions are Borel and thus the set of (\bar{u}, p) in $Z_{2,i}$ such that $\pi_{\bar{u}}$ is a covering map of M_p is Borel.

It is left to show that $F = F_{\pi_{\bar{u}}}$ can be expressed in a Borel manner. However, this amounts only to requiring that for each pair of precompact basic open $U_0, U_1 \subseteq \widehat{M}_i$, we have that $F \cap (\bar{U}_0 \times \bar{U}_1) \neq \emptyset$ iff there exists V_0, V_1 basic open with $\bar{V}_0 \subseteq U_0, \bar{V}_1 \subseteq U_1$ and $\forall q \in \mathbb{Q}^+ \exists \zeta_0 \in D \cap V_0 \exists \zeta_1 \in D \cap V_1 (d(\pi_{\bar{u}}(\zeta_0), \pi_{\bar{u}}(\zeta_1)) < q)$. \square

PROPOSITION 4.6. *There is a Borel function*

$$g: \mathbb{H}^{\mathbb{N}} \rightarrow \mathcal{D},$$

so that whenever $x = (x_n) \in \mathbb{H}^{\mathbb{N}}$ enumerates a discrete set, we have $D_{g(x)} = g(x) = \mathbb{H} \setminus \{x_n: n \in \mathbb{N}\}$.

Proof. First let A be the set of sequences $(x_n) \in \mathbb{H}^{\mathbb{N}}$ enumerating a discrete subset of \mathbb{H} . This is Borel since $(x_n) \in A$ if and only if

$$\forall n \exists \delta \in \mathbb{Q}^+ \forall m (m \neq n \Rightarrow d(x_n, x_m) > \delta).$$

For $x \notin A$, we can just let $g(x) = \mathbb{H}$. For $x \in A$, we let $g(x) = \mathbb{H} \setminus \{x_n: n \in \mathbb{N}\}$. This is a Borel function, since for any basic open $U \subset \mathbb{C}, U \subseteq g(x)$ if and only if $U \subseteq \mathbb{H}$ and no x_n is in U . \square

Recall that S_d is the space of discrete subgroups of $PSL_2(\mathbb{R})$. This is a Borel subset of $\mathcal{F}(PSL_2(\mathbb{R}))$ in the Effros Borel structure, and hence a standard Borel space.

LEMMA 5.3. *There is a Borel map $\varphi: S_d \rightarrow \mathcal{R}$ such that \mathbb{H}/G is conformally equivalent to $M_{\varphi(G)}$ for all $G \in S_d$ acting freely on \mathbb{H} .*

Proof. Note that we can indeed verify in a Borel manner whether $G \in S_d$ acts freely on \mathbb{H} : This amounts to the claim that for all basic open $U \subseteq \mathbb{H}$ we may find a finite sequence V_0, V_1, \dots, V_n of basic open sets covering \bar{U} and such that for all $i \leq n$,

$$(*) \quad g \cdot V_i \cap V_i = \emptyset \text{ for all } g \in G \text{ with } g \neq 1_G.$$

However $(*)$ is Borel, since it amounts to the assertion that for all $W \subseteq PSL_2(\mathbb{R})$ basic open not containing the identity, if $G \cap W \neq \emptyset$ then there exists $h \in W$ such that $h \cdot V_i \cap V_i = \emptyset$.

So let us just fix $G \in S_d$ acting freely on \mathbb{H} and describe $\varphi(G)$.

First we let (V_i) enumerate the basic open sets which are G -discrete, in the sense of meeting each G -orbit in at most one point, and have diameter < 1 in the hyperbolic metric ρ (see 4.B). As in the proof of 3.3, this can be used to give a chart for a representative of \mathbb{H}/G in \mathcal{R} . The one further problem is in uniformly obtaining a metric.

For any $\zeta, \xi \in \mathbb{H}$ and $g \in G$ we have

$$\inf_{h \in G} \rho(h \cdot \zeta, \xi) = \inf_{h \in G} \rho(h \cdot \zeta, g \cdot \xi),$$

and moreover this quantity is greater than zero if and only if $G \cdot \zeta \neq G \cdot \xi$. In particular

$$\inf_{h \in G} \rho(h \cdot \zeta, \xi) = \inf_{h, g \in G} \rho(h \cdot \zeta, g \cdot \xi).$$

Therefore

$$\tilde{d}(G \cdot \zeta, G \cdot \xi) = \inf_{h, g \in G} \rho(h \cdot \zeta, g \cdot \xi)$$

provides the needed metric on \mathbb{H}/G .

Thus if we let (a_i) enumerate a maximal G -discrete subset of $\mathbb{H} \cap (\mathbb{Q} + i\mathbb{Q})$ (in the sense that $G \cdot a_i \cap G \cdot a_j = \emptyset$ for all $i \neq j$), $y_{i,i'}$ = $\tilde{d}(G \cdot a_i, G \cdot a'_i)$, $A_i = \{j : G \cdot a_j \cap V_i \neq \emptyset\}$, and $\zeta_{i,j}$ to be the unique element in $V_i \cap G \cdot a_j$, if it exists, we obtain from $p = ((y_{i,i'}), (A_i), (\zeta_{i,i'}))$ an element in \mathcal{R}^n with M_p conformally equivalent to \mathbb{H}/G .

There is the further concern that all these steps can be performed in the Borel context, but this is routine and resembles earlier calculations. \square

Recall that for $x \in \mathbb{R}^N$ we let $M(x)$ be the complex manifold

$$(\mathbb{D} \times \mathbb{C}) \setminus \left(\bigcup_{n \in \mathbb{N}} \{1/(n+2)\} \times \{x_n + (n+2), x_n + (n+3), x_n + 2(n+2)\} \cup \{0\} \times \mathbb{R} \right)$$

endowed with the inherited complex structure.

LEMMA 6.3. *There is Borel map*

$$F: \mathbb{R}^N \rightarrow \mathcal{M}^2$$

such that $M(x)$ and $M_{F(x)}$ are biholomorphic for all $x \in \mathbb{R}^N$.

Proof. This follows the method of 3.3. \square

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