ILLINOIS JOURNAL OF MATHEMATICS
Volume 44, Number 2, Summer 2000

COMPRESSED POLYTOPES, INITIAL IDEALS AND COMPLETE MULTIPARTITE GRAPHS

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ABSTRACT. Convex polytopes arising from complete multipartite graphs and their toric ideals will be studied. First, it is proved that such toric ideals possess squarefree quadratic initial ideals. Second, we show that these convex polytopes are compressed and compute their f-vectors, Ehrhart polynomials and normalized volumes explicitly. Finally, all complete multipartite graphs which yield initial ideals coming from finite partially ordered sets will be classified.

Introduction

The second hypersimplex of order d is the convex polytope $\Delta(2, d)$ which is the convex hull of the configuration $\mathcal{A}_d = \{\mathbf{e}_i + \mathbf{e}_j ; 1 \leq i < j \leq d\}$ in $\mathbb{R}^d$, where $\mathbf{e}_i$ is the ith unit coordinate vector of $\mathbb{R}^d$. If G is a finite connected graph having no loop and no multiple edge on the vertex set $[d] = \{1, 2, \ldots, d\}$ with edge set $E(G)$, then we write $\mathcal{A}_G$ for the subset $\{\mathbf{e}_i + \mathbf{e}_j \in \mathcal{A}_d; \{i, j\} \in E(G)\}$ of $\mathcal{A}_d$. The edge polytope $\mathcal{P}_G$ of G is the convex hull of $\mathcal{A}_G$ in $\mathbb{R}^d$. Let $K[t_1, t_2, \ldots, t_d]$ denote the polynomial ring in d variables over a field K. The affine semigroup ring $K[G]$ which is generated by all monomials $t_i t_j$ with $\{i, j\} \in E(G)$ is called the edge ring of G. When G is the complete graph on $[d]$, its edge polytope is the second hypersimplex of order d and its edge ring is the second squarefree Veronese subring of $K[t_1, t_2, \ldots, t_d]$. In the present paper we are interested in edge polytopes and edge rings of complete multipartite graphs on $[d]$. Here, a complete multipartite graph on $[d]$ is a finite graph on $[d]$ such that, for a suitable decomposition $[d] = V_1 \cup V_2 \cup \cdots \cup V_n$ of $[d]$, its edge set consists of all $\{k, \ell\}$ with $k \in V_i$ and $\ell \in V_j$ for some $i \neq j$.

The present paper will be organized as follows. First, in Section 1, the notion of the algebra of Segre–Veronese type which generalizes both Segre products and Veronese subrings of polynomial rings will be presented. Such algebras are affine semigroup rings which possess squarefree quadratic initial ideals; in particular, these algebras are normal, Cohen–Macaulay and Koszul. The edge ring $K[G]$ of a finite connected graph G is an algebra of Segre–Veronese type if and only if G is a complete multipartite graph. Hence, the edge ring of a complete multipartite graph possesses a squarefree quadratic initial ideal and is normal, Cohen–Macaulay and Koszul.

Second, the purpose of Section 2 is to discuss the combinatorics on edge polytopes of complete multipartite graphs. We show that such a polytope is compressed, i.e.,

Received January 6, 1999; received in final form October 5, 1999.
1991 Mathematics Subject Classification. Primary 13P10; Secondary 52B20.
The first-named author was supported by JSPS Research Fellowships for Young Scientists.

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each of its reverse lexicographic triangulations is unimodular. Moreover, the f-vector, the Ehrhart polynomial together with the normalized volume of the edge polytope of a complete multipartite graph will be computed explicitly.

It is known that if R is a homogeneous semigroup ring, then R is Koszul if and only if its divisor poset (partially ordered set) Σ_R is Cohen–Macaulay. Here Σ_R is the infinite poset consisting of all monomials belonging to R, ordered by divisibility. In [15], it is proved that if R has an initial ideal which is the Stanley–Reisner ideal of a finite poset, then the divisor poset Σ_R is shellable. Moreover, in [2], it is shown that if R is extendable sequentially Koszul, then Σ_R is shellable. The second squarefree Veronese subring $R_d^{(2)}$ of $K[t_1, t_2, \ldots, t_d]$ is extendable sequentially Koszul for all $d \geq 2$, while $R_d^{(2)}$ possesses an initial ideal which is the Stanley–Reisner ideal of a finite poset if and only if $d = 2, 3, 4$. See [2] and [11]. One of the most distinguished classes of homogeneous semigroup rings having initial ideals which are Stanley–Reisner ideals of finite posets is the class of monomial ASL's (algebras with straightening laws). If, however, R is a monomial ASL, then the shellability of Σ_R follows easily (e.g., [2, Theorem 2.2]). In addition, if R is a monomial ASL, then Σ_R is chain lexicographically shellable. See [2, Theorem 2.3]. In Section 3, we will discuss the problem which complete multipartite graph yields an edge ring having an initial ideal which is the Stanley–Reisner ideal of a finite poset.

1. Algebra of Segre–Veronese type

The algebra of Segre–Veronese type which generalizes both Segre products and Veronese subrings of polynomial rings will be studied. Such algebras are affine semigroup rings which possess squarefree quadratic initial ideals; in particular, these algebras are normal, Cohen–Macaulay and Koszul.

Let K be a field and

$$K[\{t_j^{(i)}\}_{1 \leq i \leq n; 1 \leq j \leq q_i}]$$

the polynomial ring in $\sum_{i=1}^{n} q_i$ indeterminates over K. Fix an integer $N \geq 1$ and sets of integers $\{a_1, a_2, \ldots, a_n\}$, $\{b_1, b_2, \ldots, b_n\}$ and $\{c_j^{(i)}\}_{i=1,2,\ldots,n; j=1,2,\ldots,q_i}$ such that

(i) $0 \leq b_i \leq a_i$ for all $1 \leq i \leq n$;
(ii) $\sum_{i=1}^{n} b_i \leq N \leq \sum_{i=1}^{n} a_i$;
(iii) $\sum_{j=1}^{q_i} c_j^{(i)} \geq a_i$ for all $1 \leq i \leq n$.

We then write

$$\mathbf{A}(N; \{a_i, b_i, c_j^{(i)}\}_{1 \leq i \leq n; 1 \leq j \leq q_i}) \tag{1}$$

for the K-subalgebra of $K[\{t_j^{(i)}\}_{1 \leq i \leq n; 1 \leq j \leq q_i}]$ generated by all monomials

$$\prod_{i=1}^{n} \prod_{j=1}^{q_i} t_j^{(i) f_j^{(i)}}$$

such that

(i) $f_j^{(i)} \leq c_j^{(i)}$ for all $1 \leq i \leq n$ and for all $1 \leq j \leq q_i$;
(ii) $b_i \leq \sum_{j=1}^{q_i} f_j^{(i)} \leq a_i$ for all $1 \leq i \leq n$;
(iii) $\sum_{i=1}^{n} \sum_{j=1}^{q_i} f_j^{(i)} = N$.

The affine semigroup ring (1) is called an *algebra of Segre–Veronese type*.

For example, if $n = 2$, $N = 2$ and $a_i = b_i = c_j^{(i)} = 1$ for all i and j, then the affine semigroup ring (1) is the Segre product of polynomial rings $K[t_1^{(1)}, t_2^{(1)}, \ldots, t_{q_1}^{(1)}]$ and $K[t_1^{(2)}, t_2^{(2)}, \ldots, t_{q_2}^{(2)}]$. If $q_i = 1$, $a_i = N$, $b_i = 0$ and $c_1^{(i)} = N$ for all i, then the affine semigroup ring (1) coincides with the classical Nth Veronese subring of the polynomial ring $K[t_1^{(1)}, t_1^{(2)}, \ldots, t_1^{(n)}]$. Moreover, if $q_i = 1$, $a_i = 1$, $b_i = 0$ and $c_1^{(i)} = 1$ for all i, then the affine semigroup ring (1) is equal to the Nth squarefree Veronese subring of the polynomial ring $K[t_1^{(1)}, t_1^{(2)}, \ldots, t_1^{(n)}]$.

When $q_i = 1$, $a_i = c_1^{(i)}$ and $b_i = 0$ for all i, the affine semigroup ring (1) is an algebra of Veronese type which is discussed, for example, in [5] and [17].

Let $w_1, w_2, \ldots, w_\delta$ denote the minimal system of monomial generators of the algebra of Segre–Veronese type (1). Let $K[x_1, x_2, \ldots, x_\delta]$ denote the polynomial ring in δ variables over K and π the surjective homomorphism

$$\pi\colon K[x_1, x_2, \ldots, x_\delta] \to \mathbf{A}(N; \{a_i, b_i, c_j^{(i)}\}_{1 \leq i \leq n; 1 \leq j \leq q_i})$$

with $\pi(x_k) = w_k$ for all $1 \leq k \leq \delta$. The kernel of π is called the *toric ideal* (or defining ideal) of the algebra of Segre–Veronese type (1).

The technique appearing in the proof of [17, Theorem 14.2] which guarantees the existence of squarefree quadratic initial ideals of algebras of Veronese type can be applied to algebras of Segre–Veronese type in the obvious way. Hence, we immediately obtain the following result which generalizes [17, Theorem 14.2].

THEOREM 1.1. *The algebra of Segre–Veronese type* (1) *possesses a squarefree quadratic initial ideal* (*i.e., there exists a term order $<$ on the polynomial ring $K[x_1, x_2, \ldots, x_\delta]$ such that the initial ideal of the toric ideal of* (1) *with respect to $<$ is generated by squarefree quadratic monomials*). *Thus, in particular, the algebra of Segre–Veronese type* (1) *is normal, Cohen–Macaulay and Koszul.*

In the present paper, we are interested in a special kind of algebras of Segre–Veronese type, i.e., subrings of the second squarefree Veronese subring arising from complete multipartite graphs.

Let G be a finite connected graph on the vertex set $[d] = \{1, 2, \ldots, d\}$ and assume that G has no loop and no multiple edge. Let $E(G)$ denote the edge set of G. If $e = \{i, j\}$ is an edge of G joining $i \in [d]$ and $j \in [d]$, then we write $\rho(e) \in \mathbb{R}^d$ for the $(0, 1)$-vector $\mathbf{e}_i + \mathbf{e}_j$, where $\mathbf{e}_i$ is the ith unit coordinate vector of $\mathbb{R}^d$. The *edge* *polytope* $\mathcal{P}_G$ of G is the convex hull of the configuration $\mathcal{A}_G = \{\rho(e); e \in E(G)\}$ in $\mathbb{R}^d$. Let $K[\mathbf{t}] = K[t_1, t_2, \dots, t_d]$ denote the polynomial ring in d variables over a field K. The *edge ring* $K[G]$ of G is the affine semigroup ring generated by those monomials $t_i t_j$ with $\{i, j\} \in E(G)$. Let $K[\mathbf{x}] = K[\{x_{i,j}\}_{\{i,j\} \in E(G)}]$ denote the polynomial ring over K with each $\deg x_{i,j} = 1$. The *toric ideal* I_G of G is the kernel of the surjective homomorphism $\pi\colon K[\mathbf{x}] \to K[G]$ defined by $\pi(x_{i,j}) = t_i t_j$ for all $\{i, j\} \in E(G)$. We refer the reader to [12] and [14] for the detailed information about the edge polytope and the edge ring of a finite graph.

Let $q_1, q_2, \dots, q_n$ denote a sequence of positive integers with $q_1 + q_2 + \cdots + q_n = d$. Let $V_1, V_2, \dots, V_n$ denote a partition of $[d]$ (i.e., each $\emptyset \neq V_i \subset [d]$, $V_i \cap V_j = \emptyset$ if $i \neq j$, and $[d] = V_1 \cup V_2 \cup \cdots \cup V_n$) with each $\sharp(V_i) = q_i$. Here $\sharp(V_i)$ is the cardinality of V_i as a finite set. For the sake of convenience, we will assume that

$$V_i = \left\{ \sum_{j=1}^{i-1} q_j + 1, \sum_{j=1}^{i-1} q_j + 2, \cdots, \sum_{j=1}^{i-1} q_j + q_i - 1, \sum_{j=1}^{i} q_j \right\}$$

for each $1 \leq i \leq n$. The *complete multipartite graph of type* $\mathbf{q} = (q_1, q_2, \dots, q_n)$ is the finite graph $G_\mathbf{q}$ on the vertex set $[d] = V_1 \cup V_2 \cup \cdots \cup V_n$ with the edge set

$$E(G_\mathbf{q}) = \{\{k, \ell\}; k \in V_i, \ell \in V_j, 1 \leq i < j \leq n\}.$$

Note, in particular, that if $\{k, \ell\} \in E(G_\mathbf{q})$ with $k < \ell$, then $\{k', \ell'\} \in E(G_\mathbf{q})$ for any k' and ℓ' with $k' \leq k < \ell \leq \ell'$. It may be assumed that $\mathbf{q} = (q_1, q_2, \dots, q_n)$ satisfies

$$1 \leq q_1 \leq q_2 \leq \cdots \leq q_n.$$

When G is the complete graph on $[d]$, i.e., $n = d$ and each $q_i = 1$, the edge polytope is the second hypersimplex [17, p. 75] of order d and the edge ring is the second squarefree Veronese subring of $K[t_1, t_2, \dots, t_d]$.

Now, which edge ring can be an algebra of Segre–Veronese type ?

PROPOSITION 1.2. *The edge ring $K[G]$ of a finite connected graph G is an algebra of Segre–Veronese type if and only if G is a complete multipartite graph.*

Proof. Work with the same notation as in the definition of the algebra of Segre–Veronese type (1) and set $N = 2$ and each $c_j^{(i)} = 1$.

If $b_i = 2$ for some $1 \leq i \leq n$, then the algebra (1) is the second squarefree Veronese subring of $K[t_1^{(i)}, t_2^{(i)}, \dots, t_{q_i}^{(i)}]$. If $b_i = b_j = 1$ for some $i \neq j$, then the algebra (1) is the Segre product of $K[t_1^{(i)}, t_2^{(i)}, \dots, t_{q_i}^{(i)}]$ and $K[t_1^{(j)}, t_2^{(j)}, \dots, t_{q_j}^{(j)}]$.

Let us assume that $b_i = 1$ for some i and $b_j = 0$ for any $j \neq i$. If $a_i = 1$, then the algebra (1) is the edge ring of a complete bipartite graph. If $a_i \geq 2$, then the algebra (1) is the edge ring of the complete multipartite graph on the vertex set

$$\left(\bigcup_{j=1}^{q_i} \{t_j^{(i)}\} \right) \bigcup \{t_j^{(k)}; a_k \neq 0, k \neq i, 1 \leq j \leq q_k\}.$$

Suppose that each $b_i = 0$. Let I be the set of all i with $a_i \geq 2$ and J the set of all k with $a_k = 1$. Then, the algebra (1) is the edge ring of the complete multipartite graph on the vertex set

$$\left(\bigcup_{i \in I} \bigcup_{j=1}^{q_i} \{t_j^{(i)}\}\right) \cup \left(\bigcup_{k \in J} \{t_1^{(k)}, t_2^{(k)}, \ldots, t_{q_k}^{(k)}\}\right). \qquad \square$$

COROLLARY 1.3. *The edge ring of a complete multipartite graph is normal, Cohen–Macaulay and Koszul.*

In general, the term order $<$ of Theorem 1.1 can be chosen to be neither lexicographic nor reverse lexicographic. We can show, however, that the toric ideal of a complete multipartite graph possesses not only a lexicographic quadratic initial ideal but also a reverse lexicographic quadratic initial ideal.

THEOREM 1.4. *Let G be a complete multipartite graph with edge set $E(G)$ and I_G its toric ideal. Define the ordering $<$ of the indeterminates $x_{i,j}$ with $\{i, j\} \in E(G)$ by setting $x_{i,j} < x_{k,\ell}$, where $i < j$ and $k < \ell$, if and only if either (i) $i < k$ or (ii) $i = k$ and $j > \ell$. Let $<_{\text{lex}}$ denote the lexicographic term order on $K[\{x_{i,j}\}_{\{i,j\} \in E(G)}]$ induced by $<$ and $<_{\text{rev}}$ the reverse lexicographic term order on $K[\{x_{i,j}\}_{\{i,j\} \in E(G)}]$ induced by $<$. Let $\text{in}_{<_{\text{lex}}}(I_G)$ denote the initial ideal of I_G with respect to $<_{\text{lex}}$ and $\text{in}_{<_{\text{rev}}}(I_G)$ the initial ideal of I_G with respect to $<_{\text{rev}}$. Then, both initial ideals $\text{in}_{<_{\text{lex}}}(I_G)$ and $\text{in}_{<_{\text{rev}}}(I_G)$ are generated by squarefree quadratic monomials.*

Proof. First, by virtue of [17, Remark 9.2], the lexicographic initial ideal $\text{in}_{<_{\text{lex}}}(I_G)$ is generated by those squarefree quadratic monomials $x_{i,j}x_{k,\ell}$ such that either $i < j < k < \ell$ or $i < k < \ell < j$, where $\{i, j\}, \{k, \ell\} \in E(G)$.

Second, to see why the reverse lexicographic initial ideal $\text{in}_{<_{\text{rev}}}(I_G)$ is generated by squarefree quadratic monomials, we will show that the set of quadratic binomials belonging to I_G is a Gröbner basis of I_G with respect to $<_{\text{rev}}$. Let $\mathcal{G}$ denote the set of all quadratic binomials $x_{i,j}x_{k,\ell} - x_{i,\ell}x_{j,k}$ such that (i, j, k, ℓ) is a cycle of G of length 4. It follows from [14, Theorem 1.2] that $\mathcal{G}$ is a system of generators of I_G. Hence, the Buchberger criterion can be applied in order to prove that $\mathcal{G}$ is a Gröbner basis of I_G with respect to $<_{\text{rev}}$. If f and g are binomials belonging to $\mathcal{G}$ and if the initial monomials of f and g are not relatively prime, then the S-polynomial of f and g is a cubic binomial. Thus, what we must prove is that, for any cubic binomial

$$F = x_{a,b}x_{i,j}x_{k,\ell} - x_{b,i}x_{j,k}x_{\ell,a},$$

where (a, b, i, j, k, ℓ) is a cycle of G of length 6, and for any cubic binomial

$$F = x_{a,b}x_{c,a}x_{i,j} - x_{b,c}x_{a,i}x_{j,a},$$

where (a, b, c) and (a, i, j) are triangles of G having exactly one common vertex, there exist binomials $f_1, f_2, \ldots \in \mathcal{G}$ and $\{i_1, j_1\}, \{i_2, j_2\}, \ldots \in E(G)$ with

$$F = f_1 x_{i_1, j_1} + f_2 x_{i_2, j_2} + \cdots$$

such that, with respect to $<_{\text{rev}}$, each of the initial monomials of $f_1 x_{i_1, j_1}, f_2 x_{i_2, j_2}, \ldots$ is less than or equal to the initial monomial of F. The proof will be complete if we proceed case by case. See [13]. For example, if $F = x_{1,2}x_{1,3}x_{4,5} - x_{2,3}x_{1,4}x_{1,5}$, where $(1, 2, 3)$ and $(1, 4, 5)$ are triangles of G, then $\{2, 4\} \in E(G)$ since $\{2, 3\} \in E(G)$; hence $F = f x_{1,3} + g x_{1,5}$ with $f = x_{1,2}x_{4,5} - x_{1,5}x_{2,4}$ and $g = x_{1,3}x_{2,4} - x_{1,4}x_{2,3}$. □

2. Triangulations, *f*-vectors and Ehrhart polynomials

In general, let $\mathcal{P} \subset \mathbb{R}^d$ be an *integral* convex polytope, i.e., a convex polytope any of whose vertices belongs to $\mathbb{Z}^d$. An integral polytope is called *compressed* if each of its reverse lexicographic triangulations is unimodular. For example, see [16]. Moreover, an integral polytope is called *unimodular* if any of its triangulations is unimodular. It then follows that an integral polytope is unimodular if and only if each of its lexicographic triangulations is unimodular.

As before, let G be a finite connected graph on the vertex set $[d]$ having no loop and no multiple edge, and with edge set $E(G)$. Let $\mathcal{P}_G \subset \mathbb{R}^d$ be the edge polytope of G. Note that the vertex set of $\mathcal{P}_G$ is equal to $\mathcal{A}_G = \{\rho(e); e \in E(G)\}$ and that $\mathcal{P}_G \cap \mathbb{Z}^d$ coincides with $\mathcal{A}_G$. By [17, Lemma 9.5], the edge polytope $\mathcal{P}_G$ of G is unimodular if and only if any two odd cycles of G have a common vertex. Thus, in particular, the edge polytope $\mathcal{P}_G$ of a complete multipartite graph G is unimodular if and only if the type of G is one of the following:

(i) (p, q) with $1 \leq p \leq q$;
(ii) $(1, p, q)$ with $1 \leq p \leq q$;
(iii) $(1, 1, 1, p)$ with $1 \leq p$;
(iv) $(1, 1, 1, 1, 1)$.

However, it is not quite clear which edge polytopes are compressed. Our first result of this section guarantees that the edge polytope of a complete multipartite graph is compressed.

THEOREM 2.1. *The edge polytope of a complete multipartite graph is compressed.*

Proof. Let G be a complete multipartite graph with edge set $E(G)$ and $K[\mathbf{x}] = K[\{x_{i,j}\}_{\{i,j\} \in E(G)}]$ the polynomial ring over K. Fix an arbitrary reverse lexicographic term order $<$ on $K[\mathbf{x}]$ and let $\text{in}_<(I_G)$ denote the initial ideal of the toric ideal $I_G \subset K[\mathbf{x}]$ with respect to $<$. If the triangulation arising from the Stanley–Reisner ideal $\sqrt{\mathrm{in}_<(I_G)}$ is not unimodular, then it follows from [17, Lemma 9.5] that we can find two odd cycles C_1 and C_2 of G having no common vertex such that the monomial

$$\prod_{\{i,j\}\in E(C_1)\cup E(C_2)} x_{i,j} \tag{2}$$

does not belong to $\sqrt{\mathrm{in}_<(I_G)}$. Let $i_1, i_2, \ldots, i_{2s-1}$ be the vertices of the cycle C_1 with $\{i_k, i_{k+1}\} \in E(G)$ for all $1 \le k \le 2s-1$ with $i_{2s} = i_1$, and let $j_1, j_2, \ldots, j_{2t-1}$ be the vertices of the cycle C_2 with $\{j_\ell, j_{\ell+1}\} \in E(G)$ for all $1 \le \ell \le 2t-1$ with $j_{2t} = j_1$. Suppose that x_{i_2,i_3} is the weakest variable among all $x_{i,j}$'s with $\{i, j\} \in E(C_1)\cup E(C_2)$. Since G is a complete multipartite graph, either $\{i_1, j_1\}$ or $\{i_1, j_2\}$ is an edge of G. Say $\{i_1, j_1\} \in E(G)$. Then, the binomial

$$g = \prod_{k=1}^{s} x_{i_{2k-1},i_{2k}} \prod_{\ell=1}^{t} x_{i_{2\ell-1},i_{2\ell}} - {x_{i_1,j_1}}^2 \prod_{k=1}^{s-1} x_{i_{2k},i_{2k+1}} \prod_{\ell=1}^{t-1} x_{i_{2\ell},i_{2\ell+1}}$$

belongs to I_G and its initial monomial is

$$\mathrm{in}_<(g) = \prod_{k=1}^{s} x_{i_{2k-1},i_{2k}} \prod_{\ell=1}^{t} x_{i_{2\ell-1},i_{2\ell}}.$$

Now, since $\mathrm{in}_<(g)$ divides the monomial (2), the monomial (2) must belong to $\mathrm{in}_<(I_G)$. This contradiction shows that the triangulation arising from $\sqrt{\mathrm{in}_<(I_G)}$ is unimodular, as desired. □

A regular unimodular triangulation of the edge polytope of a complete multipartite graph arising from the lexicographic initial ideal of Theorem 1.2 can be constructed by imitating the technique appearing in [17, pp. 77–79].

We may call a homogeneous semigroup ring compressed if each of the reverse lexicographic initial ideals of its toric ideal is squarefree. It follows from Theorem 2.1 together with Proposition 1.2 that all algebras of Segre–Veronese type generated by squarefree quadratic monomials are compressed. There is a noncompressed algebra of Segre–Veronese type generated by squarefree cubic monomials. In fact:

Example 2.2. The affine semigroup ring $R \subset K[t_1, t_2, \ldots, t_6]$ which is generated by all squarefree cubic monomials $t_i t_j t_k$ with $1 \le i < j < k \le 6$ and $(i, j, k) \ne (4, 5, 6)$ is an algebra of Segre–Veronese type with $n = 4$, $(q_1, q_2, q_3, q_4) = (1, 1, 1, 3)$, $(a_1, a_2, a_3, a_4) = (1, 1, 1, 2)$, $(b_1, b_2, b_3, b_4) = (0, 0, 0, 0)$ and each $c_j^{(i)} = 1$. The initial ideal of its toric ideal with respect to the reverse lexicographic term order induced by the ordering $t_1t_2t_3 < t_1t_2t_4 < t_1t_2t_5 < t_1t_2t_6 < t_1t_3t_4 < \cdots < t_3t_4t_5 < t_3t_4t_6 < t_3t_5t_6$ is not squarefree. Hence, the affine semigroup ring R is noncompressed.

It seems to be a reasonable research project to find a criterion for an algebra of Segre–Veronese type generated by squarefree monomials to be compressed.

In principle, it is possible to find all facets of the edge polytope of a finite connected graph. For example, see [12, Theorem 1.7]. Here, we compute the f-vector of $\mathcal{P}_G$ in terms of the type of G.

Let $G_{\mathbf{q}}$ be the complete multipartite graph of type $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ on the vertex set $[d] = V_1 \cup V_2 \cup \cdots \cup V_n$ with each $\sharp(V_i) = q_i$ and $E(G_{\mathbf{q}})$ the edge set of $G_{\mathbf{q}}$. We know (e.g., see [12, Proposition 1.3]) that $\dim \mathcal{P}_{G_{\mathbf{q}}} = d - 1$ if $n \geq 3$, and $\dim \mathcal{P}_{G_{\mathbf{q}}} = d - 2$ if $n = 2$. Recall that $\rho(e) \in \mathbb{R}^d$ is the (0, 1)-vector $\mathbf{e}_i + \mathbf{e}_j$ if $e = \{i, j\} \in E(G_{\mathbf{q}})$, where $\mathbf{e}_i$ is the ith unit coordinate vector of $\mathbb{R}^d$. If H ($\neq G_{\mathbf{q}}$) is a subgraph of $G_{\mathbf{q}}$ with the edge set $E(H)$, then we write $\mathcal{F}_H$ for the convex hull of $\{\rho(e); e \in E(H)\}$ in $\mathbb{R}^d$. Then the next result follows from [12, Theorem 1.7].

LEMMA 2.3. (a) *If $n \geq 3$, then the subpolytope $\mathcal{F}_H$ of $\mathcal{P}_{G_{\mathbf{q}}}$ is a facet of $\mathcal{P}_{G_{\mathbf{q}}}$ if and only if either H is nonbipartite and is the induced subgraph of $G_{\mathbf{q}}$ on $[d]\backslash\{i\}$ for some $i \in [d]$, or H is the complete bipartite graph on $V_k \cup ([d]\backslash V_k)$, where $1 \leq k \leq n$.*

(b) *If $n = 2$, then the subpolytope $\mathcal{F}_H$ of $\mathcal{P}_{G_{\mathbf{q}}}$ is a facet of $\mathcal{P}_{G_{\mathbf{q}}}$ if and only if H is the induced subgraph of $G_{\mathbf{q}}$ on $[d]\backslash\{i\}$, where $i \in V_k$ with $q_k > 1$.*

COROLLARY 2.4. *The subpolytope $\mathcal{F}_H$ of $\mathcal{P}_{G_{\mathbf{q}}}$ is a face of $\mathcal{P}_{G_{\mathbf{q}}}$ if and only if one of the following holds:*

(i) *H is the complete multipartite graph on $V_1' \cup V_2' \cup \cdots \cup V_n'$, where each $V_k' \subset V_k$ and where $V_k' \neq \emptyset$ for at least three k's;*
(ii) *H is the complete bipartite graph on $V_k' \cup V'$, where $\emptyset \neq V_k' \subset V_k$ and $\emptyset \neq V' \subset [d]\backslash V_k$ for some $1 \leq k \leq n$.*

Proof. Each facet of $\mathcal{P}_{G_{\mathbf{q}}}$ is again the edge polytope of a complete multipartite graph. In general, every face of a convex polytope $\mathcal{P}$ is a face of a facet of $\mathcal{P}$. Hence, repeated applications of Lemma 2.3 enable us to obtain the desired result. □

THEOREM 2.5. *The number of i-faces (i-dimensional faces) of the edge polytope $\mathcal{P}_{G_{\mathbf{q}}}$ of the complete multipartite graph $G_{\mathbf{q}}$ of type $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ on the vertex set $[d]$ with $n \geq 2$ is $\alpha_i + \beta_i$, where*

$$\alpha_i = \sum_{k=1}^{n-1} \sum_{j=1}^{i} \binom{q_k}{j} \binom{q_{k+1} + q_{k+2} + \cdots + q_n}{i - j + 1} - \sum_{1 \leq k < \ell \leq n} \sum_{j=1}^{i} \binom{q_k}{j} \binom{q_\ell}{i - j + 1},$$

$$\beta_i = \sum_{k=1}^{n} \sum_{j=1}^{i+1} \binom{q_k}{j} \binom{d - q_k}{i - j + 2} - \sum_{1 \leq k < \ell \leq n} \sum_{j=1}^{i+1} \binom{q_k}{j} \binom{q_\ell}{i - j + 2}.$$

Proof. The number of subgraphs H of the form (i) of Corollary 2.4 with $i + 1$ vertices is α_i and the number of subgraphs H of the form (ii) of Corollary 2.4 with $i + 2$ vertices is β_i. □

We now turn to the problem of computing the Ehrhart polynomial of the edge polytope of a complete multipartite graph. If $\mathcal{P} \subset \mathbb{R}^d$ is an integral convex polytope, then we write $i(\mathcal{P}, m)$ for the number of rational points $(\xi_1, \xi_2, \ldots, \xi_d) \in \mathcal{P} \cap \mathbb{Q}^d$ with $(m\xi_1, m\xi_2, \ldots, m\xi_d) \in \mathbb{Z}^d$ for each $m = 1, 2, \ldots$; in other words,

$$i(\mathcal{P}, m) = \sharp(m\mathcal{P} \cap \mathbb{Z}^d).$$

It is known that $i(\mathcal{P}, m)$ is a polynomial in m of degree $\dim \mathcal{P}$. We call $i(\mathcal{P}, m)$ the *Ehrhart polynomial* of $\mathcal{P}$. If $\mathrm{vol}\,(\mathcal{P})$ is the normalized volume of $\mathcal{P}$, then the leading coefficient of $i(\mathcal{P}, m)$ is $\mathrm{vol}\,(\mathcal{P})/(\dim \mathcal{P})!$. We refer the reader to [10], for instance, for the detailed information about Ehrhart polynomials of convex polytopes.

THEOREM 2.6. *The Ehrhart polynomial $i(\mathcal{P}_{G_{\mathbf{q}}}, m)$ of the edge polytope $\mathcal{P}_{G_{\mathbf{q}}}$ of the complete multipartite graph $G_{\mathbf{q}}$ of type $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ on the vertex set $[d]$ with $n \geq 2$ is*

$$(3) \qquad \binom{d+2m-1}{d-1} - \sum_{k=1}^{n} \sum_{1 \leq i \leq j \leq q_k} \binom{j-i+m-1}{j-i}\binom{d-j+m-1}{d-j}.$$

Proof. It follows from [12], for example, that the Ehrhart polynomial $i(\mathcal{P}_{G_{\mathbf{q}}}, m)$ coincides with the Hilbert function of the normalization of the edge ring $K[G_{\mathbf{q}}]$ of $G_{\mathbf{q}}$. Since $K[G_{\mathbf{q}}]$ is normal, $i(\mathcal{P}_{G_{\mathbf{q}}}, m)$ is equal to the Hilbert function $\dim_K(K[G_{\mathbf{q}}])_m$ of the homogeneous K-algebra $K[G_{\mathbf{q}}] = \bigoplus_{m=0}^{\infty}(K[G_{\mathbf{q}}])_m$.

The lexicographic quadratic initial ideal of Theorem 1.4 guarantees that the set of monomials $x_{i_1,j_1}x_{i_2,j_2}\cdots x_{i_m,j_m}$ with each $\{i_r, j_r\} \in E(G)$ such that

$$(4) \qquad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq j_1 \leq j_2 \leq \cdots \leq j_m \leq d$$

is a K-basis of $(K[G_{\mathbf{q}}])_m$. How many sequences (4) with $\{i_s, j_s\} \notin E(G_{\mathbf{q}})$ for some $1 \leq s \leq m$ do we have? If we fix $1 \leq k \leq n$, then the number of sequences (4) with $\{i_s, j_s\} \notin E(G_{\mathbf{q}})$, $\{i_{s-1}, j_{s-1}\} \in E(G_{\mathbf{q}})$ and with $i_s \in V_k$, $j_s \in V_k$ for some $1 \leq s \leq m$ is

$$\sum_{1 \leq i \leq j \leq q_k} \binom{j-i+m-1}{j-i}\binom{d-j+m-1}{d-j}.$$

Since the number of sequences (4) is the binomial coefficient $\binom{d+2m-1}{d-1}$, the required formula follows immediately. □

More generally, it is possible to write down the Hilbert function of the algebra of Segre–Veronese type (1). We, however, omit the result due to the lack of usefulness.

COROLLARY 2.7. (a) *The normalized volume of the edge polytope $\mathcal{P}_{G_{\mathbf{q}}}$ of the complete multipartite graph $G_{\mathbf{q}}$ of type $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ on the vertex set $[d]$ with $n \geq 3$ is*

$$2^{d-1} - \sum_{k=1}^{n} \sum_{j=1}^{q_k} \binom{d-1}{j-1}.$$

(b) *The normalized volume of the edge polytope of the complete bipartite graph of type (p, q) is*

$$\binom{p+q-2}{p-1}.$$

Proof. (a) Since $n \geq 3$, the finite graph $G_{\mathbf{q}}$ is nonbipartite. Hence, the edge polytope $\mathcal{P}_{G_{\mathbf{q}}}$ is of dimension $d-1$ and the Ehrhart polynomial $i(\mathcal{P}_{G_{\mathbf{q}}}, m)$ is of degree $d-1$. By (3) the leading coefficient of $(d-1)!i(\mathcal{P}_{G_{\mathbf{q}}}, m)$ is $2^{d-1} - \sum_{k=1}^{n} \sum_{j=1}^{q_k} \binom{d-1}{j-1}$.

(b) If G is the complete bipartite graph of type (p, q), then the polynomial (3) turns out to be $\binom{p+m-1}{p-1}\binom{q+m-1}{q-1}$, which is a polynomial in m of degree $p+q-2$. The leading coefficient of $(p+q-2)!\binom{p+m-1}{p-1}\binom{q+m-1}{q-1}$ is $\binom{p+q-2}{p-1}$. □

Remark 2.8. In [5] the Gorenstein algebra of Veronese type is completely classified. It seems difficult to find all Gorenstein algebras of Segre–Veronese type. However, based on the technique developed in [5] together with Lemma 2.3, we can prove that the edge ring $K[G]$ of a complete multipartite graph G is Gorenstein if and only if the type of G is $(1, p)$ with $p \geq 1$, or (p, p) with $p \geq 2$, or (p, q, r) with $1 \leq p \leq q \leq r \leq 2$, or $(1, 1, 1, 1)$.

3. Semigroup rings coming from posets

Let $K[x_1, x_2, \ldots, x_\delta]$ be the polynomial ring in δ variables over a field K with each $\deg x_i = 1$. Let $\leq$ be a partial order on $[\delta] = \{1, 2, \ldots, \delta\}$ and $P = ([\delta], \leq)$ the finite poset (partially ordered set) on $[\delta]$ with the partial order $\leq$. The *Stanley–Reisner ideal* of P is the ideal of $K[x_1, x_2, \ldots, x_\delta]$ which is generated by all squarefree quadratic monomials $x_i x_j$ such that i and j are incomparable in P.

Let $K[t_1, t_2, \ldots, t_d]$ be the polynomial ring in d variables over K and R a homogeneous semigroup ring with the minimal system of monomial generators $w_1, w_2, \ldots, w_\delta$ with each $w_i \in K[t_1, t_2, \ldots, t_d]$. The *divisor poset* of R is the infinite poset Σ_R consisting of all monomials belonging to R, ordered by divisibility. It is known that R is Koszul if and only if Σ_R is Cohen–Macaulay. For example, see [15, Corollary 2.2]. Let I denote the toric ideal of R, i.e., I is the kernel of the surjective homomorphism $\pi\colon K[x_1, x_2, \ldots, x_\delta] \to R$ defined by $\pi(x_i) = w_i$ for all $1 \leq i \leq \delta$. If R is Koszul, then I is generated by quadratic binomials. Moreover, if I possesses an initial ideal generated by quadratic monomials, then R is Koszul; e.g., see [4]. We say that R

comes from a poset if its toric ideal I possesses an initial ideal which is the Stanley–Reisner ideal of a finite poset. In [15] it is proved that if R comes from a poset, then Σ_R is shellable. Moreover, in [2] it is proved that if R is extendable sequentially Koszul, then Σ_R is shellable. For example, the squarefree second Veronese subring $R_d^{(2)}$ of $K[t_1, t_2, \dots, t_d]$ is extendable sequentially Koszul for all $d \geq 2$, while $R_d^{(2)}$ comes from a poset if and only if $d = 2, 3, 4$. See [2] and [11]. One of the most familiar examples of homogeneous semigroup rings coming from posets is the monomial ASL (algebra with straightening laws). See [3], [6] and [10] for detailed information about ASL's.

As before, let $R \subset K[t_1, t_2, \dots, t_d]$ be a homogeneous semigroup ring with the minimal system of monomial generators $w_1, w_2, \dots, w_\delta$. Then, we say that R is a *monomial ASL* if there exists a partial order $\leq$ on $\{w_1, w_2, \dots, w_\delta\}$ satisfying the following conditions:

(ASL-1) The set of all monomials $w_{i_1} w_{i_2} \cdots$ of R with $w_{i_1} \leq w_{i_2} \leq \cdots$ (such a monomial is called *standard* with respect to the partial order $\leq$) is a K-basis of R;

(ASL-2) If w_i and w_j are incomparable in the partial order $\leq$ and if $w_k w_\ell$, where $w_k \leq w_\ell$, is a unique standard monomial (whose existence and uniqueness follow from (ASL-1)) with $w_i w_j = w_k w_\ell$, then we have $w_k \leq w_i$ and $w_k \leq w_j$.

The toric ideal of a monomial ASL possesses a reverse lexicographic initial ideal which is the Stanley–Reisner ideal of a finite poset. It is known that the divisor poset of a monomial ASL is chain lexicographically shellable. Moreover, in general, a homogeneous semigroup ring whose divisor poset is chain lexicographically shellable is extendable sequentially Koszul. We refer the reader to [1] and [2] for further information about monomial ASL's and extendable sequentially Koszul semigroup rings.

Before stating the main result of the present section, we will discuss some examples of monomial ASL's.

Example 3.1. (a) A homogeneous semigroup ring R is called *trivial* (cf. [8]) if, starting with polynomial rings, R is obtained by repeated applications of Segre products and tensor products. Every trivial semigroup ring is a monomial ASL. (In fact, every trivial semigroup ring belongs to the class of monomial ASL's arising from finite posets discussed below.)

(b) Let $P = ([d], \leq_P)$ be an arbitrary poset on the finite set $[d] = \{1, 2, \dots, d\}$ and $K[s, t_1, t_2, \dots, t_d]$ the polynomial ring in $d + 1$ variables over a field K. Recall that a poset ideal of P is a subset I of $[d]$ such that $i \in I$ and $j \in [d]$ with $j \leq_P i$ implies $j \in I$. The empty subset can be a poset ideal of P. Let $J(P)$ denote the set of all poset ideals of P. For each $I \in J(P)$, we set $u_I = s \prod_{i \in I} t_i \in K[s, t_1, t_2, \dots, t_d]$. In [9], it is proved that the homogeneous semigroup ring $K[\{u_I\}_{I \in J(P)}]$ is a monomial ASL with respect to the partial order $\leq$ on $\{u_I\}_{I \in J(P)}$ defined by $u_I \leq u_{I'}$ if and only if $I \subset I'$. The relation required in (ASL-2) is of the form $u_I u_{I'} = u_{I \cap I'} u_{I \cup I'}$. The finite poset $(\{u_I\}_{I \in J(P)}, \leq)$ is a distributive lattice and $u_{I \cap I'}$ is the meet of u_I and $u_{I'}$; $u_{I \cup I'}$ is the join of u_I and $u_{I'}$. See [1] and [8] for related topics.

Let $K[t_1, t_2, \dots, t_d]$ denote the polynomial ring in d variables over a field K with each $\deg t_i = 1$. Let $A_d^{(q)}$ denote the qth Veronese subring of $K[t_1, t_2, \dots, t_d]$ and let $R_d^{(q)}$ denote the qth squarefree Veronese subring of $K[t_1, t_2, \dots, t_d]$. Thus, $A_d^{(q)}$ is generated by the $\binom{d+q-1}{d-1}$ monomials of degree q of $K[t_1, t_2, \dots, t_d]$ and $R_d^{(q)}$ is generated by the $\binom{d}{q}$ squarefree monomials of degree q of $K[t_1, t_2, \dots, t_d]$. It is known [15, Theorem 4.2] that $A_d^{(q)}$ comes from a poset for any d and any q. However, it is proved in [11, Theorem 2.3] that $R_d^{(q)}$ with $2 \le q < d$ comes from a poset if and only if either $q = 2$ and $d = 3, 4$, or $q \ge 3$ and $d = q + 1$. We recall from the proof of [15, Theorem 4.2] the poset from which $A_d^{(q)}$ comes.

Let $\Omega_d^{(q)}$ denote the set of all sequences $(i_1, i_2, \dots, i_q) \in \mathbb{Z}^q$ with $1 \le i_1 \le i_2 \le \cdots \le i_q \le d$. We introduce the partial order $\le_d^{(q)}$ on $\Omega_d^{(q)}$ by setting $(i_1, i_2, \dots, i_q) \le_d^{(q)} (j_1, j_2, \dots, j_q)$ if $i_{2k-1} \le j_{2k-1}$ and $j_{2k} \le i_{2k}$ for all k. We then identify each sequence $(i_1, i_2, \dots, i_q) \in \Omega_d^{(q)}$ with the monomial $t_{i_1} t_{i_2} \cdots t_{i_q} \in A_d^{(q)}$. Now, the proof of [15, Theorem 4.2] guarantees that $A_d^{(q)}$ comes from the poset $\Omega_d^{(q)}$; in other words, the set of standard monomials with respect to $\Omega_d^{(q)}$ is a K-basis of $A_d^{(q)}$. However, the poset $\Omega_d^{(q)}$ possesses at least two minimal elements if $d \ge 2$ and $q \ge 3$. In fact, if $d \ge 2$ and if $q \ge 3$ is odd, then both $(1, 1, \dots, 1)$ and $(1, 2, 2, \dots, 2)$ are minimal elements of $\Omega_d^{(q)}$, and if $d \ge 2$ and if $q \ge 3$ is even, then both $(1, 1, \dots, 1, d)$ and $(1, 2, 2, \dots, 2, d)$ are minimal elements of $\Omega_d^{(q)}$. Hence, the axiom (ASL-2) fails to hold for $A_d^{(q)}$ and $\Omega_d^{(q)}$ if $d \ge 2$ and $q \ge 3$.

PROPOSITION 3.2. (a) *The Veronese subring $A_d^{(q)}$ with $d \ge 2$ and $q \ge 2$ is a monomial ASL if and only if $q = 2$.*

(b) *The divisor poset of the Veronese subring $A_d^{(q)}$ is locally semimodular for any $d \ge 2$ and any $q \ge 2$.*

Proof. (a) First, we show that if $d \ge 2$ and $q = 2$, then $A_d^{(2)}$ is a monomial ASL on $\Omega_d^{(2)}$. In fact, if (i, j) and (k, ℓ) are incomparable in $\Omega_d^{(2)}$, then $(i, j)(k, \ell) = (i', j')(k', \ell')$, where $i' \le k' \le \ell' \le j'$ with $\{i, j, k, \ell\} = \{i', j', k', \ell'\}$, which is the required relation in (ASL-2).

Second, to see why $A_d^{(q)}$ with $d \ge 2$ cannot be a monomial ASL if $q \ge 3$, it is enough to prove that the initial ideal of the toric ideal of $A_d^{(q)}$ with respect to any reverse lexicographic term order cannot be squarefree if $d \ge 2$ and $q \ge 3$. Since the monomials $t_1^q, t_1^{q-1}t_2, t_1^{q-2}t_2^2$ and $t_1^{q-3}t_2^3$ belong to $A_d^{(q)}$ if $d \ge 2$ and $q \ge 3$, the binomials $x_2^2 - x_1x_3$ and $x_3^2 - x_2x_4$ belong to the toric ideal of $A_d^{(q)}$. Hence, either x_2^2 or x_3^2 must belong to the minimal system of monomial generators of the initial ideal of the toric ideal of $A_d^{(q)}$ with respect to any reverse lexicographic term order.

(b) It is known [8, Proposition 2.3 (a)] that $A_d^{(q)}$ is strongly Koszul for any $d \ge 2$ and any $q \ge 2$. Hence, its divisor poset is locally semimodular [8, Proposition 1.4]. □

We are now in the position to discuss the problem which edge rings of complete multipartite graphs come from posets.

THEOREM 3.3. *The edge ring $K[G]$ of a complete multipartite graph G comes from a poset if and only if the type of G is one of the following*:

(i) (p, q) *with* $1 \leq p \leq q$;
(ii) $(1, p, q)$ *with* $1 \leq p \leq q$;
(iii) $(1, 1, p, q)$ *with* $1 \leq p \leq q$.

Moreover, if the type of G is one of the above, then $K[G]$ is a monomial ASL.

Proof. First of all, it is known [11, Proposition 2.2] that if G is a finite graph such that the edge ring $K[G]$ comes from a poset, then the edge ring $K[G']$ of any induced subgraph G' of G also comes from a poset. Moreover, if G is the complete graph on the vertex set $[d]$, then its edge ring comes from a poset if and only if $d = 2, 3, 4$. See [11, Theorem 2.3]. Thus, if G is a complete multipartite graph of type $(q_1, q_2, \ldots, q_n)$ and if its edge ring $K[G]$ comes from a poset, then $n \leq 4$.

Let G denote the complete multipartite graph of type $(2, 2, 2)$. The explicit computation of all regular triangulations of the edge polytope $\mathcal{P}_G$ with the computer program PUNTOS by De Loera guarantees that the toric ideal I_G possesses 24 quadratic initial ideals (up to symmetry). Each of them is squarefree. However, none of them satisfies the well-known combinatorial condition, (e.g., [7]) for a squarefree quadratic monomial ideal to be the Stanley–Reisner ideal of a finite poset. Hence, the edge ring $K[G]$ *does not* come from a poset.

It follows that if the edge ring $K[G]$ of a complete multipartite graph G of type $(q_1, q_2, \ldots, q_n)$ comes from a poset, then $n \leq 4$ and at most two of q_i's can be $q_i \geq 2$. Hence, if the edge ring $K[G]$ of a complete multipartite graph G comes from a poset, then the type of G is (p, q), or $(1, p, q)$, or $(1, 1, p, q)$.

We now prove that the edge ring of the complete multipartite graph of type $(1, 1, p, q)$ is a monomial ASL. The case of the complete multipartite graph of type (p, q) or $(1, p, q)$ can be done similarly. Let G be the complete multipartite graph on the vertex set $[p+q]$ with the partition $[p+q] = V_1 \cup V_2 \cup V_3 \cup V_4$, where

$$V_1 = \{1\}, V_2 = \{2, 3, \ldots, p-1\}, V_3 = \{p\}, V_4 = \{p+1, p+2, \ldots, p+q\}.$$

Let Ω_G denote the set of variables $x_{i,j}$ with $i < j$ and $\{i, j\} \in E(G)$. Let $\mathbb{N}^2$ denote the infinite planar distributive lattice consisting of all pairs (i, j) of nonnegative integers with the partial order defined by $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. We then regard Ω_G as a subposet of $\mathbb{N}^2$ via the injective map ω: $\Omega_G \to \mathbb{N}^2$ defined as follows: If $x_{i,j} \in \Omega_G$ with $(i, j) \notin \{(1, 2), (1, 3), \ldots, (1, p-1)\}$, then $\omega(x_{i,j}) = (i, j)$; if $x_{i,j} \in \Omega_G$ with $i = 1$ and $2 \leq j \leq p-1$, then $\omega(x_{i,j}) = (j, p+q+1)$. For example, if $p = 5$ and $q = 4$, then Ω_G is the poset with the Hasse diagram of Figure 1.

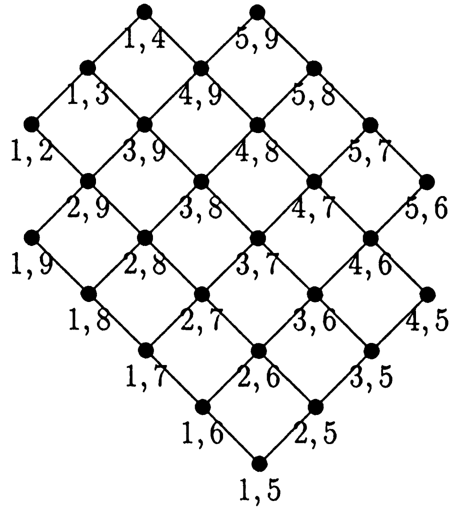

Figure 1. Hasse diagram of Ω_G with $p = 5$ and $q = 4$.

If $(i, j) \notin \{(1,2), (1,3), \ldots, (1, p-1)\}$ and if $2 \leq j' \leq p-1$, then $x_{i,j} \leq x_{1,j'}$ in Ω_G if and only if $1 \leq i \leq j' \leq j$ since $j' \leq p-1$ and $j \geq p$. If neither (i, j) nor (i', j') belongs to $\{(1,2), (1,3), \ldots, (1, p-1)\}$, then $x_{i,j} \leq x_{i',j'}$ in Ω_G if and only if $i \leq i' \leq j \leq j'$ since $i, i' \leq p$ and $j, j' \geq p$. Thus, $x_{i,j}$ and $x_{i',j'}$ with $i \leq i'$ are comparable in Ω_G if and only if $i \leq i' \leq j \leq j'$. In other words, $x_{i,j}$ and $x_{i',j'}$ with $i \leq i'$ are incomparable in Ω_G if and only if either $i < j < i' < j'$ or $i < i' < j' < j$. Now, the first paragraph of the proof of Theorem 1.4 guarantees that there exists a term order on the polynomial ring $K[\{x_{i,j}\}_{\{i,j\} \in E(G)}]$ such that the initial ideal of I_G is generated by those squarefree quadratic monomials $x_{i,j}x_{k,\ell}$, where $\{i, j\}, \{k, \ell\} \in E(G)$, such that either $i < j < k < \ell$ or $i < k < \ell < j$. This initial ideal coincides with the Stanley–Reisner ideal of the finite poset Ω_G.

It remains to show that the edge ring $K[G]$ is a monomial ASL on Ω_G. Since the set of standard monomials with respect to the above initial ideal is equal to the set of standard monomials with respect to the partial order on Ω_G, the axiom (ASL-1) is satisfied. In order to prove the axiom (ASL-2), suppose that $x_{i,j}$ and $x_{i',j'}$ with $i \leq i'$ are incomparable in Ω_G. If $i < j < i' < j'$, then $i = 1$ and $2 \leq j \leq p-1$ and $x_{i,j}x_{i',j'} = x_{i,i'}x_{j,j'}$ with $x_{j,j'} < x_{i,j}$ and $x_{j,j'} < x_{i',j'}$. If $i < i' < j' < j$, then neither (i, j) nor (i', j') belongs to $\{(1,2), (1,3), \ldots, (1, p-1)\}$ and $x_{i,j}x_{i',j'} = x_{i,j'}x_{i',j}$ with $x_{i,j'} < x_{i,j}$ and $x_{i,j'} < x_{i',j'}$. Hence, the axiom (ASL-2) is satisfied. □

Remark 3.4. If a homogeneous semigroup ring R is generated by squarefree monomials and if its toric ideal I possesses a quadratic initial ideal $in_<(I)$, then $\text{in}_<(I)$ must be generated by squarefree monomials. This obvious fact is, however, essential in [14] for the construction of a Koszul semigroup ring having no quadratic Gröbner basis. Hence, a homogeneous semigroup ring R generated by squarefree monomials is quasi-poset [15, p. 384] if and only if R comes from a poset.

Conjecture 3.5. (a) The edge ring of a complete multipartite graph is extendable sequentially Koszul.

(b) (follows from (a)) The divisor poset of the edge ring of a complete multipartite graph is shellable.

We conclude the present paper with a discussion of edge rings of complete multipartite graphs which are strongly Koszul.

Let R be a homogeneous semigroup ring with the minimal system of monomial generators $w_1, w_2, \ldots, w_\delta$. Then, R is called *strongly Koszul* if the ideals $(w_i) \cap (w_j)$ are generated in degree 2 for all $i \neq j$. Every strongly Koszul semigroup ring is extendable sequentially Koszul. It is known [8, Proposition 1.4] that the divisor poset of a homogeneous semigroup ring R is locally semimodular if and only if R is strongly Koszul. The edge ring $K[G]$ of a connected bipartite graph G is strongly Koszul if and only if G is a complete bipartite graph [8, Theorem 4.5].

PROPOSITION 3.6. *The edge ring $K[G]$ of a complete multipartite graph G is strongly Koszul if and only if the type of G is (p, q) or $(1, 1, p)$ or $(1, 1, 1, 1)$.*

Proof. If G is either a complete bipartite graph or the complete graph with 4 vertices, then the edge ring $K[G]$ is strongly Koszul. See [8, Example 1.6]. If G' is the complete multipartite graph of type $(1, 1, p)$, then the edge ring $K[G']$ is isomorphic to the polynomial ring in one variable over the edge ring of the complete bipartite graph of type $(2, p)$. Hence, $K[G']$ is strongly Koszul. This proves the "if" part of the proposition.

It follows from [11, Corollary 1.6] that if G is a finite graph such that the edge ring $K[G]$ is strongly Koszul and if G' is any induced subgraph of G, then $K[G']$ is again strongly Koszul. Thus, in order to prove the "only if" part of the proposition, it is enough to show that if the type of a complete multipartite graph G is $(1, 2, 2)$ or $(1, 1, 1, 2)$ or $(1, 1, 1, 1, 1)$, then $K[G]$ is not strongly Koszul. Suppose that the type of a complete multipartite graph G is $(1, 2, 2)$ or $(1, 1, 1, 2)$ or $(1, 1, 1, 1, 1)$. Then, the finite graph in Fig. 2 is a subgraph of G. Hence, it may be assumed that the monomials

$$u = t_1t_2,\ u' = t_2t_3,\ v = t_1t_3,\ v' = t_1t_4,\ w = t_4t_5,\ w' = t_1t_5$$

belong to $K[G]$. Note that $uvw = u'v'w' = {t_1}^2t_2t_3t_4t_5$. If the ideal $(w) \cap (u')$ is generated in degree 2, then $(w) \cap (u')$ must be generated by wu' $(= t_2t_3t_4t_5)$. Now, the monomial ${t_1}^2t_2t_3t_4t_5$ is contained in $(w) \cap (u')$. However, in $K[G]$, $t_2t_3t_4t_5$ cannot divide ${t_1}^2t_2t_3t_4t_5$. Hence, $(w) \cap (u')$ cannot be generated in degree 2. Thus, $K[G]$ is not strongly Koszul. □

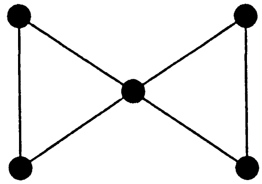

Figure 2

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