ILLINOIS JOURNAL OF MATHEMATICS
Volume 44, Number 4, Winter 2000

AN INEQUALITY FOR p-ORTHOGONAL SUMS IN NON-COMMUTATIVE L_p

GILLES PISIER

ABSTRACT. We give an alternate proof of one of the inequalities proved recently for martingales (= sums of martingale differences) in a non-commutative L_p-space, with $1 < p < \infty$, by Q. Xu and the author. This new approach is restricted to p an even integer, but it yields a constant which is $O(p)$ when $p \to \infty$ and it applies to a much more general kind of sum which we call p-orthogonal. We use mainly combinatorial tools, namely the Möbius inversion formula for the lattice of partitions of a p-element set.

0. Introduction

In a recent paper [PX], Quanhua Xu and the author have proved non-commutative versions of the Burkholder-Gundy classical inequalities (see [BG], [B1]–[B4]) relating the L_p-norms of a martingale with those of its square function ($1 < p < \infty$). We will continue this investigation here. Our objective is two-fold. First we will improve the order of growth of the constant in the main inequality from [PX] when $p \to \infty$. We obtain a constant which is $O(p)$ when $p \to \infty$, thus yielding the "sharp" order of growth. Sharp constants themselves are known in the classical-commutative-case, see [B3] and [B4, §11], but they seem out of reach of our method.

Secondly, we wish to extend the inequality from martingales to a much broader class of sums in non-commutative L_p-spaces: the p-orthogonal sums, which are defined as follows.

Let (M, τ) be a von Neumann algebra equipped with a standard (= faithful, normal) trace with $\tau(1) = 1$, and let $L_p(\tau)$ be the associated "non-commutative" L_p-spaces. (Of course, if M is commutative, we recover the classical L_p associated to a probability space.) Let $p \geq 2$ be an even integer. A family $d = (d_i)_{i \in I}$ is called p-orthogonal if, for any *injective* function g: $[1, 2, \ldots, p] \to I$, we have

$$\tau(d^*_{g(1)} d_{g(2)} d^*_{g(3)} d_{g(4)} \ldots d^*_{g(p-1)} d_{g(p)}) = 0.$$

In the commutative case, i.e., for classical random variables, this notion is very close to that of a "multiplicative sequence" already considered in the literature (see Remark 2.4 below for more details).

Let us assume I finite for simplicity. We will denote simply by $\| \quad \|_p$ the norm in $L_p(\tau)$.

Received June 2, 1999; received in final form August 24, 1999.
1991 Mathematics Subject Classification. Primary 60B99, 60G46, 60G48.
Supported in part by the National Science Foundation and by the Texas Advanced Research Program.

Then the following inequality, which is our main result, holds:

$$\left\| \sum_{i\in I} d_i \right\|_{L_p(\tau)} \le \frac{3\pi}{2} p\, S(d, p) \tag{0.1}$$

where we have set

$$S(d, p) = \max \left\{ \left\| \left(\sum d_i^* d_i \right)^{1/2} \right\|_p , \quad \left\| \left(\sum d_i d_i^* \right)^{1/2} \right\|_p \right\}. \tag{0.2}$$

Clearly, any martingale difference sequence is p-orthogonal, but the class of p-orthogonal sums includes a broader class of sums which appear rather naturally in Harmonic Analysis. For instance, let $\Lambda \subset G$ be a subset of a discrete group with unit element e. We call Λ p-dissociate if for any choice $t_1, t_2, \ldots, t_p$ of p *distinct* points in Λ we have

$$t_1^{-1} t_2 t_3^{-1} t_4 \ldots t_{p-1}^{-1} t_p \neq e.$$

See [Ru] for examples of this in the Abelian case. Then let $\lambda\colon G \to B(\ell_2(G))$ be the left regular representation of G, let $\mathcal{M}$ be the von Neumann algebra generated by λ and let τ_G be the usual normalized trace on $\mathcal{M}$ defined by

$$\tau_G(x) = \langle x\delta_e, \delta_e \rangle.$$

Let $(\delta_t)_{t\in G}$ be the canonical basis of $\ell_2(G)$.

With this notation (and with τ as before), for any function $x\colon \Lambda \to L_p(\tau)$ the family

$$d_t = \lambda(t) \otimes x(t)$$

is p-orthogonal in $L_p(\tau_G \times \tau)$. Therefore (0.1) holds in this case too for any finite subset $I \subset \Lambda$. More generally, a family $(\Lambda_i)_{i\in I}$ of disjoint subsets of Λ will be called p-dissociate if every family $(t_i)_{i\in I}$ with $t_i \in \Lambda_i$ for all i in I is itself p-dissociate. Then assuming, say, that x is finitely supported, if we define

$$d_i = \sum_{t\in\Lambda_i} \lambda(t) \otimes x(t)$$

we again obtain a p-orthogonal sum so that (0.1) holds in this case too. For instance in the case $G = \mathbb{Z}$ and $\Lambda_i = [2^i, 2^{i+1}[$, treating the cases of $\{\Lambda_i \mid i \text{ even}\}$ and $\{\Lambda_i \mid i \text{ odd}\}$ separately, we can recover from (0.1) one of the classical Littlewood-Paley inequalities for Fourier series:

$$\left\| \sum_{n>0} a_n e^{int} \right\|_p \le C_p \|\underline{S}\|_p$$

where

$$\underline{S} = \left(\sum_{k\geq 0} \left| \sum_{2^k \leq n < 2^{k+1}} a_n e^{int} \right|^2 \right)^{1/2}$$

and where, say, we assume that $(a_n)_{n>0}$ is a finitely supported sequence of scalars.

A surprising feature of our proof of the martingale inequalities (or their extensions) is that we use very elementary tools. Indeed, in the non-commutative setting which is our main motivation, most of the usual techniques such as stopping times or maximal inequalities are unavailable, or apparently inefficient. Therefore, we must use only Hölder's inequalities and certain identities. For example when $p = 4$ we are using an identity of the form

$$\begin{aligned}\left(\sum d_i\right)^4 &= \Sigma + 6\sum d_i^4 - 8\left(\sum d_i^3\right)\left(\sum d_i\right)\\ &\quad - 3\left(\sum d_i^2\right)^2 + 6\left(\sum d_i^2\right)\left(\sum d_i\right)^2.\end{aligned}$$

where

$$\Sigma = \sum_{i_1,i_2,i_3,i_4 \text{ all distinct}} d_{i_1} d_{i_2} d_{i_3} d_{i_4}.$$

More generally, for any even integer p, there is an analogous identity for $(\sum d_i)^p$ in which the coefficients appearing (such as $6, -8, -3, 6$ when $p = 4$) can be explicitly computed using the Möbius inversion formula, classical in the combinatorics of partitions (cf. [R1], [R2], [A]). In particular, there are explicit formulae (due to Schützenberger, see Theorem 1.2 below) for these coefficients, which lead to suprisingly good bounds for the constants in our inequalities.

*Remark.*0.1. Many examples of non-commutative martingales can be given using (non-commutative) Harmonic Analysis. Let G be a discrete group, and let $\lambda_G \colon G \to \ell_2(G)$ be its left regular representation. The von Neumann algebra of G is defined as $M = \lambda_G(G)''$, and it can be equipped with the standard trace τ defined by $\tau_G(x) = \langle x\delta_e, \delta_e\rangle$. Let $G_n (n \in \mathbb{N})$ be a non-decreasing sequence of subgroups, and let $M_n = \lambda_G(G_n)''$. Then, denoting by $\mathbb{E}_n$ the (contractive) conditional expectation from M to M_n (which is also contractive on $L_p(\tau_G)$ whenever $1 \leq p < \infty$), for any f in $L_p(\tau_G)$, the sequence $d_n = \mathbb{E}_n f - \mathbb{E}_{n-1} f$ is a martingale difference sequence, hence satisfies (0.1).

*Remark.*0.2. As explained in [PX], the "free group filtration" is a typical example to which the preceding point applies. By this we mean the case when $G = \mathbf{F}_\infty$, the free group with countably many generators denoted by $\{g_0, g_1, g_2, \ldots\}$, $G_n \subset G$ is the subgroup generated by $\{g_0, g_1, \ldots, g_n\}$ and again $M_n = \lambda_G(G_n)''$. We will use this example below in one of the proofs. We could consider more generally the filtration associated to a free product of a countable collection of groups.

Remark.0.3. Another example is the "free-Gaussian" analog of the preceding. Let (M, τ) be a von Neumann algebra equipped with a standard normalized trace. Let $(x_n)_{n\geq 0}$ be a free semi-circular family in Voiculescu's sense [VDN] in (M, τ), and let M_n be the von Neumann algebra generated in M by $\{x_0, \ldots, x_n\}$. Then again (M_n) is an interesting example to which (0.1) applies. This case was recently studied by Biane and Speicher [BS]. Their main result gives evidence that, for martingales relative to the free group filtration and its free-Gaussian analog, the constant appearing in (0.1) might actually be *bounded* when $p \to \infty$, but this remains open.

Acknowledgement. I thank the referee for his careful reading of the manuscript.

1. Möbius inversion

We will make crucial use of some well known ideas from the combinatorial theory of partitions, which can be found, for instance, in Rota's texts [R1], [R2] or in the book [A]. We denote by P_n the lattice of all partitions of $[1, \ldots, n]$, equipped with the following order: we write $\sigma \leq \pi$ (or equivalently $\pi \geq \sigma$) when every "block" of the partition σ is contained in some block of π. Let $\dot{0}$ and $\dot{1}$ be respectively the minimal and maximal elements in P_n, so that $\dot{0}$ is the partition into n singletons and $\dot{1}$ the partition formed of the single set $\{1, \ldots, n\}$. We denote by $\nu(\pi)$ the number of blocks of π (so that $\nu(\dot{0}) = n$ and $\nu(\dot{1}) = 1$).

For any π in P_n and any $i = 1, 2, \ldots, n$, we denote by $r_i(\pi)$ the number of blocks (possibly $= 0$) of π of cardinality i. In particular, we have $\sum_1^n i r_i(\pi) = n$ and $\sum_1^n r_i(\pi) = \nu(\pi)$.

Given two partitions σ, π in P_n with $\sigma \leq \pi$ we denote by $\mu(\sigma, \pi)$ the Möbius function, which has the following fundamental property.

PROPOSITION 1.1. *Let V be a vector space. Consider two functions Φ: $P_n \to V$ and Ψ: $P_n \to V$.*

(i) *If*

$$\Psi(\sigma) = \sum_{\pi \leq \sigma} \Phi(\pi),$$

then

$$\Phi(\sigma) = \sum_{\pi \leq \sigma} \mu(\pi, \sigma)\Psi(\pi).$$

(ii) *If*

$$\Psi(\sigma) = \sum_{\pi \geq \sigma} \Phi(\pi),$$

then

$$\Phi(\sigma) = \sum_{\pi \geq \sigma} \mu(\sigma, \pi)\Psi(\pi).$$

(iii) *In particular,*

$$\sum_{0 \leq \pi \leq \sigma} \mu(\pi, \sigma) = 0 \quad \forall \sigma \neq \dot{0}.$$

Remark. (iii) follows from (i) applied with Φ equal to the delta function at $\dot{0}$ (i.e., $\Phi(\pi) = 0 \, \forall \pi \neq \dot{0}$ and $\Phi(\dot{0}) = 1$) and $\Psi \equiv 1$.

We also recall Schützenberger's theorem (see [A] or [R1-2]):

THEOREM 1.2. *For any π we have*

$$\mu(\dot{0}, \pi) = \prod_{i=1}^{n} [(-1)^{i-1}(i-1)!]^{r_i(\pi)},$$

and consequently

(1.1)
$$\sum_{\pi \in P_n} |\mu(\dot{0}, \pi)| = n!.$$

We now apply these results to set the stage for the questions of interest to us. Let $E_1, \ldots, E_n, V$ be vector spaces equipped with a multilinear form (i.e., a "product")

$$\varphi\colon E_1 \times \cdots \times E_n \to V.$$

Let I be a finite set. For each $k = 1, 2, \ldots, n$ and $i \in I$, we give ourselves elements $d_i(k) \in E_k$, and we form the sum

$$F_k = \sum_{i \in I} d_i(k).$$

Then we are interested in "computing" or "expanding" in a specific manner the quantity

$$\varphi(F_1, \ldots, F_n).$$

We can start by writing, obviously,

$$\varphi(F_1, \ldots, F_n) = \sum_{g} \varphi(d_{g(1)}(1), \ldots, d_{g(n)}(n))$$

where the sum runs over all functions $g\colon [1, 2, \ldots, n] \to I$. Let $\pi(g)$ be the partition associated to g, namely the partition obtained from $\bigcup_{i \in I} g^{-1}(\{i\})$ after deletion of all the empty blocks. We can write

$$\varphi(F_1, \ldots, F_n) = \sum_{\sigma \in P_n} \Phi(\sigma)$$

where $\Phi(\sigma) = \sum_{g:\ \pi(g)=\sigma} \varphi(d_{g(1)}(1), \ldots, d_{g(n)}(n))$. By Theorem 1.1, if we let $\Psi(\sigma) = \sum_{\pi \geq \sigma} \Phi(\pi)$ we can write using (ii) and (iii) in Proposition 1.1:

$$\begin{aligned}
\varphi(F_1, \ldots, F_n) &= \Phi(\dot{0}) + \sum_{\dot{0}<\sigma} \Phi(\sigma) \\
&= \Phi(\dot{0}) + \sum_{\dot{0}<\sigma} \sum_{\pi \geq \sigma} \mu(\sigma, \pi) \Psi(\pi) \\
&= \Phi(\dot{0}) + \sum_{\dot{0}<\pi} \Psi(\pi) \cdot \sum_{\dot{0}<\sigma \leq \pi} \mu(\sigma, \pi) \\
&= \Phi(\dot{0}) - \sum_{\dot{0}<\pi} \Psi(\pi) \mu(\dot{0}, \pi).
\end{aligned}$$

Recapitulating, we state:

COROLLARY 1.3. *The following identity holds*:

$$\varphi(F_1, \ldots, F_n) = \Phi(\dot{0}) - \sum_{0<\pi} \Psi(\pi) \mu(\dot{0}, \pi)$$

where

$$\Phi(\dot{0}) = \sum_{g \text{ injective}} \varphi(d_{g(1)}(1), \ldots, d_{g(n)}(n))$$

and

$$\Psi(\pi) = \sum_{g:\ \pi(g) \geq \pi} \varphi(d_{g(1)}(1), \ldots, d_{g(n)}(n)).$$

2. The commutative case

Although the main point of this paper is the non-commutative case, we prefer to present the proof first in the classical setting. This will make it much easier for the reader to follow the arguments in the next sections. Note that although many results similar to our Theorem 2.1 below exist in the literature (e.g., see [St] and Remark 2.4 below), we could not quite find a reference for the same result.

Let (Ω, m) be any measure space and let $p = 2k$ be an even integer. Let $(d_i)_{i \in I}$ be a finite sequence in $L_p = L_p(m)$. We will say that $(d_i)_{i \in I}$ is *p-orthogonal* if for any injective map $g\colon I \to [1, \ldots, p]$ we have

$$\int \bar{d}_{g(1)} d_{g(2)} \bar{d}_{g(3)} \ldots \bar{d}_{g(p-1)} d_{g(p)} dm = 0. \tag{2.1}$$

Clearly, if $p = 2$ we recover the usual orthogonality in L_2. Throughout this section, we will let

$$S = \left(\sum_{i \in I} |d_i|^2 \right)^{1/2}.$$

It is easy to check that any martingale difference sequence in L_p is p-orthogonal (consider the largest value of g, say $g(i) = n$ and take the conditional expectation of index $n-1$, before the integral in (2.1)).

THEOREM 2.1. *Let $(d_i)_{i\in I}$ be a p-orthogonal finite sequence in $L_p = L_p(\Omega, m)$. Then for all even integers $p = 2k$, we have*

$$A_p\|S\|_p - \left(\sum_{i\in I}\|d_i\|_p^p\right)^{1/p} \le \left\|\sum_{i\in I} d_i\right\|_p \le 2p\|S\|_p \tag{2.2}$$

where $0 < A_p \le 1$ is a constant depending only on p.

It is well known that a random variable f on a probability space is exponentially integrable, i.e.,

$$\exists\, \delta > 0 \text{ such that } \int \exp(\delta|f|)dP < \infty$$

iff $f \in L_p$ for any even integer $p > 0$ and

$$\exists\, K \text{ such that } \|f\|_p \le Kp \quad \forall p > 0 \text{ even integer.}$$

Moreover, the corresponding norms are equivalent. Thus we have:

COROLLARY 2.2. *Let $(d_i)_{i\in I}$ be a (countable) family of random variables on a probability space (Ω, P) which are p-orthogonal for any even integer $p = 2k$. Then, if the "square function" $S = (\sum |d_i|^2)^{1/2}$ is in the unit ball of L_∞, we have*

$$\int \exp\left(\delta\left|\sum d_i\right|\right) dP \le 2$$

where $\delta > 0$ is a numerical constant (independent of the family (d_i)).

Proof of Theorem 2.1. For simplicity we restrict ourselves to the $\mathbb{R}$-valued case. We apply the combinatorics in §1 to the multilinear form

$$\varphi\colon L_p \times \cdots \times L_p \to \mathbb{R}$$

defined by $\varphi(x_1, x_2, \ldots, x_{p-1}, x_p) = \int x_1x_2\ldots x_{p-1}x_p\, dm$. The hypothesis in Theorem 2.1 guarantees that $\Phi(\dot{0}) = 0$. Let $f = \sum_{i\in I} d_i$. Applying Corollary 1.3, we thus obtain

$$\|f\|_p^p = -\sum_{\dot{0}<\pi} \mu(\dot{0}, \pi)\Psi(\pi) \tag{2.3}$$

where

$$\Psi(\pi) = \int \prod_{j=1}^{p}\left(\sum_{i\in I} d_i^j\right)^{r_j(\pi)} d\mu.$$

If $j \geq 2$, then $\left|\sum d_i^j\right|^{1/j} \leq S$, so by Hölder's inequality $\left(\frac{r_1(\pi)}{p} + \frac{p-r_1(\pi)}{p} = 1\right)$,

$$|\Psi(\pi)| \leq \int |f|^{r_1(\pi)} S^{p-r_1(\pi)} d\mu \leq \|f\|_p^{r_1(\pi)} \|S\|_p^{p-r_1(\pi)}.$$

Thus we obtain

$$\|f\|_p^p \leq \sum_{\dot{0}<\pi} |\mu(\dot{0},\pi)| \; \|f\|_p^{r_1(\pi)} \|S\|_p^{p-r_1(\pi)}.$$

Note that $\dot{0} < \pi$ implies $r_1(\pi) \leq p-2$, hence the last sum can be rewritten as

$$\sum_{0\leq r\leq p-2} \|f\|_p^r \|S\|_p^{p-r} a_r \quad \text{with } a_r = \sum_{r_1(\pi)=r} |\mu(\dot{0},\pi)|.$$

A moment of thought shows that $a_r = \binom{p}{r} b_r$ where b_r is the sum of $|\mu(\dot{0},\sigma)|$ over all partitions σ of $[1,\dots,p-r]$ without any singleton. A fortiori, by (1.1), we have

$$b_r \leq (p-r)!.$$

Thus we finally obtain

$$\|f\|_p^p \leq \sum_{0\leq r\leq p-2} \|f\|_p^r \|S\|_p^{p-r} \binom{p}{r} (p-r)!.$$

Therefore, using the sublemma below, we conclude that

$$\|f\|_p \leq 2p\|S\|_p.$$

SUBLEMMA 2.3. *Let x, y be positive numbers such that*

$$x^p \leq \sum_{0\leq r<p} x^r y^{p-r} \binom{p}{r} (p-r)!$$

Then $x \leq 2py$.

Proof. Let $t = y/x$. We have

$$1 \leq \sum_{0\leq r<p} \binom{p}{r} t^{p-r} (p-r)!.$$

Using $\int_0^\infty s^{p-r} e^{-s} ds = (p-r)!$ and $\int_0^\infty e^{-s} ds = 1$, we obtain

$$1 \leq \int_0^\infty [(1+ts)^p - 1] e^{-s} ds = \int_0^\infty (1+ts)^p e^{-s} ds - 1$$

whence $2 \leq \int_0^\infty \exp(pts - s) ds$.

Therefore if $pt < 1$ this implies $2 \leq (1-pt)^{-1}$ hence $\frac{1}{t} \leq 2p$ (and if $pt \geq 1$, then $\frac{1}{t} \leq p$ which is even better). $\square$

We now turn to the reverse inequality.

With the same notation as before, we now "isolate" the terms in (2.3) corresponding to the partitions π such that $r_2(\pi) = p/2$, i.e., π is a partition of $[1, \ldots, p]$ into $p/2$ pairs. Let α_p be the number of such partitions. For such a π, by Theorem 1.2 we have $\mu(\dot{0}, \pi) = (-1)^{p/2}$ and $\Psi(\pi) = \int \left(\sum |d_i|^2\right)^{p/2} dm = \|S\|_p^p$. Thus we obtain

$$|f\|_p^p = \alpha_p(-1)^{p/2+1}\|S\|_p^p - \sum{}'\mu(\dot{0}, \pi)\Psi(\pi) \tag{2.4}$$

where the symbol $\sum'$ means that we sum over all π with $r_1(\pi) \leq p-2$ and $r_2(\pi) < p/2$. A simple calculation shows that

$$\alpha_p = p![2^{p/2}(p/2)!]^{-1}.$$

We can write

$$\sum{}'\mu(\dot{0}, \pi)\Psi(\pi) = \sum_{0 \leq r \leq p-2} C(r) \tag{2.5}$$

where $C(r) = \sum_{\substack{r_1(\pi)=r \\ r_2(\pi)<p/2}} \mu(\dot{0}, \pi)\Psi(\pi)$.

By arguing as above, we obtain

$$|C(r)| \leq \binom{p}{r}(p-r)!\|f\|_p^r S^{p-r}.$$

But now, this estimation will be sufficiently efficient for our purposes only if $r > 0$; the term $C(0)$ has to be estimated separately. We have

$$\begin{aligned} |C(0)| \leq & \sum_\lambda \operatorname{card}(\pi \mid r_j(\pi) = \lambda_j, \forall j \geq 0)\Pi((i-1)!)^{\lambda_i} \\ & \cdot \int \left(\sum d_i^2\right)^{\lambda_2} \left(\sum |d_i|^3\right)^{\lambda_3} \cdots \left(\sum |d_i|^p\right)^{\lambda_p} dm, \end{aligned}$$

where the sum runs over all integers $\lambda_j \geq 0$ such that $p = \lambda_1 + 2\lambda_2 + \cdots + p\lambda_p$ with $\lambda_2 < p/2$ and $\lambda_1 = 0$.

Since $2 < 3 \leq p$, we can write $\frac{1}{3} = \frac{1-\theta}{2} + \frac{\theta}{p}$ with $\theta > 0$. Hence, by Hölder,

$$\left\| \left(\sum |d_i|^3\right)^{1/3} \right\|_p \leq \|S\|_p^{1-\theta} \left(\sum \|d_i\|_p^p\right)^{\theta/p}.$$

Let $h = \left(\sum_{i \in I} \|d_i\|_p^p\right)^{1/p}$. Since $\lambda_2 < p/2$, we have $2\lambda_2 \leq p-2$ and since we may as well assume $h \leq \|S\|_p$ (otherwise the left side of (2.2) is negative), we again obtain, by Hölder,

$$\begin{aligned} \int \left(\sum |d_i|^2\right)^{\lambda_2} \cdots \left(\sum |d_i|^p\right)^{\lambda_p} dm & \leq \|S\|_p^{2\lambda_2} \left\| \left(\sum |d_i|^3\right)^{1/3} \right\|_p^{p-2\lambda_2} \\ & \leq \|S\|_p^{2\lambda_2+(1-\theta)(p-2\lambda_2)} \cdot h^{\theta(p-2\lambda_2)} \\ & \leq \|S\|_p^{p-2\theta} h^{2\theta}. \end{aligned}$$

Thus, returning to (2.4) and (2.5) we can write

$$\|f\|_p^p \geq \alpha_p \|S\|_p^p - \sum_{0<r<p-1} |C(r)| - |C(0)|$$

which implies

$$\alpha_p \|S\|_p^p \leq \|f\|_p^p + \sum_{0<r<p-1} \frac{p!}{r!} \|f\|_p^r S^{p-r} + \beta_p \|S\|_p^{p-2\theta} h^{2\theta}$$

where β_p is a constant depending only on p. Clearly, since $\theta > 0$, this last estimate shows that the ratio $\|S\|_p \cdot [\max\{\|f\|_p, h\}]^{-1}$ must be bounded above by a constant depending only on p. This yields the left side of (2.1). $\square$

Remark 2.4. The literature contains numerous attempts to generalize orthogonality. For instance, in Stout's book [St] a sequence of (real valued) random variables is called "multiplicative" (resp. "multiplicative of order r") if it admits moments of all order (resp. of all order $\leq r$) and is p-orthogonal for all p (resp. for all $p \leq r$). We are aware of works by Azuma (1967), Serfling (1969), Dharmadhikari and Jogdeo (1969) (for which we refer to [St] for precise references) which all relate to the notion of p-orthogonality, but we could not find results like Theorem 2.1 in the literature, although it might be known. One notable exception is the paper [Se] (see also [LS]) which contains a statement ([Se, Th. 2.1]) similar to the right side of (2.2), namely it is proved there that there is a constant A such that for any p-orthogonal family $(d_i)_{i\in\mathbb{N}}$ and any n, we have

$$\left\|\sum_1^n d_i\right\|_p \leq A n^{1/2} \sup_{i\in\mathbb{N}} \|d_i\|_p. \tag{2.6}$$

Note that (2.6) follows also from the right side of (2.2). The basic idea of the proof of (2.6) in [Se] turns out to be essentially the same as the one used above for the right side of (2.2), but the dependence of A with respect to p (or the connection with the combinatorics of partitions) does not appear in [Se]. (I am very grateful to Prof. Serfling for kindly communicating to me a copy of this paper upon request, to allow a comparison with the above results.)

Remark 2.5. As a corollary, we obtain a proof of the classical Burkholder-Gundy inequalities, which say that $\|S\|_p$ and $\left\|\sum d_n\right\|_p$ are equivalent whenever $d = (d_n)$ is a martingale difference sequence. Indeed, as already mentioned, these are p-orthogonal. Moreover, the inequality $\left(\sum \|d_n\|_p^p\right)^{1/p} \leq 2\left\|\sum d_n\right\|_p$ is elementary (by interpolation between $p = 2$ and $p = \infty$). Therefore, (2.2) implies in this case that for any choices of signs $\varepsilon_n = \pm 1$ we have

$$\left\|\sum \varepsilon_n d_n\right\|_p \leq C_p \left\|\sum d_n\right\|_p. \tag{2.7}$$

Finally interpolation and duality starting from (2.7) allow us to pass from p an even integer to the whole range $1 < p < \infty$.

Note there is a well known very classical proof due to Paley [Pa] (for dyadic martingales), which is also based on the case when p is an even integer, but Paley's proof uses the "martingale assumption" several times (and not merely p-orthogonality); moreover he uses the maximal inequalities, which do not seem to have a counterpart for non-commutative martingales.

Remark 2.6. Note that we cannot have a lower bound $A_p\|S\|_p \le \left\|\sum d_i\right\|_p$ for general p-orthogonal sums. Indeed, just taking a pair d_1, d_2 and the rest equal to zero, we see that when $p = 4$ this would imply that $\|d_1\|_p \le A_p^{-1}\|d_1 + d_2\|_p$ which is clearly absurd without any assumption on the pair d_1, d_2. (Note in particular that p-orthogonality does not even imply linear independence!.)

3. The non-commutative case

Let M be a von Neumann algebra equipped with a faithful normal and normalized trace τ. Let $1 \le p < \infty$. The space $L_p(M,\tau)$ (or simply $L_p(\tau)$) is defined as the completion of M with respect to the norm $\|x\|_p = \tau(|x|^p)^{1/p}$ (here of course $|x| = (x^*x)^{1/2}$). It is natural, say by convention, to set $L_\infty(\tau) = M$ equipped with the operator norm.

Now if p is an even integer we say that a finite sequence $(d_i)_{i\in I}$ in $L_p(\tau)$ is p-orthogonal if, for any *injective* map $g\colon [1,\ldots,p] \to I$, we have

$$\tau(d^*_{g(1)}d_{g(2)}\ldots d^*_{g(p-1)}d_{g(p)}) = 0.$$

Observe that p-orthogonality is inherited by subfamilies, and also, that if the cardinality of I is $< p$ then any family $d = (d_i)_{i\in I}$ is p-orthogonal, but this is actually irrelevant for our purposes, since we are only interested in the case when I is large compared with p.

Of course if M is commutative, then (M,τ) can be identified with $L_\infty(\Omega, m)$ for some measure space (Ω, m) and $\tau(x) = \int x dm$, so that we recover the notion introduced in the preceding section. The main result of this paper is the following non-commutative version of Theorem 2.1.

THEOREM 3.1. *Let (M,τ) be as above. Let $p > 2$ be an even integer. Then for any p-orthogonal finite sequence $(d_i)_{i\in I}$ in $L_p(\tau)$, we have*

$$\left\|\sum_{i\in I} d_i\right\|_{L_p(\tau)} \le \frac{3\pi}{2} p\|S\|_{L_p} \tag{3.1}$$

where the "square function" S is defined as

$$S = \left(\sum_{i\in I} d_i^* d_i + d_i d_i^*\right)^{1/2} . \tag{3.2}$$

In particular, when I is infinite, if S converges (strong operator topology) to a bounded operator in the unit ball of M, and if $(d_i)_{i\in I}$ is p-orthogonal for all p, then the series $\sum d_i$ obviously converges in $L_2(\tau)$ and its sum satisfies

$$\tau\left(\exp\left(\delta\left|\sum d_i\right|\right)\right) \le 2$$

where $\delta > 0$ is a numerical constant (independent of the family (d_i)).

Proof. Let $f = \sum_{i\in I} d_i$. As before, we can write

$$\tau[(f^*f)^{p/2}] = -\sum_{\dot{0}<\pi} \mu(\dot{0},\pi)\Psi(\pi)$$

where Φ and Ψ are now defined as follows:

$$\Phi(\sigma) = \sum_{g:\,\pi(g)=\sigma} \tau(d_{g(1)}^* d_{g(2)} \dots d_{g(p-1)}^* d_{g(p)}),$$
$$\Psi(\pi) = \sum_{\sigma\ge\pi} \Phi(\sigma),$$

or equivalently,

$$\Psi(\pi) = \sum_{g:\,\pi(g)\ge\sigma} \tau(d_{g(1)}^* d_{g(2)} \dots d_{g(p-1)}^* d_{g(p)}).$$

A quick inspection of the proof of Theorem 2.1 shows that all we need is the next statement.

SUBLEMMA 3.2. *For any partition π, we have*

$$|\Psi(\pi)| \le (\alpha\|S\|_p)^{p-r_1(\pi)}\|f\|_p^{r_1(\pi)},$$

where $\alpha = \frac{3\pi}{4}$.

Indeed, using this and arguing as for Theorem 2.1, we obtain

$$\|f\|_p^p \le \sum_{0\le r<p} \|f\|_p^r(\alpha\|S\|_p)^{p-r}\binom{p}{r}(p-r)!,$$

hence by Sublemma 2.3, we conclude that

$$\|f\|_p \le 2\alpha p\|S\|_p.$$

This shows (3.1). The last assertion in Theorem 3.1 is then deduced from this exactly as Corollary 2.2 was deduced from Theorem 2.1. We leave the details to the reader. □

Remark. The inequality (3.1) probably admits a converse (analogous to the left side of (2.1)), but we could not prove it. The difficulty lies in the fact that (when, say, $p = 4$) terms such as

$$\psi = \sum_{ij} \tau(d_i^* d_j d_i^* d_j)$$

may be negative in the non-commutative case. For instance, if $(d_i)_{1\le i\le n}$ is a family of anti-commuting self-adjoint unitaries (i.e., a spin system) then $d_i^* d_j d_i^* d_j = -I$ for all $i \neq j$ and it is equal to I otherwise. Hence, in this case $\psi = n - (n^2 - n) = 2n - n^2$.

Remark. The above proof actually shows that $\|f\|_p \le 2\alpha p S(d, p)$, with $S(d, p)$ as defined in (0.2).

To prove Sublemma 3.2, we need several more lemmas. In the first one, we denote by $\mathbf{F}_I$ the free group with free generators $(g_i)_{i\in I}$ and by φ the normalized trace on the von Neumann algebra of $\mathbf{F}_I$ (essentially as in Remark 0.2).

SUBLEMMA 3.3. *Fix $p \ge 2$ and let $\pi \in P_p$. Let B_1 be the union of all the singletons of π, and let B_2 be the complement of B_1 in $[1, \ldots, p]$. Let $f_k = \sum_{i\in I} d_i(k)$ be a (finite) sum in $L_p(\tau)$. Let $\tilde{f}_k = \sum_{i\in I} \lambda(g_i) \otimes d_i(k)$ in $L_p(\varphi \times \tau)$. Then, for a suitable discrete group G, there are elements $F_1, \ldots, F_p$ in $L_p(\tau_G \times \tau)$ satisfying*

$$\|F_k\|_p = \|\tilde{f}_k\|_p \forall k \in B_2 \quad \textit{and} \quad \|F_k\|_p = \|f_k\|_p \forall k \in B_1, \tag{3.3}$$

and such that

$$\sum_{\pi(g)\ge\pi} \tau(d_{g(1)}(1) \ldots d_{g(p)}(p)) = (\tau_G \otimes \tau)[F_1 F_2 \ldots F_p]. \tag{3.4}$$

Proof. First consider the case when π has only one block $[1, \ldots, p]$, i.e., we want to rewrite

$$\psi = \sum_{i\in I} \tau(d_i(1) \ldots d_i(p)).$$

Then if $p = 2$ this is easy; we can write

$$\begin{aligned} \psi &= \sum_{i,j\in I} \varphi(\lambda(g_i)^* \lambda(g_j)) \tau(d_i(1) d_j(2)) \\ &= (\varphi \times \tau)[F_1 F_2] \end{aligned}$$

where

$$F_1 = \sum \lambda(g_i)^* \otimes d_i(1), \quad F_2 = \sum \lambda(g_j) \otimes d_j(2),$$

and we obtain the announced result.

Now assume that π has one block $[1, \dots, p]$ but that p is arbitrary. Let

$$\begin{aligned} F_1 &= \sum \lambda(g_i)^* \otimes 1 \otimes 1 \dots 1 \otimes d_i(1), \\ F_2 &= \sum \lambda(g_i) \otimes \lambda(g_i)^* \otimes 1 \cdots \otimes 1 \otimes d_i(2), \\ F_3 &= \sum 1 \otimes \lambda(g_i) \otimes \lambda(g_i)^* \otimes \cdots \otimes 1 \otimes d_i(3), \end{aligned}$$

and so on, until

$$\begin{aligned} F_{p-1} &= \sum 1 \otimes \cdots \otimes 1 \otimes \lambda(g_i) \otimes \lambda(g_i)^* \otimes d_i(p-1), \\ F_p &= \sum 1 \otimes \cdots \otimes 1 \otimes \lambda(g_i) \otimes d_i(p). \end{aligned}$$

Then it is easy to check that (3.3) holds in $L_p(\tau_G \times \tau)$ where G is a product of suitably many copies of the free group $\mathbb{F}_I$. Moreover, we clearly have $\psi = (\tau_G \otimes \tau)[F_1 F_2 \dots F_p]$. In addition, we have produced a group G and families $(\xi_i^1)_{i\in I}, \dots, (\xi_i^p)_{i\in I}$ in $VN(G)$ such that, for any map $g\colon [1, \dots, p] \to I$, we have $\tau_G(\xi^1_{g(1)} \dots \xi^p_{g(p)}) \neq 0$ if and only if $g(i) = g(j)\ \forall\, i, j$ and in that case the non-zero value is equal to 1.

It is now easy to see the recipe for the general case.

Let $A_1, \dots, A_\nu$ be the blocks of the partition π with more than one element. We will introduce discrete groups $G_1, \dots, G_\nu$ and their product $G = G_1 \times \cdots \times G_\nu$. Let $VN(G)$ denote the von Neumann algebra of G, generated by the left regular representation λ_G. We will identify λ_G with $\lambda_{G_1} \otimes \cdots \otimes \lambda_{G_\nu}$ and $VN(G)$ with $VN(G_1)\overline{\otimes} \cdots \overline{\otimes} VN(G_\nu)$.
For each q with $1 \le q \le \nu$ the previous argument (applied to each block separately) produces elements $(\xi_i^q)_{i\in I}$ in $VN(G_q)$ such that for any function $g\colon A_q \to I$, $\tau_{G_q}\left(\prod_{a\in A_q} \xi^q_{g(a)}\right) = 1$ iff g takes one single value only and $= 0$ otherwise. (Here the product sign is meant to respect the order of the elements in A_q.)
Then we define

$$F_k \in VN(G_1) \otimes \cdots \otimes VN(G_\nu) \otimes L_p(\tau)$$

as follows:

$$\forall\, k \in A_1 \qquad F_k = \sum_{i\in I} \xi_i^1 \otimes 1 \otimes \cdots \otimes 1 \otimes d_i(k),$$

$$\forall\, k \in A_2 \qquad F_k = \sum_{i\in I} 1 \otimes \xi_i^2 \otimes 1 \cdots \otimes d_i(k),$$

$$\forall\, k \in A_\nu \qquad F_k = \sum_{i\in I} 1 \otimes \cdots \otimes \xi_i^\nu \otimes d_i(k).$$

Finally, if $k \notin A_1 \cup \cdots \cup A_\nu$ (i.e., k belongs to some singleton block of the partition π) we set

$$F_k = 1 \otimes \cdots \otimes 1 \otimes f_k.$$

Then it is easy to check that (3.4) holds. Finally, going back to the definition of $(\xi_i^q)_{i\in I}$ we see that (3.3) holds. Indeed, it is well known that (analogous to Fell's absorption principle) we have

$$\left\|\sum \lambda(g_i)^* \otimes \lambda(g_i) \otimes 1 \otimes d_i\right\|_p = \left\|\sum \lambda(g_i) \otimes d_i\right\|_p.$$

The latter identity can be checked easily in our case by expanding the p-th powers of the sums on both sides and observing that the corresponding moments are pairwise identical. We leave this to the reader. □

LEMMA 3.4. *Let $p \geq 2$ be an even integer. For any $d = (d_i)_{i\in I}$ in $L_p(\tau)$, we have*

$$\left\|\sum_{i\in I} \lambda(g_i) \otimes d_i\right\|_p \leq \frac{3\pi}{4} S(d, p). \tag{3.5}$$

We will deduce this from the next result. The inequality (3.6) below is due to Buchholz [Bu2]; we include a slightly different argument. ((3.7) is well known.)

LEMMA 3.5. *Let $p \geq 2$ be an even integer. Let $(c_i)_{i\in I}$ be a free circular family in Voiculescu's sense (cf. [VDN]) normalized so that $\varphi(|c_i|^2) = 1$ and $\|c_i\|_\infty = 2$. Then for all $d = (d_i)_{i\in I}$ in $L_p(\tau)$ we have*

$$\left\|\sum_{i\in I} c_i \otimes d_i\right\|_p \leq K_p S(d, p) \tag{3.6}$$

where $K_p = \left[\binom{p}{p/2}\frac{1}{1+p/2}\right]^{1/p} \leq 2$. Moreover, we also have

$$\left\|\sum_{i\in I} \lambda(g_i) \otimes d_i\right\|_p \leq 3\pi/8 \left\|\sum c_i \otimes d_i\right\|_p. \tag{3.7}$$

Proof. Let $p = 2q$. By [Sp1] (see also [BSp] and [HT]), we can write

$$\left\|\sum c_i \otimes d_i\right\|_p^p = \sum_{\pi\in S_q^{nc}} \sum_{i_1 i_2 \ldots i_q \in I} \tau(d_{i_1}^* d_{i_{\pi(1)}} \ldots d_{i_q}^* d_{i_{\pi(q)}})$$

where the first sum runs over a certain subset S_q^{nc} of the set S_q of all permutations of $[1, \ldots, q]$. This subset is defined as follows. We consider the sequence of numbers $\Omega = [1, \pi(1), 2, \pi(2), \ldots, q, \pi(q)]$. We will associate to π a partition of $[1, 2, \ldots, 2q]$ into disjoint pairs like this: Let $1 \le i < j \le 2q$. Then we say that the two-point set $[i, j]$ belongs to the partition if, in Ω, we find the same number at both the i-th and the j-th place. Clearly this is indeed a partition of $[1, \ldots, 2q]$ into pairs composed of an odd and an even integer. We will denote by S_q^{nc} the set of permutations π such that the associated partition just defined is non-crossing (cf. [K, Sp2]). It can be shown by a counting argument (cf. [K]) that $\mathrm{card}(S_q^{\mathrm{nc}}) = \binom{2q}{q}\frac{1}{q+1}$ (Catalan number). Hence we have

$$\left\|\sum c_i \otimes d_i\right\|_p \le K_p \gamma$$

where γ is the positive number defined by

$$\gamma^p = \max_{\pi \in S_q^{\mathrm{nc}}} \left\{ \left| \sum_{i_1, \ldots, i_q} \tau(d_{i_1}^* d_{i_{\pi(1)}} \ldots d_{i_q}^* d_{i_{\pi(q)}}) \right| \right\}.$$

Thus the proof of (3.6) can be easily completed using Lemma 3.6 below (perhaps of some independent interest). To check (3.7) we can note that by Voiculescu's results, the family $(c_i)_{i \in I}$ has the same distribution as a family of the form $(u_i|c_i|)_{i \in I}$ where $(u_i)_{i \in I}$ and $(c_i)_{i \in I}$ are $*$-free and where $(u_i)_{i \in I}$ and $(\lambda(g_i))_{i \in I}$ have the same $*$-distribution (in the sense of [VDN]). Let $\delta = \varphi(|c_i|)$ (independent of I). A simple computation shows that $\delta = 8/3\pi$. In addition, note that $(u_i \varphi(|c_i|))_{i \in I}$ can be viewed as obtained by a suitable conditional expectation from $(u_i|c_i|)_{i \in I}$. Hence we can write

$$\delta \left\|\sum u_i \otimes d_i\right\|_p \le \left\|\sum u_i |c_i| \otimes d_i\right\|_p = \left\|\sum c_i \otimes d_i\right\|_p$$

which yields (3.7). □

Proof of Sublemma 3.2. We apply Hölder's inequality to the right side of (3.4); then we use (3.3) and (3.5) to obtain Sublemma 3.2. □

LEMMA 3.6. *Let $(d_i(k))_{i \in I}$, $k = 1, 2, \ldots, p$ (with $p = 2q$ as above) be families of elements in $L_p(\tau)$. Then, for all π in S_p^{nc}, we have*

$$\left| \sum_{i_1 i_2 \ldots i_q \in I} \tau(d_{i_1}(1) d_{i_{\pi(1)}}(2) \ldots d_{i_q}(p-1) d_{i_{\pi(q)}}(p)) \right| \le S_1 S_2 \ldots S_p \tag{3.8}$$

where $S_k = S((d_i(k))_{i \in I}, p)$. More generally, for any $t \ge 1$, we have

$$\left\| \sum_{i_1 \ldots i_q} d_{i_1}(1) d_{i_{\pi(1)}}(2) \ldots d_{i_q}(p-1) d_{i_{\pi(q)}}(p) \right\|_t \le \prod_{k=1}^{p} S((d_i(k))_{i \in I}, pt). \tag{3.9}$$

Proof. Note that when $t = 1$, (3.9) obviously implies (3.8). We will prove (3.9) (for all $t \geq 1$) by induction on q. The case $q = 1$ is very easy since it is well known that for all $t \geq 1$,

$$(3.10)\quad \left\| \sum d_i(1) d_i(2) \right\|_t \leq \left\| \left(\sum d_i(1) d_i(1)^* \right)^{1/2} \right\|_{2t} \cdot \left\| \left(\sum d_i(2)^* d_i(2) \right)^{1/2} \right\|_{2t}.$$

Assume that (3.9) has been proved (for all $t \geq 1$) for the value $q - 1$. Let us show that it also holds for q. By definition of S_q^{nc}, the partition of $[1, \ldots, 2q]$ into pairs associated to π is non-crossing. This implies that this partition admits an interval $[k, k+1]$ as one of its blocks. Moreover if we delete this block the resulting partition of the remaining set (with the induced ordering) is still non-crossing. Let $x = \sum_{i_1 \ldots i_q} d_{i_1}(1) d_{i_{\pi(1)}}(2) \ldots d_{i_q}(p-1) d_{i_{\pi(q)}}(p)$. Thus we can write

$$x = \sum_\alpha a_\alpha \sum_{i \in I} d_i(k) d_i(k+1) b_\alpha;$$

hence

$$(3.11)\quad \|x\|_t \leq \left\| \sum_{i \in I} d_i(k) d_i(k+1) \right\|_{qt} \cdot C$$

where

$$C = \sup \left\{ \left\| \sum a_\alpha T b_\alpha \right\|_t \;\middle|\; \|T\|_{qt} \leq 1 \right\}.$$

Now, by the induction hypothesis we know that for any $s \geq 1$, for any u with $\|u\|_\infty \leq 1$, we have

$$\left\| \sum_\alpha a_\alpha u b_\alpha \right\|_s \leq C',$$

with $C' = \prod_{\xi \notin [k,k+1]} S((d_i(\xi))_{i \in I}, (p-2)s)$. Thus the linear mapping v defined by

$$v(y) = \sum_\alpha a_\alpha y b_\alpha$$

is bounded from L_∞ into L_s with norm $\leq C'$. Since the partition corresponding to $\sum b_\alpha u a_\alpha$ is obviously non-crossing also, we have the same bound for ${}^t v(y) = \sum b_\alpha y a_\alpha$, or equivalently we know that v is bounded with norm $\leq C'$ from $L_{s'}$ to L_1. By interpolation, for any $0 < \theta < 1$, it follows that v is also bounded from L_a to L_b where

$$\frac{1}{a} = \frac{1-\theta}{\infty} + \frac{\theta}{s'}, \quad \frac{1}{b} = \frac{1-\theta}{s} + \frac{\theta}{1}.$$

If we choose s so that $\frac{1}{s} = \frac{1}{t}\left[1 - \frac{1}{q}\right]$. Then imposing $b = t$, we find θ determined by $\theta(1 - 1/s) = \frac{1}{b} - \frac{1}{s} = \frac{1}{t} - \frac{1}{s}$. Then the value of a is given by $\frac{1}{a} = \frac{\theta}{s'} = \theta\left(1 - \frac{1}{s}\right) =$

$\frac{1}{t} - \frac{1}{s} = \frac{1}{qt}$. Thus we conclude that v is bounded from L_{qt} to L_t with norm $\leq C'$. In other words, we have established that

$$C \leq C'.$$

Note that $(p-2)s = 2(q-1)s = 2qt = pt$. Moreover, by (3.10) (applied in L_{qt} instead of L_t) we have

$$\left\| \sum d_i(k) d_i(k+1) \right\|_{qt} \leq C''$$

with

$$C'' = \left\| \left(\sum d_i(k) d_i(k)^* \right)^{1/2} \right\|_{pt} \cdot \left\| \left(\sum d_i(k+1)^* d_i(k+1) \right)^{1/2} \right\|_{pt}.$$

Hence we can finally deduce from (3.11) that $\|x\|_t \leq CC'' \leq C'C''$ and since $(p-2)s = pt$ we find that $C'C''$ is less or equal to the right side of (3.9). □

Remark. The analogs of Proposition 1.1 and Theorem 1.2 for the lattice of non-crossing partitions are proved in [Sp2]. Thus we can combine this with the same argument as above if the function $\sigma \to \Phi(\sigma)$ is supported by the set of non-crossing partitions, and the resulting constants will remain bounded when p tends to ∞. However, we could not find a significant application of this idea.

4. Applications to harmonic analysis

The results of this section can be viewed as a continuation of a series of investigations devoted to Fourier series with coefficients in a non-commutative L_p-space, such as [TJ], [BP], [LP], [LPP], [X].

As explained in the introduction, our main inequality applies to p-dissociate partitions $\Lambda = \bigcup_{i \in I} \Lambda_i$ of a subset Λ in a discrete group G. The inequality in Theorem 4.1 below is closely related (and partly motivated) by the recent papers [H1-2] on the so-called $\Lambda(p)_{cb}$-sets, which are a certain non-commutative version of Rudin's classical $\Lambda(p)$-sets (cf. [Ru]). The basic examples of such sets are the p-dissociate ones. However, in the quest for examples of the "largest possible" sets satisfying such inequalities, the next result turns out to be more efficient and more flexible (in particular in the analysis of $\Lambda(p)_{cb}$-sets constructed as random subsets of a given set), even though its assumptions become more complicated than the condition of being p-dissociate.

THEOREM 4.1. *Let $1 = \sum_{j \in J} P_j$ be an orthogonal decomposition of the identity of $L_2(\tau)$. Let $p = 2q$ be an even integer > 2. Let $d = (d_i)_{i \in I}$ be a finite family in $L_p(\tau)$. We set $x^\omega = x^*$ if q is odd and $x^\omega = x$ if q is even. Let F be the* *set of all* injective *functions* $g\colon [1, 2, \ldots, q] \to I$. *For any* g *in* F, *we let* $x_g = d^*_{g(1)} d_{g(2)} d^*_{g(3)} \ldots d^{\omega}_{g(q)}$. *We define*

$$N(d) = \sup_{j \in J} \mathit{card}\{g \in F \mid P_j x_g \neq 0\}.$$

Then we have

$$\left\| \sum_{i \in I} d_i \right\|_p \leq \left[(4N(d))^{1/p} + p \cdot \frac{9\pi}{8} \right] S(d, p).$$

Proof. Since the argument is essentially the same as in [H2] modulo the combinatorics of §1, we will only sketch the proof.

Let $f = \sum_{i \in I} d_i$. We have

$$\|f\|_p^q = \|f^* f \ldots f^{\omega}\|_2.$$

Developing this product as in §1 but with $n = q$ this time, $V = L_2(\tau)$ and φ the product mapping, we obtain

$$f^* f f^* \ldots f^{\omega} = \Phi(\dot{0}) - \sum_{\dot{0} < \pi \in P_q} \mu(\dot{0}, \pi) \Psi(\pi) \tag{4.1}$$

where $\Phi(\sigma) = \sum_{\pi(g) = \sigma} x_g$. Using Sublemma 3.3 and (3.5) with p replaced by q (recall $\alpha = \frac{3\pi}{4}$), we obtain

$$\|\Psi(\pi)\|_2 \leq \|f\|_p^{r_1(\pi)} (\alpha \|S\|_p)^{q - r_1(\pi)}. \tag{4.2}$$

On the other hand, we can write

$$\begin{aligned}
\|\Phi(\dot{0})\|_2^2 &= \sum_{j \in J} \|P_j \Phi(\dot{0})\|_2^2 \\
&= \sum_{j \in J} \left\| \sum_{g \in F} P_j x_g \right\|_2^2 \\
&\leq \sum_{j \in J} \sum_{g \in F} \|P_j x_g\|_2^2 N(d) \\
&\leq \sum_{g \in F} \|x_g\|_2^2 N(d) \\
&\leq \sum_{g_1, \ldots, g_q \in I} \tau(d^*_{g(1)} \ldots d^{\omega}_{g(q)} (d^{\omega}_{g(q)})^* \ldots d_{g(1)}) \cdot N(d);
\end{aligned}$$

hence by a special case of Lemma 3.6, we have

$$\|\Phi(\dot{0})\|_2^2 \leq N(d) S(d, p)^{2q}. \tag{4.3}$$

Thus, arguing as in the proof of Theorem 2.1, combining (4.1), (4.2) and (4.3), we finally obtain

$$\|f\|_p^q \le N(d)^{1/2} S(d,p)^q + \sum_{0\le s<q} \binom{q}{s} (q-s)!\|f\|_p^s (\alpha S(d,p))^{q-s};$$

hence, if we now set $y = \frac{S(d,p)}{\|f\|_p}$, we have

$$1 \le N(d)^{1/2} y^q + \sum_{0\le s<q} \binom{q}{s} (q-s)!(\alpha y)^{q-s}.$$

We claim that $y \ge \min\left\{\left(\frac{1}{2N(d)^{1/2}}\right)^{1/q}, \frac{1}{3q\alpha}\right\}$. Indeed, if $y < (2N(d)^{1/2})^{-1/q}$ then, as in the proof of Sublemma 2.3, we have

$$3/2 \le \int_0^\infty (1+\alpha t y)^q e^{-t}\, dt$$

which, if $qy\alpha < 1$, yields $3/2 \le (1-qy\alpha)^{-1}$ whence $y \ge \frac{1}{3q\alpha}$; otherwise $qy\alpha \ge 1$ which also implies $y \ge \frac{1}{3q\alpha}$. Thus we conclude as announced that a fortiori we have

$$1/y \le (2N(d)^{1/2})^{1/q} + 3q\alpha. \qquad \square$$

COROLLARY 4.2 [H2]. *Let $\Lambda \subset G$ be a subset of a discrete group G. Let $p = 2q$ be an even integer > 2. For any t in G, let $N_q(t, \Lambda)$ be the number of q-tuples $(t_1, \dots, t_q)$ of mutually distinct elements of Λ such that*

$$t = t_1^{-1} t_2 t_3^{-1} t_4 \dots t_q^{\omega}.$$

We assume that

$$N_q(\Lambda) = \sup_{t\in G} N_q(t,\Lambda) < \infty.$$

Then, for any finitely supported family $a = (a_t)_{t\in\Lambda}$ in a non-commutative L_p-space associated to a semi-finite trace T, we have

$$\left\|\sum_{t\in\Lambda} \lambda(t)\otimes a_t\right\|_{L_p(\tau_G\times T)} \le \left[(4N_q(\Lambda))^{1/p} + p\frac{9\pi}{8}\right] S(a,p).$$

Proof. We apply the previous result to $\tau = \tau_G \times T$ so that $L_2(\tau) = L_2(\tau_G) \otimes_2 L_2(T)$ and to the $\perp$ decomposition $L_2(\tau) = \bigoplus_{t\in G} H_t$ with $H_t = \lambda(t)\otimes L_2(T)$ and $I = \Lambda$. Clearly, if we set $d_t = \lambda(t)\otimes a_t$, $t \in \Lambda$ we find $N(d) \le N_q(\Lambda)$ and the result follows since $S(d,p) = S(a,p)$. $\square$

5. Tensor products of Banach spaces

The main idea exploited above can also be used in a very abstract setting, which we briefly indicate in this section. Let $E_1, \dots, E_p$ be Banach spaces and let $E_1 \widehat{\otimes} \cdots \widehat{\otimes} E_p$ be their projective tensor product equipped with its projective norm denoted by $\| \quad \|_\wedge$ (e.g., see [DF]).

For each $k = 1, 2, \dots, p$ consider a finite sum

$$f_k = \sum_{i \in I} d_i(k)$$

where $d_i(k)$ are elements of E_k.

Now let $(\varepsilon_i)_{i \in I}$ be a sequence of independent ± 1-valued random variables on a probability space (Ω, P) with $P(\varepsilon_i = \pm 1) = 1/2$, as usual.

Note. The family $(\varepsilon_i)_{i \in I}$ is the Abelian counterpart of the family $(\lambda(g_i))_{i \in I}$ used above.

We wish to develop the tensor product

$$f_1 \otimes \cdots \otimes f_p$$

in the Banach space $E_1 \widehat{\otimes} \cdots \widehat{\otimes} E_p$. We will use the notation in §1 applied to the canonical multilinear mapping φ: $E_1 \times \cdots \times E_p \to E_1 \widehat{\otimes} \cdots \widehat{\otimes} E_p$. Hence we now have

$$\Phi(\dot{0}) = \sum_g d_{g(1)}(1) \otimes \cdots \otimes d_{g(p)}(p)$$

where the sum runs over all injective maps g: $[1, 2, \dots, p] \to I$. Let π be a partition of $[1, \dots, p]$. Using the random variables $(\varepsilon_i)_{i \in I}$ instead of $(\lambda(g_i))_{i \in I}$ in the preceding section, it is easy to adapt the proof of Sublemma 3.2 to obtain the following result:

Let $A \subset [1, \dots, p]$ be the union of the singletons of the partition π (note that the cardinality of A is at most $p - 2$, unless $\pi = \dot{0}$) and, as before, let

$$\Psi(\pi) = \sum_{g\colon \pi(g) \geq \pi} d_{g(1)}(1) \otimes \cdots \otimes d_{g(p)}(p).$$

Then we have

$$f_1 \otimes \cdots \otimes f_p = \Phi(\dot{0}) - \sum_{0 < \pi} \Psi(\pi) \mu(\dot{0}, \pi)$$

and

$$\|\Psi(\pi)\|_\wedge \leq \prod_{k \in A} \|f_k\| \cdot \prod_{k \notin A} S_k$$

where

$$S_k = \left(\mathbb{E} \left\| \sum_{i \in I} \varepsilon_i d_i(k) \right\|^p \right)^{1/p}.$$

We can now state the main result of this section.

THEOREM 5.1. *With the above notation, we have*

$$\left\| f_1 \otimes \cdots \otimes f_p - \sum_{\substack{g\colon [1,\ldots,p]\to I \\ g \text{ injective}}} d_{g(1)} \otimes \cdots \otimes d_{g(p)} \right\|_{\wedge} \leq \sum_{\substack{A\subset[1,\ldots,p] \\ |A|\leq p-2}} \prod_{k\in A} \|f_k\| \cdot \prod_{k\notin A} S_k \cdot (p-|A|)!.$$

In the particular case $E_1 = E_2 = \cdots = E_p = E$ we obtain:

COROLLARY 5.2. *Let $f = \sum_{i\in I} d_i$ be a finite sum in a Banach space E. Let $f^{\otimes p} = f \otimes \cdots \otimes f$ (p-times). Then*

$$\left\| f^{\otimes p} - \sum_{\substack{g\colon [1,\ldots,p]\to I \\ g \text{ injective}}} d_{g(1)} \otimes \cdots \otimes d_{g(p)} \right\|_{\wedge} \leq \sum_{0\leq s\leq p-2} \binom{p}{s} (p-s)! \|f\|^s S^{p-s}$$

where $S = \left(\mathbb{E} \left\| \sum_{i\in I} \varepsilon_i d_i \right\|^p\right)^{1/p}$.

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Department of Mathematics, Texas A&M University, College Station, TX 77843.

Equipe d'Analyse, Case 186, 75252, Université Paris VI, Paris Cedex 05, France
gip@ccr.jussieu.fr