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ON NONSINGULAR CHACON TRANSFORMATIONS

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ABSTRACT. We construct nonsingular Chacon transformations with 2-cuts of type III_λ, for every $0 \le \lambda \le 1$, and type II_∞ and show that their 2-fold Cartesian product is ergodic.

1. Introduction

Chacon's transformation with 2-cuts (i.e., with three copies of the n^{th} column in the $(n+1)^{\text{st}}$ column for all $n \ge 0$) as described in [F], p. 86, has been shown to enjoy several interesting properties. For instance, it is a (finite measure preserving) rank one weakly mixing transformation that is not mixing [C], [F], has a trivial centralizer and no non-trival factors [J], and enjoys the minimal self-joinings property [JRS].

In this paper we study nonsingular analogues of Chacon's transformation (with 2-cuts). Some nonsigular analogues are known in [RS] for the case of unbounded cuts and [JS] for the case of 2-cuts. For the transformations in [JS], at every stage of the construction each level is cut into three subintervals at a fixed ratio $1 : \lambda : 1$, for a constant $0 < \lambda < 0$, so that the resulting transformation is of type III_λ and has no nontrivial factors. Also, type III_1 can be obtained in a similar manner, but types III_0 and II_∞ are not available due to technical reasons.

Using different methods from those above (nonsingular joinings and coding techniques), we analyze nonsingular Chacon transformations with 2-cuts and variable ratios, that is, at the n^{th} stage of the construction each level is cut into three intervals with the ratio $1 : \lambda_n : 1$. We show that if the λ_n's are suitably controlled then even type II_∞ and III_0 nonsigular Chacon transformations with ergodic 2-fold Cartesian product are available (Theorem 4.2, Proposition 5.1 and Section 6).

We note that as far as nonsingular rank one transformations admitting unbounded cuts, types II_∞ and III_0 with nonsingular minimal self-joinings are known in [RS]. Also, the type III_λ transformations of [JS] have been studied further in [JS2] and have been shown to be power weakly mixing in [AFS2].

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2. The pseudo-metric d_A

We let $\mathbf{X} = (X, \mathfrak{B}, \mu)$ denote a Lebesgue probability space. A nonsingular transformation is a measurable invertible map $T\colon (X, \mathfrak{B}, \mu) \hookleftarrow$ such that $\mu(A) = 0$ if and only if $\mu(T(A)) = 0$.

We consider the pseudo metric $d_A(x, y)$ derived from any $T \times T$-invariant subset A of $X \times X$. For each $x \in X$ we define a subset A_x of X by

$$A_x = \{y \in X \mid (x, y) \in A\}.$$

Then we see that

$$\begin{aligned} A_{Tx} &= \{y \mid (Tx, y) \in A\} \\ &= \{y \mid (x, T^{-1}y) \in A\} \\ &= T(A_x). \end{aligned}$$

The pseudo metric d_A on the space X is defined by

$$d_A(x, y) = \mu(A_x \Delta A_y).$$

As observed in [R2], if T is an ergodic probability-preserving transformation and if a measurable subset A is $(T \times T)$-invariant, then the pseudo-metric d_A is an isometry for T, i.e., $d_A(Tx, Ty) = d_A(x, y)$. However, our transformations are nonsingular and this does not necessarily follow; but if the Radon-Nikodym derivatives of powers of a nonsingular transformation are carefully considered we will see that the pseudo-metric still can be used to show ergodicity of Cartesian products.

PROPOSITION 2.1. *Let $A \subset X$ with $\mu(A) > 0$ and let $\varepsilon > 0$. Then X is covered by a countable number of ε-balls with respect to the metric d_A.*

Proof. Let $\{E_i\}_{i \geq 1}$ be a countable basis for $(X, \mathfrak{B}, \mu)$. Then for all $\varepsilon > 0$,

$$X = \bigcup_{i \geq 1} \{x \in X\colon\ \mu(E_i \Delta A_x) < \varepsilon/2\}.$$

Let $B_\varepsilon(x) = \{y \in X\colon\ d_A(x, y) \leq \varepsilon\}$ and choose and fix $x_i \in E_i$, for $i \geq 1$. Then

$$X = \bigcup_{i \geq 1} B_\varepsilon(x_i). \qquad \square$$

Remark. If an ergodic finite measure-preserving transformation T acts on X and if $A \subset X \times X$ is $(T \times T)$-invariant, then one can show that (X, d_A) is totally bounded (see [R2]).

3. Chacon maps with variable cuts

Let $\{\theta_n\}_{n\geq 0}$ be a sequence of positive numbers with $\theta_n \leq 1$, and let $\{n_i\}_{i\geq 1}$ and $\{m_i\}_{i\geq 0}$ be sequences of positive integers.

Let $M_0 = 0$, $N_i = M_i + m_i$, $M_{i+1} = N_i + n_i$, and for all $j \geq 0$,

$$\lambda_j = \begin{cases} 1, & \text{if } M_k \leq j < N_k, k \geq 0, \\ \theta_k, & \text{if } N_k \leq j < M_{k+1}, k \geq 0. \end{cases}$$

We define a family of nonsingular Chacon transformations as follows. First set

$$\alpha = \frac{1}{1 + \sum_{j\geq 0} \frac{\lambda_j}{(2+\lambda_0)\cdots(2+\lambda_j)}}.$$

Let $I(1,0) = [0,\alpha)$ and set $C_0 = \{I(1,0)\}$, $h_0 = 1$. Assuming that column C_n of height h_n (i.e., consisting of h_n intervals) is defined, we will define inductively column C_{n+1} of height $h_{n+1} = 3h_n + 1$.

For the sake of clarity in the exposition we will first describe how to obtain column C_1. Decompose $I(1,0)$ into three disjoint intervals $I_0(1,0), I_1(1,0), I_2(1,0)$ from left to right with lengths

$$|I_0(1,0)| = |I_2(1,0)| = \frac{\alpha}{2+\lambda_0} \quad \text{and} \quad |I_1(1,0)| = \frac{\alpha\lambda_0}{2+\lambda_0}.$$

Then add a new interval S_1, abutting with $[0,\alpha)$, called a *spacer*, over $I_1(1,0)$ of the same length as $I_1(1,0)$. Stack the four intervals from bottom to top in the order

$$I_0(1,0), I_1(1,0), S_1, I_2(1,0)$$

and rename them $I(1,1), I(2,1), I(3,1), I(4,1)$. Note that $I(4,1)$ is a measure-preserving copy of $I(1,1)$ for any value of λ_0. Then $C_1 = \{I(j,1)\colon j = 1,\dots,4\}$. Finally write $B(1) = I(1,1), T(1) = I(4,1)$ and $C(1) = \bigcup_{j=1}^{4} I(j,1)$. Now assume that column C_n of height h_n has been defined. We also write $C(n) = \bigcup_{j=1}^{h_n} I(j,n)$.This partially defines an injective transformation T by the affine map

$$T\colon I(j,n) \to I(j+1,n),$$

for $1 \leq j < h_n$. Now decompose each level $I(j,n)$, $j = 1,\dots,h_n$, of C_n into intervals $I_k(j,n)$, for $k = 0,1,2$ such that

$$|I_0(j,n)| = |I_2(j,n)| = \left(\frac{\alpha}{2+\lambda_n}\right)\mu(I(j,n)),$$

$$|I_1(j,n)| = \left(\frac{\alpha\lambda_n}{2+\lambda_n}\right)\mu(I(j,n)).$$

Let S_{n+1} be a new interval which abuts with $C(n)$ over the interval $I_1(h_n, n)$ of the same length as $I_1(h_n, n)$. Thus

$$\mu(S_{n+1}) = \frac{\lambda_n \alpha}{(2+\lambda_0)(2+\lambda_1)\cdots(2+\lambda_n)}.$$

We observe that the maximal length of levels in column C_n is

$$\frac{\alpha}{(2+\lambda_0)(2+\lambda_1)\cdots(2+\lambda_n)}.$$

Let $C_{n,0} = \{I_0(j,n)\colon 1 \le j \le h_n\}$, the left sub-column of C_n, $C_{n,1} = \{I_1(j,n)\colon 1 \le j \le h_n\}$, the middle sub-column, and $C_{n,2} = \{I_2(j,n)\colon 1 \le j \le h_n\}$, the right sub-column. We write $C_i(n) = \cup_{j=1}^{h_n} I_i(j,n)$. Stack the sub-columns from left to right and extend the transformation by the affine maps

$$\begin{aligned} T\colon I_0(h_n, n) &\to I_1(1, n), \\ T\colon I_1(h_n, n) &\to S_{n+1}, \\ T\colon S_{n+1} &\to I_2(1, n). \end{aligned}$$

This defines column C_{n+1}. Rename the $h_{n+1} = 3h_n + 1$ intervals in C_{n+1} as $I(1, n+1), \ldots, I(h_{n+1}, n+1)$. In the limit this defines a nonsingular transformation $T_{\bar\lambda}$ on [0, 1), where $\bar\lambda = \{\lambda_j\}$. To simplify the notation we will write T for $T_{\bar\lambda}$. Finally write

$$B(n) = \bigcup_{j=1}^{h_{n-1}} I_0(j, n-1), \quad T(n) = \bigcup_{j=1}^{h_{n-1}} I_1(j, n-1).$$

It follows that T^{2h_n+1} maps $B(n+1)$ onto $T(n+1)$ in a measure-preserving way independent of the values of λ_i.

4. Ergodic Cartesian product

We will use the following lemma whose proof is straightfoward and is omitted.

LEMMA 4.1. *Let (Y, m) be a pobability space, let $0 < \varepsilon < 1$ and let A and B be disjoint sets of positive measure. Set*

$$C = A \cup B.$$

If a measurable set D satisfies

$$m(C \cap D) > (1-\epsilon) m(C)$$

then

$$m(A \cap D) > \left(1 - \varepsilon\left(1 + \frac{m(B)}{m(A)}\right)\right) m(A).$$

THEOREM 4.2. *Let T be the nonsingular Chacon transformation with variable cuts. Then if $m_k \to \infty$ as $k \to \infty$ the transformation $T \times T$ is ergodic.*

The proof will be based on a series of lemmas. In what follows we fix a $(T \times T)$-invariant set A of positive measure. What we have to show is that $\mu(A_x) = 1$ for a.e. $x \in X$. Fix $\frac{1}{6} > \varepsilon > 0$ and take $0 < \delta = \delta(\varepsilon) < \varepsilon$ such that if $B \subset X$ and $\mu(B) < \delta$ then $\mu(T^{\pm 1}B) < \varepsilon$.

LEMMA 4.3. *Let $D \subset X$, $\mu(D) > 0$ be such that if $x, y \in D$ then $d_A(x, y) < \delta$. Then there exists an integer $i \geq 1$ and measurable subsets E and E' of D of positive measure such that*

$$T^{h_n} E \cup T^{h_n+1} E' \subset D \text{ where } n = M_i,$$

and

$$\sum_{j=M_i+1}^{\infty} \mu(S_j) < \delta.$$

Proof. Let ξ_k, $k \geq 1$, denote the finite partition of X defined by

$$\xi_k = \left\{ I(j,k) \quad (j = 1, \ldots, h_k),\, S_{k+1},\, \bigcup_{i \geq k+2} S_i \right\}.$$

This defines a refining sequence of partitions converging to the point partition. For each $x \in X$ let $\xi_k(x)$ denote the element in ξ_k containing x. By the Martingale convergence theorem, for a.e. $x \in X$,

$$\mu(D|\xi_k)(x) = \frac{\mu(D \cap \xi_k(x))}{\mu(\xi_k(x))} \to 1_D(x) \quad \text{as} \quad k \to \infty.$$

We claim that for a.e. $x \in X$ and for infinitely many i,

$$x \in C_1(M_i).$$

This is observed as follows. We know that for a.e. $x \in X$, x is eventually in $C(k)$. For any $k \geq 1$, the sets

$$C(k) \cap C_1(\ell), \quad \ell \geq k$$

are independent under the measure induced by μ on $C(k)$. This is because for any $\epsilon_i \in \{0, 1\}$, $0 \leq i \leq \ell - k$ we have

$$\frac{\mu(C(k) \cap \bigcap_{i=0}^{\ell-k} C_1(k+i)^{\epsilon_i} \cap C_1(\ell+1))}{\mu(C(k) \cap \bigcap_{i=0}^{\ell-k} C_1(k+i)^{\epsilon_i})} = \frac{\lambda_{\ell+1}}{2 + \lambda_{\ell+1}},$$

where A^ϵ denotes A if $\epsilon = 1$ and A^c if $\epsilon = 0$. Then this implies

$$\begin{aligned}\frac{\mu(C(k)\cap\bigcap_{i=0}^{\ell-k}C_1(k+i))}{\mu(C(k))} &= \frac{\mu(C(k)\cap\bigcap_{i=0}^{\ell-k}C_1(k+i))}{\mu(C(k)\cap\bigcap_{i=0}^{\ell-k-1}C_1(k+i))}\\ &\quad\times\frac{\mu(C(k)\cap\bigcap_{i=0}^{\ell-k-1}C_1(k+i))}{\mu(C(k)\cap\bigcap_{i=0}^{\ell-k-2}C_1(k+i))}\\ &\quad\times\cdots\times\frac{\mu(C(k)\cap C_1(k))}{\mu(C(k))}\\ &= \frac{\lambda_\ell}{2+\lambda_\ell}\frac{\lambda_{\ell-1}}{2+\lambda_{\ell-1}}\cdots\frac{\lambda_k}{2+\lambda_k}\\ &= \frac{\mu(C(k)\cap C_1(\ell))}{\mu(C(k))}\frac{\mu(C(k)\cap C_1(\ell-1))}{\mu(C(k))}\\ &\quad\cdots\frac{\mu(C(k)\cap C_1(k))}{\mu(C(k))}\end{aligned}$$

Moreover, we have

$$\sum_{i:\, M_i>k}\frac{\mu\{x\in C(k)\colon x\in C_1(M_i)\}}{\mu(C(k))} = \sum_{i:\, M_i>k}\frac{1}{3} = \infty.$$

Then by Borel-Cantelli's lemma we see that for a.e. $z \in C(k)$, and for infinitely many i

$$z \in C_1(M_i)$$

and hence for a.e. $x \in X$, and for infinitely many i,

$$x \in C_1(M_i).$$

We fix such $x \in D$ for which the Martingale Convergence Theorem was applied so that there exists an integer $L = L(x) \geq 1$ such that

$$\frac{\mu(D\cap\xi_k(x))}{\mu(\xi_k(x))} > 1-\delta \quad \text{for all } k \geq L. \tag{1}$$

Now we choose and fix $i \geq 1$ so that

1. $x \in C_1(n)$, where $n = M_i$,
2. $M_i \geq L$,
3. $(\frac{1}{3})^{m_i-1} + (\frac{1}{3})^{m_0+m_1+\cdots+m_{i-1}} < \delta$.

We note that the maximal length of levels in column C_{M_i} is less than $(\frac{1}{3})^{m_0+m_1+\cdots+m_{i-1}}$. From the nature of the spacers,

$$\sum_{j\geq M_i+1}\mu(S_j) < \mu(I(h_{M_i}, M_i)).$$

Now

$$\mu(I(h_{M_i}, M_i)) \leq \left(\frac{1}{3}\right)^{m_0+m_1+\cdots+m_{i-1}},$$

and therefore

$$\sum_{j \geq M_i+1} \mu(S_j) < \delta.$$

Now we define the sets E and E'. The interval $\xi_{n+1}(x)$ is $I_1(j_n, n)$ for some $1 \leq j_n \leq h_n$. We set

$$E = T^{-h_n}(D \cap I_1(j_n, n)) \cap D \cap I_0(j_n, n),$$

and

$$E' = D \cap I_1(j_n, n) \cap T^{-h_n-1}(D \cap I_2(j_n, n)).$$

It is clear that

$$T^{h_n} E \cup T^{h_n+1} E' \subset D.$$

Now we show that E and E' have positive measure. It follows from (1) that

$$\mu(D \cap I_1(j_n, n)) > (1-\delta)\mu(I_1(j_n, n)).$$

Now the Radon-Nikodym derivatives $\frac{d\mu T^{-h_n}}{d\mu}$ and $\frac{d\mu T^{h_n+1}}{d\mu}$ are constant on the subset $I_1(j_n, n)$ and

$$T^{-h_n}(I_1(j_n, n)) = I_0(j_n, n),$$

$$T^{h_n+1}(I_1(j_n, n)) = I_2(j_n, n).$$

Thus

$$\frac{\mu(T^{-h_n}(D \cap I_1(j_n, n)))}{\mu(I_0(j_n, n))} = \frac{\mu(D \cap I_1(j_n, n))}{\mu(I_1(j_n, n))} > 1-\delta > \frac{1}{2},$$

and

$$\frac{\mu(T^{h_n+1}(D \cap I_1(j_n, n)))}{\mu(I_2(j_n, n))} = \frac{\mu(D \cap I_1(j_n, n))}{\mu(I_1(j_n, n))} > 1-\delta > \frac{1}{2}.$$

We note that

$$\mu(D \cap I(j_n, n)) > (1-\delta)\mu(I(j_n, n)).$$

Apply Lemma 4.1 to $I_0(j_n, n)$ and $I_1(j_n, n) \cup I_2(j_n, n)$ instead of A and B. We see that

$$\begin{aligned} \mu(I_0(j_n, n) \cap D) &> \left[1 - \delta\left\{1 + \frac{\mu(I_1(j_n, n) \cup I_2(j_n, n))}{\mu(I_1(j_n, n))}\right\}\right]\mu(I_0(j_n, n) && (2)\\ &= (1-3\delta)\mu(I_0(j_n, n)) && (3)\\ &> \frac{1}{2}\mu(I_0(j_n, n)), && (4) \end{aligned}$$

and

$$\mu(I_2(j_n, n) \cap D) > (1 - 3\delta)\mu(I_2(j_n, n)) \quad (5)$$
$$> \frac{1}{2}\mu(I_2(j_n, n)). \quad (6)$$

Hence,

$$\mu(E) > 0 \quad \text{and} \quad \mu(E') > 0. \qquad \square$$

PROPOSITION 4.4. *Let i be as in Lemma 4.3 and set $n = M_i$. If $B \subset X$ satisfies $\mu(B) < \delta$ then $\mu(T^{h_n+1}B) < 4\varepsilon$.*

Proof. Let V denote $B \cap C(n)$. We will show that

$$\mu(T^{h_n+1}V) < 3\varepsilon$$

and

$$\mu\left(T^{h_n+1}\left(\bigcup_{j\geq n+1} S_j\right)\right) < \varepsilon.$$

For $1 \leq k \leq m_i$, define

$$Z_k = \bigcap_{j=n}^{n+k-1} C_2(j) \text{ and } W_k = \bigcap_{j=n}^{n+k-1} C_0(j).$$

First we mention that for any $\ell > 0$, the restriction

$$T^{-2h_\ell-1}\colon C_2(\ell) \to C_0(\ell)$$

is a measure-preserving bijection. We refer to Figure 1. This yields the following three facts. First, for any $k \geq 1$, the restriction of

$$T^{h_n-h_{n+k}} = T^{-2h_n-1} \circ T^{-2h_{n+1}-1} \circ \cdots \circ T^{-2h_{n+k-1}-1}$$

to the subset Z_k is a measure-preserving map onto W_k. This is observed as follows. When $k = 1$, since $T^{2h_n-1}\colon C_2(n) \to C_0(n)$ is measure-preserving, so is $T^{h_n-h_{n+1}} = T^{-2h_n-1}\colon Z_2 \to W_2$. We assume $T^{h_n-h_{n+k-1}}\colon Z_{k-1} \to W_{k-1}$ is measure-preserving. We note that

$$T^{h_n-h_{n+k}} = T^{h_n-h_{n+k-1}} \circ T^{-2h_{n+k-1}}.$$

We know that

$$Z_k \subset C_2(n + k - 1)$$

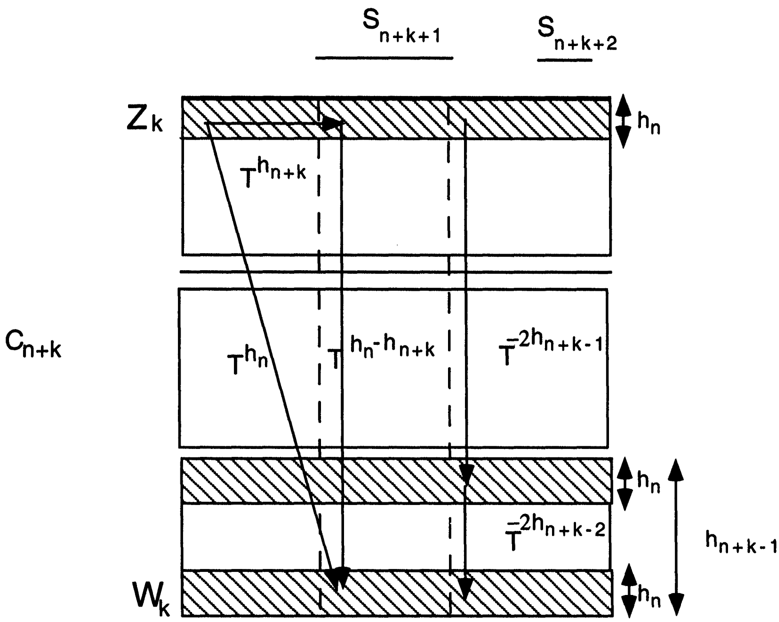

Figure 1. Column C_{n+k}

and

$$T^{-2h_{n+k-1}-1}\colon\ C_2(n+k-1) \to C_0(n+k-1)$$

is measure-preserving. Hence the restriction of $T^{-2h_{n+k-1}-1}$ to Z_k is measure-preserving. In addition,

$$T^{-2h_{n+k-1}-1} Z_k \subset Z_{k-1}.$$

By the inductive hypothesis,

$$T^{h_n - h_{n+k-1}}\colon\ T^{-2h_{n+k-1}-1} Z_k \to W_k$$

is measure-preserving.

Next, let us decompose the set V into three pieces in the C_n−column:

$$V_j = V \cap C_j(n), \quad j = 0, 1, 2.$$

For $1 \le k < m_i$, define inductively,

$$V_{\underbrace{2\cdots 2}_{k \text{ times}} j} = V_{\underbrace{2\cdots 2}_{k \text{ times}}} \cap C_j(n+k),\ j = 0, 1, 2.$$

Then

$$V_{\underbrace{2\cdots 2}_{k \text{ times}} j} \subset Z_k \cap C_j(n+k).$$

We write $T^{h_n} = T^{h_{n+k}} \circ T^{h_n - h_{n+k}}$ and observe that when $k < m_i$, $T^{h_{n+k}}$ is measure-preserving on $C_0(n+k)$, and we have already seen that $T^{h_n - h_{n+k}}$ is measure-preserving from Z_k to W_k. Therefore

$$\mu(T^{h_n} V_{\underbrace{2\cdots 2}_{k \text{ times}} 0}) = \mu(V_{\underbrace{2\cdots 2}_{\ell \text{ times}} 0}), \quad 1 \le k < m_i,$$

and similarly,

$$\begin{aligned}
\mu(T^{h_n+1} V_{\underbrace{2\cdots 2}_{k \text{ times}} 1}) &= \mu(T^{h_n - h_{n+k}}(T^{h_{n+k}+1} V_{\underbrace{2\cdots 2}_{k \text{ times}} 1})) \\
&= \mu(T^{h_{n+k}+1} V_{\underbrace{2\cdots 2}_{k \text{ times}} 1}) \\
&= \mu(V_{\underbrace{2\cdots 2}_{k \text{ times}} 1}).
\end{aligned}$$

The third fact, which can be observed from the figure, is that for all $k \ge 1$,

$$T^{h_n}(Z_k) \subset W_k \cup \left(\bigcup_{j=n+k+1}^{\infty} S_j \right).$$

Now, for $1 \le k < m_i$,

$$\mu(W_k) \le \left(\frac{1}{3}\right)^k.$$

Set $k = m_i - 1$. Then

$$\mu(T^{h_n} Z_{m_i - 1}) \le \left(\frac{1}{3}\right)^{m_i - 1} + \left(\frac{1}{3}\right)^{m_0 + \cdots + m_{i-1}} < \delta.$$

Therefore

$$\mu(T T^{h_n}(Z_{m_i - 1})) < \varepsilon.$$

Now we are ready to evaluate

$$\begin{aligned}
\mu(T^{h_n+1} V) &= \mu(T^{h_n+1} V_0) + \mu(T^{h_n+1} V_1) + \mu(T^{h_n+1} V_2) \\
&= \mu(T(T^{h_n} V_0)) + \mu(V_1) + \mu(T^{h_n+1} V_{20}) + \mu(T^{h_n+1} V_{21}) \\
&\quad + \mu(T^{h_n+1} V_{22}) \\
&= \mu(T(T^{h_n} V_0 \cup T^{h_n} V_{20})) + \mu(V_1 \cup V_{21}) + \mu(T^{h_n+1} V_{22})
\end{aligned}$$

$$\vdots$$

$$= \mu\left(T\left(\bigcup_{\ell=0}^{m_i-1} T^{h_n} V_{\underbrace{2\cdots2}_{\ell \text{ times}}0}\right)\right) + \mu\left(\bigcup_{\ell=0}^{m_i-1} V_{\underbrace{2\cdots2}_{\ell \text{ times}}1}) + \mu(T^{h_n+1} V_{\underbrace{2\cdots2}_{m_i \text{ times}}}\right)$$
$$< \varepsilon + \delta + \varepsilon < 3\varepsilon.$$

Here we used the fact that the sets $V_{\underbrace{2\cdots2}_{k \text{ times}}0}$, $1 \le k \le m_i - 1$, (respectively $V_{\underbrace{2\cdots2}_{k \text{ times}}1}$) are disjoint and so

$$\mu\left(\bigcup_{k=0}^{m_i-1} T^{h_n} V_{\underbrace{2\cdots2}_{k \text{ times}}0}\right) = \mu\left(\bigcup_{k=0}^{m_i-1} V_{\underbrace{2\cdots2}_{k \text{ times}}0}\right) < \mu(V) < \delta,$$

and $V_{\underbrace{2\cdots2}_{m_i \text{ times}}} \subset Z_{m_i-1}$. Finally we evaluate $\mu(T^{h_n+1}\bigcup_{j\ge n+1} S_j)$. Note that

$$T^{h_n+1}\left(\bigcup_{j\ge n+1} S_j\right) \subset I(1,n) \cup \left(\bigcup_{j\ge n+2} S_j\right)$$

and hence

$$\mu\left(T^{h_n+1}\bigcup_{j\ge n+1} S_j\right) < \left(\frac{1}{3}\right)^{m_{i-1}} + \left(\frac{1}{3}\right)^{m_0+m_1+\cdots+m_{i-1}}$$
$$< \delta < \varepsilon. \qquad \square$$

COROLLARY 4.5. *Let E be as in Lemma 4.3 and let $x \in E$. Then $d_A(x, Tx) < 7\varepsilon$.*

Proof. Let E' and i be as in Lemma 4.3 and set $n = M_i$. Let $x \in E$ and $x' \in E'$. Then

$$\begin{aligned} d_A(x, Tx) &\le d_A(x, T^{h_n}x) + d_A(T^{h_n}x, T^{h_n+1}x) + \mu(T(A_{T^{h_n}x} \triangle A_x))) \\ &< \delta + d_A(T^{h_n}x, T^{h_n+1}x) + \varepsilon. \end{aligned}$$

Moreover,

$$\begin{aligned} d_A(T^{h_n}x, T^{h_n+1}x) &\le d(T^{h_n}x, T^{h_n+1}x') + d(T^{h_n+1}x', T^{h_n+1}x) \\ &< \delta + \mu(T^{h_n+1}(A_{x'} \triangle A_x)) \\ &= \delta + 4\varepsilon < 5\varepsilon, \text{ using Proposition 4.4.} \end{aligned}$$

Therefore,

$$d_A(x, Tx) < \delta + 5\varepsilon + \varepsilon < 7\varepsilon. \qquad \square$$

Proof of Theorem 4.2. We have showed that for any $\varepsilon > 0$ and any measurable subset D of X of positive measure such that

$$d_A(x, y) < \varepsilon, \quad \text{for all } x \in D,$$

there exists a measurable set $E \subset D$ of positive measure such that

$$d_A(x, Tx) < 7\varepsilon, \quad \text{for all } x \in E.$$

In fact we show

$$d_A(x, Tx) < 7\varepsilon \quad \text{for a.e. } x \in X.$$

Otherwise assume that the set

$$F = \{x \in X \colon d_A(x, Tx) \geq 7\varepsilon\}$$

satisfies $\mu(F) > 0$. Using Proposition 2.1, there is a measurable subset D of F of positive measure such that

$$d_A(x, y) < \varepsilon \quad \text{for all } x, y \in D.$$

On the other hand by Corollary 4.5, we have a measurable subset E of D of positive measure such that

$$d_A(x, Tx) < 7\varepsilon \quad \text{for all } x \in E,$$

which contradicts $E \subset F$.

Now since $\varepsilon > 0$ is arbitrary,

$$d_A(x, Tx) = 0 \quad \text{for a.e. } x \in X,$$

and this means

$$d_A(x, T^n x) = 0 \quad \text{for all } n \text{ and a.e.} x \in X.$$

Finally, for any $\varepsilon > 0$ take a measurable set $D \subset X$ of positive measure such that

$$d_A(x, y) < \varepsilon, \quad \text{for all } x, y \in D.$$

Then for a.e. $x \in X$ and a.e. $y \in X$, it follows from ergodicity of T that there exist integers n and m such that

$$T^n x \in D \quad \text{and} \quad T^m y \in D.$$

Then

$$d_A(x, y) \leq d_A(x, T^n x) + d_A(T^n x, T^m y) + d_A(T^m y, y) < \varepsilon.$$

Since ε is arbitrary, we see that $d_A(x, y) = 0$ for a.e. $x \in X$ and $y \in X$. This means that there is a measurable set $F \subset X$ of positive measure such that

$$\mu(A_x \Delta F) = \mu(A_{Tx} \Delta F) = 0 \quad \text{a.e. } x \in X.$$

This means,

$$F = A_{Tx} = T(A_x) = T(F) (\operatorname{mod} \mu).$$

Then it follows from the ergodicity of T that $\mu(F) = 1$. □

Finally we observe that the proof of Lemma 4.3 also obtains the following proposition for a more general class of measures.

PROPOSITION 4.6. *Let T be a nonsingular Chacon transformation such that at stage i of the construction column C_i is cut in the ratio $1 : \lambda_i : \gamma_i$. If the series*

$$\sum_{i \geq 0} \frac{\min\{1, \lambda_i, \gamma_i\}}{1 + \lambda_i + \gamma_i}$$

diverges then T has no L^∞ eigenvalue other than 1.

Proof. Let f be an L^∞ function such that $f(Tx) = \beta f(x)$ a.e. x. We may assume $|f| = 1$ and $|\beta| = 1$. Let $\varepsilon > 0$ and choose a complex number, c, and a measurable subset of positive measure, $D \subset X$, such that

$$|f(x) - c| < \varepsilon \quad \text{for all } x \in D.$$

If we use the same techniques as in the proof of Lemma 4.3, we see that the following holds.

There exists $n \geq 1$ and E_n, E_n' in D such that

$$T^{h_n} E_n \cup T^{h_n+1} E_n' \subset D.$$

Let $x \in E_n$ and $x' \in E_n'$. Then,

$$\begin{aligned} |\beta^{h_n} - 1| &= |f(T^{h_n} x) - f(x)| < 2\varepsilon, \\ |\beta^{h_n+1} - 1| &= |f(T^{h_n+1} x') - f(x')| < 2\varepsilon. \end{aligned}$$

Thus

$$|\beta^{h_n} - \beta^{h_n+1}| < 4\varepsilon, \text{ and } |\beta - 1| = |\beta^{h_n+1} - \beta^{h_n}| < 4\varepsilon.$$

Since ε is arbitrary, $\beta = 1$. □

5. Type II_∞ transformations

In the previous sections we saw that the time intervals $(M_k, N_k - 1]$, $k \geq 1$, where all the ratios λ_j are 1, play the important role in the ergodicity of $T \times T$, under the condition that the length of the intervals $m_k = N_k - M_k$, tend to infinity as $k \to \infty$. In this section we control the lengths n_k of the other time intervals $(N_k, M_{k+1}]$, $k \geq 1$, together with the ratios $\lambda_j = \theta_k$, $j \in (N_k, M_{k+1}]$, so that T is type II_∞. We thus obtain a new class of II_∞ transformations with ergodic Cartesian products.

We first note that the orbit equivalence class of T is the same as that of the induced transformation of T on the subset $[0, \alpha] \subset X$, which is a product type odometer.

Now, for a sequence of positive numbers θ_k, $k \geq 1$, with $\theta_k \leq 1$, and a sequence of positive integers n_k, $k \geq 1$, consider the infinite product probability measure space

$$(Y, m) = \left(\Pi_{k\geq 1}\Pi_{N_k<i\leq M_{k+1}}\{0, 1\}, \Pi_{k\geq 1}\Pi_{N_k<i\leq M_{k+1}}\left\{\frac{2}{2+\theta_k}, \frac{\theta_k}{2+\theta_k}\right\}\right).$$

Suppose m is an atomic measure, that is,

$$\sum_{k\geq 1} n_k\theta_k < \infty.$$

Then Y is an infinite countable set up to a null set. In other words, the odometer acting on (Y, m) is of type I_∞. Hence, the odometer defined on

$$(X_\infty, \mu_\infty) = \left(\prod_{k\geq 1}\prod_{N_k<i\leq M_{k+1}}\{0, 1, 2\}, \prod_{k\geq 1}\prod_{N_k<i\leq M_{k+1}}\left\{\frac{1}{2+\theta_k}, \frac{\theta_k}{2+\theta_k}, \frac{1}{2+\theta_k}\right\}\right)$$

is of type II_∞.

Meawhile, the odometer defined on

$$(X_1, \mu_1) = \left(\prod_{k\geq 1}\prod_{M_k<i\leq N_k}\{0, 1, 2\}, \prod_{k\geq 1}\prod_{M_k<i\leq N_k}\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}\right)$$

is of type II_1, and the induced transformation of T on the subset $[0, \alpha)$ is orbit equivalent with the product of each group generated by odometers, which is of type II_∞, Thus we have:

PROPOSITION 5.1. *If*

$$\sum_{k\geq 1} n_k\theta_k < \infty$$

then the Chacon transformation T is of type II_∞.

6. Type III_λ, $0 \le \lambda \le 1$, Chacon transformations

We will see that if the parameters θ_k and n_k of the transformation T are suitably chosen type III_λ, $0 \le \lambda \le 1$, orbit equivalence classes of T are available. As mentioned earlier, type III_λ for $\lambda \ne 0$ were already obtained in [JS].

For $0 < \lambda < 1$, set

$$n_k = 1 \quad \text{and} \quad \theta_k = \frac{\lambda}{2+\lambda}, k \ge 1.$$

For the type III_1 example, let λ_1 and λ_2 in (0, 1) be such that $\log(\lambda_1)/\log(\lambda_2)$ is irrational. For $k \ge 1$, set

$$\begin{aligned} n_k &= 1, \\ \theta_k &= \frac{\lambda_1}{2+\lambda_1} \text{ if } k \text{ is odd}, \\ \theta_k &= \frac{\lambda_2}{2+\lambda_2} \text{ if } k \text{ is even}. \end{aligned}$$

Then the transformation is type III_1.

For type III_0, fix $0 < \lambda < 1$ and for $k \ge 1$, set

$$\theta_k = \frac{\lambda^{2^k}}{2+\lambda^{2^k}}$$

and let $\{n_k\}_{k\ge1}$ be a sequence of positive integers satisfying

$$\sum_{k=1}^{\infty} n_k \lambda^{2^k} = \infty.$$

Then by [HOO], p. 126, the transformation is of type III_0.

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