ILLINOIS JOURNAL OF MATHEMATICS
Volume 44, Number 4, Winter 2000

IDEALS ATTAINING A GIVEN HILBERT FUNCTION

MATTHEW J. RODRIGUEZ

ABSTRACT. We improve the result of Charalambous and Evans [C-E] to show that the Betti number sequence in their example of incomparable minimals among the resolutions for a fixed Hilbert function is indeed minimal. Their example was dependent upon the graded betti numbers. We give an example of a finite length Hilbert function and two cyclic finite length modules attaining the Hilbert function for which the betti number sequences are incomparable, i.e., independent of the grading.

Given a graded module M over a polynomial ring $R = k[x_0, \ldots, x_n]$, Hilbert [Hi] showed how to obtain the Hilbert function from the (projective) resolution. Progress in the opposite direction, i.e., in obtaining the resolutions of modules attaining a fixed Hilbert function, has been scarce. We present some further results. Fixing a Hilbert function, many different projective resolutions may occur for the graded modules. We denote a typical resolution,

$$0 \to \oplus_d F^{n_d}[-d] \to \cdots \to \oplus_d F^{0_d}[-d] \to M \to 0,$$

as $\beta = (\beta_{m_d})$ where the β_{m_d} = rank F^{m_d} are the graded Betti numbers of M. We may partially order two resolutions $\alpha \leq \beta$ if $\alpha_m \leq \beta_m$ for all m (coarse ordering on Betti numbers) or if $\alpha_{m_d} \leq \beta_{m_d}$ for all m and d (fine ordering on graded Betti numbers). For now, we will use the latter partial ordering. In 1993, Hulett [Hu] and Bigatti [B] independently showed that a unique largest element in this partial ordering exists in characteristic zero, and in 1996 Pardue [P] improved the result to all characteristics. However, there was no reason to believe that a unique smallest element should always exist. Charalambous and Evans [C-E] proved through counterexample that a unique minimal resolution need not exist in the fine partial ordering, weaker than a result independent of the grading. In this paper, we first restate and then strengthen their theorem by proving that the Betti number sequence is in fact smallest. Then, we also prove the desired result of the existence of incomparable minimals among the resolutions of graded modules with a fixed Hilbert function, i.e., the existence of those which are incomparable in the coarse partial ordering of Betti numbers. Once again, we concentrate on the finite cyclic case.

THEOREM 0. [C-E]. *Let $R = k[x_0, \ldots, x_n], n \geq 2$. Then there exists a Hilbert series for a cyclic finite length R-module and two incomparable smallest sets of graded Betti numbers for that Hilbert series.*

Received February 15, 1999; received in final form May 11, 2000.
1991 Mathematics Subject Classification. Primary 18G10 ; Secondary 13D10, 13D40, 14C05.
The author is indebted to Professor E. Graham Evans Jr. for his instruction and advice during and after this project, and thanks to the Summer Research Oppoirtunities Program for their support.

The counterexample provided for the Hilbert function $H_M(s) = \{1, 3, 4, 2, 1\}$ (where we write a finite Hilbert function as a sequence of its nonzero terms, usually beginning in degree zero) was the two modules R/I_1 and R/I_2 where $I_1 = (x_0x_2, x_1x_2, x_0^3, x_1^3, x_2^3)$ and $I_2 = (x_1^2, x_2^2, x_0^2x_1, x_0^2x_2, x_0^5)$. The betti diagrams for these modules are given below

R/I_1:	total:	1	5	6	2
	0:	1	.	.	.
	1:	.	2	1	.
	2:	.	3	4	1
	3:	.	.	.	.
	4:	.	.	1	1
R/I_2:	total:	1	5	6	2
	0:	1	.	.	.
	1:	.	2	.	.
	2:	.	2	4	.
	3:	.	.	.	1
	4:	.	1	2	1.

Note that the degrees of the elements in the socle as well as those of the second syzygies differ in the two modules and a device of Stanley [S] shows that no attainable smaller resolutions lies beneath both of them. However, since the Betti number sequence in the two cases are identical we consider these two modules to be weakly (finely) incomparable.

THEOREM 1. *Let* $R = k[x_0, \dots, x_n]$ *and* $H_M(s) = \{1, 3, 4, 2, 1\}$. *Then the unique smallest Betti number sequence is* 1, 5, 6, 2.

Proof. Of course we use Theorem 0 to establish existence and need only demonstrate the sequence's minimality. This is a brief calculation using facts about almost complete intersections and Gorenstein ideals. First, the Hilbert function of a Gorenstein ideal is symmetric and we use the contrapositive. Second, an almost complete intersection would contain an R sequence of generators f_1, f_2, f_3 and a fourth generator f_4 where the ideal (f_1, f_2, f_3): (f_4) is Gorenstein. To obtain the Hilbert function of the module we subtract the Hilbert function of $R/(f_1, f_2, f_3)$ from the given one for $R/(f_1, f_2, f_3, f_4)$ and observe that this difference is necessarily symmetric. Surveying all possible combinations of the degrees of the generators for an R sequence (with the obvious upper bound of five) we see that no such difference occurs and hence no almost complete intersections exist. Thus, a minimum of five generators is required. We remark that since the Hilbert function is asymmetric the number of socle elements for a module attaining it is greater than one. Therefore, since the alternating sum of the betti number sequence is zero, we have shown that 1, 5, 6, 2 is the smallest Betti number sequence. □

Remark. A sample of the above calculations should clarify the argument. Consider an R sequence of generators in degrees 2, 3, and 5 with Hilbert function $J_M(s) = \{1, 3, 5, 6, 6, 5, 3, 1\}$. The difference $J_M(s) - H_M(s)$ is {1, 4, 5, 5, 3, 1} (omitting the leading zeroes) representing a Hilbert function beginning in degree two that is clearly asymmetric.

Similar examples abound in $k[x_0, x_1, x_2]$ and we conjecture that no strongly (coarsely) incomparable resolutions for a given finite length Hilbert function exist in this ring. Thus, we extend our search to $R = k[x_0, x_1, x_2, x_3]$ and prove the desired result.

LEMMA 2. *Let* $R = k[x_0, \ldots, x_n]$ *and* $H_M(s) = \{1, 4, 8, 10, 8, 3, 1\}$. *Then the only possible R sequences leading to an almost complete intersection yielding* $H_M(s)$ *are* 2, 2, 3, 4 *and* 2, 3, 3, 4 *in which case the fifth generators are of degree* 3 *and* 2 *respectively.*

Proof. Some observations about $H_M(s)$ are helpful. The Hilbert function of R is $\{1, 4, 10, 20, 35, \ldots\}$; hence this first disagrees with $H_M(s)$ in degree two and this difference requires two quadratic generators for the ideal. These two generators cut at most $4 \times 2 = 8$ from the degree three term of the Hilbert function leaving $20 - 8 = 12$. However, in order to match the degree three term of $H_M(s)$, two more cubic generators are needed. Of course we cannot use all four of them in the R sequence, for then their Hilbert function is $\{1, 4, 8, 10, 8, 4, 1\}$ indicating that we must supply a fifth generator in degree five in which case the generator would multiply nontrivially in degree six and the Hilbert function would vanish there. Thus, we may take only three of the four quadratic and cubic generators and are left to examine the cases 2, 2, 3, d and 2, 3, 3, d where d $\leq$ 7. Now, we apply the argument from Theorem 1 and notice when the difference of $H_M(s)$ and the Hilbert function for an R sequence yields a symmetric sequence denoting the Hilbert function of the quotient ideal. Calculation shows this occurs only for the R sequences 2, 2, 3, 4 and 2, 3, 3, 4 with Hilbert functions {1, 3, 5, 3, 1} and $\{1, 4, 8, 11, 8, 4, 1\}$ (note that these Hilbert functions begin in degree 3 and 2) respectively. □

THEOREM 3. *Let* $\operatorname{char} k = 5$ *and* $R = k[x_0, x_1, x_2, x_3]$. *Then there exists a (finite length) Hilbert series,* $\{1, 4, 8, 10, 8, 3, 1\}$, *for a (cyclic finite length) R-module and two incomparable sequences of Betti numbers corresponding to modules attaining it.*

Remark. We can easily generalize the theorem to $R = k[x_0, \ldots, x_n]$ as in the proof of Theorem 0 by introducing a linear term into the ideals for each additional variable. Also, the characteristic condition is needed to avoid trouble in cases such as characteristic 2. The theorem holds in characteristic 0.

Remark. Here are a few comments about the method of this research. The discovery of I_1 and I_2 in the proof below was a delicate and tedious process. First, code was written in Macaulay 2 to search for incomparable minimals among the resolutions of monomial ideals and the results were the 7-generated I_1 seen below and a 6-generated one. This provided bounds for our search, though we could not rule out the existence of a 5-generated (nonmonomial) ideal with a smaller resolution. Thus, we performed a second search for 5-generated ideals with the given Hilbert function, taking three pure powers of the variables in the degrees of the R sequence (either 2, 2, 3 or 2, 3, 3), a random quadratic or cubic binomial, and a homogeneous fourth degree polynomial. We obtained thousands of ideals attaining the Hilbert function, all of which had the same Betti diagram leading to the result.

Proof. We first provide two candidate modules for incomparability and remark that the ideal with more generators has fewer socle elements, hence exhibiting the requisite strong incomparability. Let $M_1 = R/I_1$ and $M_2 = R/I_2$ where

$$I_1 = (x_0^2, x_1^2, x_0x_2^2, x_3^3, x_1x_2^3, x_1x_2^2x_3^2, x_3^5)$$

and

$$I_2 = (x_0^2, x_1^2, x_2^3, x_1x_2^2 - x_1x_3^2, x_0x_1x_2x_3 + x_0x_2^2x_3 + x_0x_1x_3^2 - x_0x_2x_3^2 - x_2^2x_3^2 + x_0x_3^3 - x_1x_3^3 + x_3^4).$$

(There is no special significance in the choice of I_2, thousands of similar ideals were found with the same Betti diagram.) Their Betti diagrams are given below.

R/I_1:	total:	1	7	15	12	3
	0:	1	.	.	.	.
	1:	.	2	.	.	.
	2:	.	2	2	.	.
	3:	.	1	5	2	.
	4:	.	2	7	8	1
	5:	.	.	.	.	1
	6:	.	.	1	2	1

R/I_2:	total:	1	5	12	12	4
	0:	1	.	.	.	.
	1:	.	2	.	.	.
	2:	.	2	2	.	.
	3:	.	1	3	1	.
	4:	.	.	6	9	3
	5:	.	.	1	1	.
	6:	.	.	.	1	1

To complete the proof, we show that the socle of any almost complete intersection with the Hilbert function must have at least four elements. Applying Lemma 2, we are reduced to considering those two cases of R sequences. We note that in either case the Betti number sequence is 1, 5, 12, 12, 4. For such an ideal $(f_1, f_2, f_3, f_4, f_5)$ to be an almost complete intersection, the ideal (f_1, f_2, f_3, f_4): (f_5) must be Gorenstein. Thus we first examine such ideals for the case of an R sequence with generators in degrees 2, 2, 3, and 4. For this purpose, consider the generating function

$$F(t) = \frac{\sum_{\beta}(-1)^j \beta_{ij} t^j}{(1-t)^n}$$

which is a polynomial in our finite case. Multiplying through by $(1-t)^4$ gives the polynomial in the numerator, e.g., $(1+3t+5t^2+3t^3+t^4)(1-t)^4 = 1-t-t^2-3t^3+8t^4-3t^5-t^6-t^7+t^8$. This polynomial is translated into a Betti diagram containing the coefficients where the sign changes signal a shift of syzygies (from j to $j \pm 1$) and the exponent denotes the degree of the syzygies, hence the following diagram:

total:	1	5	8	5	1
0:	1	1	.	.	.
1:	.	1	.	.	.
2:	.	3	8	3	.
3:	.	.	.	1	.
4:	.	.	.	1	1

Note that the sum along each upward sloping diagonal is the corresponding coefficient in the polynomial. We call the sum of the coefficients the *Stanley bound* in that any module attaining the Hilbert function must have at least this number as the sum of its (graded) Betti numbers [S2, p. 62]. We may calculate the same sequence of sums from the Betti diagrams of the two modules (we must sum the diagonals of the diagrams since we do not have the polynomial at hand) and taking the sum over min {sum along the corresponding diagonal}. Ideally, the sum would be below the Stanley bound and without further argument we could conclude that there exists no module with a resolution beneath them both. Observe in this case that the Stanley bound is greatly exceeded indicating the amount of delicacy needed in the proof.

In the above diagram, the presence of a linear and quadratic generator requires a relation in degree three and then a fourth generator in degree three to compensate and yield the correct Hilbert function, giving a revised smallest diagram:

total:	1	6	10	6	1
0:	1	1	.	.	.
1:	.	1	1	.	.
2:	.	4	8	4	.
3:	.	.	1	1	.
4:	.	.	.	1	1

We now appeal to a mapping cone argument. Consider the mapping cone in this case (where we have simplified notation in listing only the degrees of the generators and exponents to indicate repeated degrees):

$$\begin{array}{ccccccccc}
\{11\} & \longrightarrow & \{7,8,9^2\} & \longrightarrow & \{4,5^2,6^2,7\} & \longrightarrow & \{2^2,3,4\} & \longrightarrow & \{0\} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\{11\} & \longrightarrow & \{8^4,9,10\} & \longrightarrow & \{6,7^8,9\} & \longrightarrow & \{4,5,6^4\} & \longrightarrow & \{3\}.
\end{array}$$

This, of course, need not be minimal, for instance the element in degree eleven is certainly superfluous, and one of degree eight and another of degree nine may be unnecessary. Nonetheless, in the worst case in which all possible cancellations do occur the fourth syzygy will still have a rank of four, hence the fourth Betti number in the example is indeed minimal. Consider the other case of an R sequence in degrees 2, 3, 3, and 4. If an R sequence in degrees 2, 2, 3, and 4 also exists then by the previous case we are done. Suppose not. Then there are two quadratic generators, only one of which is in the R sequence. Since these two cannot form an R sequence they must have a common linear divisor and the ideal has a linear generator. On the other hand, we found that the Hilbert function of the quotient ideal was $\{1, 4, 8, 11, 8, , 4, 1\}$. Observe that the degree one term is four, indicating that there are no linear generators. Therefore, every almost complete intersection has an R sequence in degrees 2, 2, 3, and 4. □

Note that while we have discovered an example of incomparable minimals among the resolutions for a given Hilbert function we have not (in the above theorem) found the minimal Betti number sequences which may lie under either of them; i.e., we have not eliminated the possibility of a module with betti number sequence 1, 5, 10, 10, 4 for instance. Much further study is required before any general results may be presented, for to this point progress has been only in finding examples to match our hypotheses. Of particular interest is gaining knowledge of the frequency with which strong incomparability occurs for finite Hilbert functions. A slightly simpler matter would be to find multiple (more than two) incomparables for the same Hilbert function.

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Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720
mjrodrig@math.berkeley.edu