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SEMIGLOBAL RESULTS FOR $\overline{\partial}_b$ ON WEAKLY CONVEX HYPERSURFACES IN $\mathbb{C}^n$

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ABSTRACT. Let M be a smoothly bounded weakly convex hypersurface in $\mathbb{C}^n$ ($n \geq 4$), $0 \in M$. Let $q \in \mathbb{N}$, $1 \leq q < n-2$. We derive a homotopy formula for $\overline{\partial}_b$ without shrinking on certain submanifolds of M (no finite type condition is assumed for M).

Introduction

Let Ω be a bounded weakly convex domain in $\mathbb{C}^n$ ($n \geq 4$) with smooth boundary $M =: \partial\Omega$. Let $q \in \mathbb{N}$, $1 \leq q < n-2$. For each q, we wish to describe the open subsets ω_q of M (with smooth boundary) for which homotopy formulas hold for $\overline{\partial}_b$ without shrinking. By this we mean there exist continuous, integral operators $\mathcal{E}_q$, $\mathcal{F}_{q+1}$, defined on $\overline{\omega_q}$, such that for all $\lambda \in \mathbb{N}$, $f \in C^\lambda_{(0,q)}(\overline{\omega_q})$, $\overline{\partial}_b f \in C^\lambda_{(0,q+1)}(\overline{\omega_q})$, the following equation is satisfied in ω_q:

$$f = \overline{\partial}_b \mathcal{E}_q f + \mathcal{F}_{q+1} \overline{\partial}_b f.$$

Existence of homotopy formulas of the above type yields solvability results for the tangential C-R operator without shrinking. Such results are of special interest on their own, since by the work of Kuranishi [4] and Webster [12] they are linked to the local embeddability question of abstract CR structures.

The first local homotopy formula for $\overline{\partial}_b$ without shrinking was obtained by Henkin [2], for smooth strongly pseudoconvex hypersurfaces in $\mathbb{C}^n$ ($n \geq 3$). Homotopy formulas for $\overline{\partial}_b$ that satisfy C^λ-estimates were obtained by Schaal [9] for certain submanifolds of complex ellipsoids. Local solvability for $\overline{\partial}_b$ on weakly pseudoconvex boundaries of finite type has been studied by Shaw [10], [11]. Recently, Michel-Shaw [7] obtained homotopy formulas for the tangential C-R operator on certain submanifolds of smooth weakly pseudoconvex hypersurfaces in $\mathbb{C}^n$. As an application, they obtained C^∞-regularity up to the boundary for local solutions to smooth $\overline{\partial}_b$-closed forms.

In this paper, we derive a homotopy formula for $\overline{\partial}_b$ without shrinking on certain smoothly bounded subsets of M (no finite type condition is assumed for M). More precisely we show the following:

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THEOREM. *Let M be the boundary of a smoothly bounded weakly convex domain Ω in $\mathbb{C}^n$ ($n \geq 4$). Let $q \in \mathbb{N}$, $1 \leq q < n-2$, $k \in \mathbb{N}$, $1 \leq k \leq n-2-q$. Assume that there exists a smooth domain $\mathcal{D}_0$ in $\mathbb{C}^n$ described by*

$$\mathcal{D}_0 = \{z \in \mathbb{C}^n \mid r(z_{n-k+1}, \ldots, z_n) < 0\}$$

where r is smooth, real-valued, convex function of $z_{n-k+1}, \ldots, z_n, \bar{z}_{n-k+1}, \ldots, \bar{z}_n$ such that $\partial\mathcal{D}_0$, M intersect real transversally. Let $\omega = M \cap \mathcal{D}_0$. Then, for each $\ell \in \mathbb{N}$, there exist continuous integral operators K_q^ℓ, K_{q+1}^ℓ, defined on $\overline{\omega}$, such that for all $f \in C^{3(\ell+2n-1)}_{(0,q)}(\overline{\omega})$, $\bar{\partial}_b f \in C^{3(\ell+2n-1)}_{(0,q+1)}(\overline{\omega})$ we have

$$f = \bar{\partial}_b K_q^\ell f + K_{q+1}^\ell \bar{\partial}_b f \quad \text{in } \omega.$$

More precisely we have

$$\|K_q^\ell f\|_{C^\ell_{(0,q-1)}(\overline{\omega})} \leq c_{\ell,n}(\|f\|_{C^{3(\ell+2n-1)}_{(0,q)}(\overline{\omega})} + \|\bar{\partial}_b f\|_{C^{3(\ell+2n-1)}_{(0,q+1)}(\overline{\omega})})$$

where $c_{\ell,n}$ is a positive constant.

Arguing in a similar manner as in Michel-Shaw [7] we obtain the following:

COROLLARY. *Let $M, \mathcal{D}_0, q, k$ be as in the theorem. Given $f \in C^\infty_{(0,q)}(\overline{\omega})$, $\bar{\partial}_b f = 0$ in ω, there exists $u \in C^\infty_{(0,q-1)}(\overline{\omega})$ such that $\bar{\partial}_b u = f$ in ω.*

The paper is organized as follows. In Section 1, we define the kernels and the operators needed for the homotopy formula. Section 2 is devoted mainly to the derivation of the homotopy formula. The starting point will be a jump formula for f. The choice of $M, \mathcal{D}_0$ will allow us to construct Leray maps and hence obtain Leray-Koppelman formulas for the Bochner-Martinelli integrals of f, f^+, f^-. Unfortunately the integral operators thus created become quite singular as $z \to \partial\omega$. To overcome the difficulty we shall exploit an idea of Lieb-Range and replace the boundary integrals by nonboundary ones such that the integrand forms vanish to high enough order on $\overline{\omega}$. This can be achieved by an application of Stokes' theorem for piecewise smooth domains. The growth of the integrand forms will cancel the singularities of the kernels and it will allow us to obtain the desired homotopy formula.

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1. Preliminaries

Let Ω be a smooth, bounded weakly convex domain in $\mathbb{C}^n$ ($n \geq 4$), described by a defining function ρ: $W \to \mathbb{R}$, where W is an open neighborhood of $\partial\Omega$, $d\rho|_{\partial\Omega} \neq 0$

and

$$\Omega = \{z \in W \mid \rho(z) < 0\}.$$

Without loss of generality we shall use as ρ the distance function

$$\rho(z) = \begin{cases} d(z, \partial\Omega) & \text{for } z \in \mathbb{C}^n \setminus \overline{\Omega} \\ -d(z, \partial\Omega) & \text{for } z \in \overline{\Omega}. \end{cases}$$

We define

$$\mathcal{D}_0 = \{z \in \mathbb{C}^n \mid r(z_{n-k+1}, \dots, z_n) < 0\}$$

where r is smooth, real-valued, convex function of $z_{n-k+1}, \dots, z_n, \overline{z}_{n-k+1}, \dots, \overline{z}_n$ such that $dr|_{\partial\mathcal{D}_0} \neq 0$, $dr \wedge d\rho \neq 0$ on $\partial\mathcal{D}_0 \cap \partial\Omega$.

Let $\omega =: \partial\Omega \cap \mathcal{D}_0$. We define

$$\Omega^- = \{z \in W \mid \rho < 0, r < 0\}, \quad \Omega^+ = \{z \in W \mid \rho > 0, r < 0\}$$
$$S_1 = \{z \in W \mid \rho = 0, r \leq 0\} =: \omega, \quad S_{12} = \{z \in W \mid \rho = 0 = r\}$$
$$\widetilde{S_1} = \{z \in W \mid -\rho = 0, r \leq 0\}, \quad \widetilde{S_{12}} = \{z \in W \mid -\rho = 0 = r\}.$$

Construction of the Leray maps. We define smooth $\mathbb{C}^n$-valued maps

$$P(\zeta, z) =: (p_1(\zeta, z), \dots, p_n(\zeta, z)) : W \times \mathbb{C}^n \to \mathbb{C}^n,$$

$$p_j(\zeta, z) =: \frac{\partial\rho}{\partial\zeta_j}(\zeta),$$

$$Q(\zeta, z) =: (q_1(\zeta, z), \dots, q_n(\zeta, z)) \colon \mathbb{C}^n \times W \to \mathbb{C}^n,$$

$$q_j(\zeta, z) =: -\frac{\partial\rho}{\partial z_j}(z),$$

$$R(\zeta, z) =: (r_1(\zeta, z), \dots, r_n(\zeta, z)) : \mathbb{C}^n \times \mathbb{C}^n,$$

$$r_j(\zeta, z) = \begin{cases} 0 & \text{when } 1 \leq j \leq n-k, \\ \dfrac{\partial r}{\partial\zeta_j}(\zeta) & \text{when } n-k+1 \leq j \leq n. \end{cases}$$

We set

$$\Phi(\zeta, z) = \langle P(\zeta, z), \zeta - z\rangle, \quad \Psi(\zeta, z) = \langle Q(\zeta, z), \zeta - z\rangle,$$
$$X(\zeta, z) =: \langle R(\zeta, z), \zeta - z\rangle, \quad \Phi_0(\zeta, z) = |\zeta - z|^2.$$

LEMMA 1.1. *There exists $W' \subset W$ open neighborhood of $\partial\Omega$ such that:*

(i) *For $z \in W' \cap \overline{\Omega}$, $\zeta \in W' \cap \mathbb{C}^n \setminus \overline{\Omega}$ we have*

$$2\mathrm{Re}\,\Phi(\zeta, z) \geq \rho(\zeta) - \rho(z).$$

(ii) *For $z \in W' \cap \mathbb{C}^n \setminus \overline{\Omega}$, $\zeta \in W' \cap \overline{\Omega}$ we have*

$$2\mathrm{Re}\,\Psi(\zeta, z) \geq \rho(z) - \rho(\zeta).$$

Proof. The reader may look at Chaumat-Chollet (Proposition 4) [1]. It is a well-known fact of the convexity of $\overline{\Omega}$.

LEMMA 1.2. *For $\zeta, z \in \mathbb{C}^n$ we have*

$$2\mathrm{Re}\,X(\zeta, z) \geq r(\zeta) - r(z).$$

Proof. It is based on the third inequality of (2.1.12) in Theorem 2.1.22 in Hörmander [3]. We define

$$\eta_1(\zeta, z) =: \frac{\langle P(\zeta, z), d\zeta\rangle}{\Phi(\zeta, z)}, \quad \eta_2(\zeta, z) =: \frac{\langle Q(\zeta, z), d\zeta\rangle}{\Psi(\zeta, z)},$$
$$\eta_3(\zeta, z) =: \frac{\langle R(\zeta, z), d\zeta\rangle}{X(\zeta, z)}, \quad \eta_0(\zeta, z) =: \frac{\langle \overline{\zeta} - \overline{z}, d\zeta\rangle}{|\zeta - z|^2}.$$

Remarks. 1. η_1 is holomorphic in z and well defined for $z \in W' \cap \overline{\Omega^-}$, $\zeta \in W' \cap \Omega^+$.

2. η_2 is holomorphic in ζ and well defined for $z \in W' \cap \Omega^+$, $\zeta \in W' \cap \overline{\Omega^-}$.

3. η_3 is holomorphic in $z, \zeta_1, \ldots, \zeta_{n-k}$.

The singularities of Φ, Ψ appear when $(\zeta, z) \in \partial\Omega \times \partial\Omega$. Those of X appear when $(\zeta, z) \in \partial\mathcal{D}_0 \times \partial\mathcal{D}_0$.

Construction of the kernels. Let

$$\Delta =: \left\{\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4 \mid \lambda_i \geq 0, \quad \sum_{i=0}^{i=3} \lambda_i = 1\right\}.$$

For any $\emptyset \neq K$ ordered subset of $\{0, 1, 2, 3\}$ we set

$$\Delta_K =: \{\lambda \in \Delta \mid \lambda_j = 0 \quad j \notin K\}.$$

Whenever Δ_K contains only one point we regard it as positively oriented. If $K = (k_1, .., k_m)$, $m \geq 2$, we orient Δ_K inductively such that

$$\partial\Delta_K = \sum_{\nu=1}^{m} (-1)^{\nu+1} \Delta_{k_1..\widehat{k_\nu}..k_m}$$

where the symbol $\widehat{k_\nu}$ means that k_ν is omitted. For $\lambda \in \Delta_K$ we set

$$\eta_K = \sum_{j \in K} \lambda_j \eta_j$$

and for $0 \leq q \leq n-1$ we define

$$D_{n,q}(\eta_K)(\zeta, z, \lambda) = c_{n,q}\, \eta_K \wedge ((\overline{\partial}_\zeta + d_\lambda)\eta_K)^{n-1-q} \wedge (\overline{\partial}_z \eta_K)^q$$

where
$$c_{n,q} = (2\pi i)^{-n}(-1)^{\frac{q(q-1)}{2}} \binom{n-1}{q}$$

and $d\lambda$ denotes the total differential on Δ_K.

These double differential forms are the generalized Cauchy Fantappié forms. If we set $D_{n,n}(\eta_K) = 0 = D_{n,-1}(\eta_K)$, then the above forms satisfy

$$(\overline{\partial}_\zeta + d_\lambda) D_{n,q}(\eta_K) = (-1)^q \overline{\partial}_z D_{n,q-1}(\eta_K).$$

If $|K| = m$ then $\dim \Delta_K = m-1$. By integrating

$$\int_{\Delta_K} D_{n,q}(\eta_K)(\zeta, z, \lambda)$$

we obtain linear combinations of terms of the form

$$\eta_{k_1} \wedge .. \wedge \eta_{k_m} \wedge (\overline{\partial}_\zeta \eta_{k_1})^{i_1} \wedge \cdots \wedge (\overline{\partial}_\zeta \eta_{k_m})^{i_m} \wedge (\overline{\partial}_z \eta_{k_1})^{j_1} \wedge \cdots (\overline{\partial}_z \eta_{k_m})^{j_m}$$

where $i_1 + \cdots + i_m = n - q - m$ and $j_1 + \cdots + j_m = q$.

2. Derivation of the homotopy formula

Let $f \in C^0_{(0,q)}(\overline{\omega})$. We set

$$f^-(z) = \int_{(\zeta,\lambda) \in S_1 \times \Delta_0} f(\zeta) \wedge D_{(n,q)}(\eta_0)(\zeta, z, \lambda) \quad \text{for } z \in \Omega^-,$$

$$f^+(z) = \int_{(\zeta,\lambda) \in S_1 \times \Delta_0} f(\zeta) \wedge D_{(n,q)}(\eta_0)(\zeta, z, \lambda) \quad \text{for } z \in \Omega^+.$$

We have the following jump lemma.

LEMMA 2.1. *Let $f \in C^1_{(0,q)}(\overline{\omega})$. Then for all $\phi \in C^\infty_{(n,n-q-1)}(\mathbb{C}^n)$ we have*

$$\text{(2.1.1)} \quad \lim_{\epsilon \to 0} \int_{z \in \omega} [f^-(z - \epsilon\nu(z)) - f^+(z + \epsilon\nu(z))] \wedge \phi(z) = \int_{z \in \omega} f(z) \wedge \phi(z)$$

where $\nu(z)$ is the outward unit normal to $\partial\Omega$ at the point $z \in \omega$.

Proof. See Theorem 3.8 in Kytmanov [5].

LEMMA 2.2. (i) *For* $z \in \Omega^-$, *we have*

$$f^-(z) = \bar{\partial}_z \int_{(\zeta,\lambda)\in S_1\times\Delta_{01}} f(\zeta) \wedge D_{n,q-1}(\eta_{01})(\zeta, z, \lambda) + \int_{(\zeta,\lambda)\in S_1\times\Delta_{01}} \bar{\partial} f(\zeta) \wedge D_{n,q}(\eta_{01})(\zeta, z, \lambda) - \int_{(\zeta,\lambda)\in S_{12}\times\Delta_{01}} f(\zeta) \wedge D_{n,q}(\eta_{01})(\zeta, z, \lambda). \tag{2.2.1}$$

(ii) *For* $z \in \Omega^+$, *we have*

$$f^+(z) = \bar{\partial}_z \int_{(\zeta,\lambda)\in S_1\times\Delta_{02}} f(\zeta) \wedge D_{n,q-1}(\eta_{02})(\zeta, z, \lambda) + \int_{(\zeta,\lambda)\in S_1\times\Delta_{02}} \bar{\partial} f(\zeta) \wedge D_{n,q}(\eta_{02})(\zeta, z, \lambda) - \int_{(\zeta,\lambda)\in S_{12}\times\Delta_{02}} f(\zeta) \wedge D_{n,q}(\eta_{02})(\zeta, z, \lambda). \tag{2.2.2}$$

Proof. (i) For $z \in \Omega^-$ we apply Stokes' Theorem to $d_{\zeta,\lambda}(f(\zeta)\wedge D_{n,q}(\eta_{01})(\zeta, z, \lambda))$ on $S_1 \times \Delta_{01}$. We shall have

$$f^-(z) = \bar{\partial}_z \int_{(\zeta,\lambda)\in S_1\times\Delta_{01}} f(\zeta) \wedge D_{n,q-1}(\eta_{01})(\zeta, z, \lambda) + \int_{(\zeta,\lambda)\in S_1\times\Delta_{01}} \bar{\partial} f(\zeta) \wedge D_{n,q}(\eta_{01})(\zeta, z, \lambda) - \int_{(\zeta,\lambda)\in S_{12}\times\Delta_{01}} f(\zeta) \wedge D_{n,q}(\eta_{01})(\zeta, z, \lambda) + \int_{(\zeta,\lambda)\in S_1\times\Delta_1} f(\zeta) \wedge D_{n,q}(\eta_1)(\zeta, z, \lambda).$$

Taking into account that η_1 is holomorphic in z, $q \geq 1$ we can easily show that for $z \in \Omega^-$, $\zeta \in S_1$ we have

$$\int_{\Delta_1} D_{n,q}(\eta_1)(\zeta, z, \lambda) = 0.$$

Hence we can obtain (i). Similarly by applying Stokes' theorem for $z \in \Omega^+$ to $d_{\zeta,\lambda}(f(\zeta) \wedge D_{n,q}(\eta_{02}(\zeta, z, \lambda))$ on $S_1 \times \Delta_{02}$ and taking into account that η_2 is holomorphic in ζ and $n - q - 1 \geq 2 \geq 1$ we can obtain (ii).

We wish to further analyze

$$\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{01}} f(\zeta)\wedge D_{n,q}(\eta_{01})(\zeta,z,\lambda) \quad \text{for } z\in\Omega^-,$$

$$\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{02}} f(\zeta)\wedge D_{n,q}(\eta_{02})(\zeta,z,\lambda) \quad \text{for } z\in\Omega^+.$$

Following Henkin's idea we shall put into the picture the Cauchy-Fantappié form η_3.

LEMMA 2.3. (i) *For* $z\in\Omega^-$, *we have*

$$\begin{aligned}(2.3.1)\quad -\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{01}} f(\zeta)\wedge D_{n,q}(\eta_{01})(\zeta,z,\lambda) &= -\overline{\partial}_z\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{013}} f(\zeta)\wedge D_{n,q-1}(\eta_{013})(\zeta,z,\lambda)\\ &\quad -\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{013}} \overline{\partial}f(\zeta)\wedge D_{n,q}(\eta_{013})(\zeta,z,\lambda)\\ &\quad -\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{03}} f(\zeta)\wedge D_{n,q}(\eta_{03})(\zeta,z,\lambda)\end{aligned}$$

(ii) *For* $z\in\Omega^+$, *we have*

$$\begin{aligned}(2.3.2)\quad -\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{02}} f(\zeta)\wedge D_{n,q}(\eta_{02})(\zeta,z,\lambda) &= -\overline{\partial}_z\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{02}} f(\zeta)\wedge D_{n,q-1}(\eta_{023})(\zeta,z,\lambda)\\ &\quad -\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{023}} \overline{\partial}f(\zeta)\wedge D_{n,q}(\eta_{023})(\zeta,z,\lambda)\\ &\quad -\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{03}} f(\zeta)\wedge D_{n,q}(\eta_{03})(\zeta,z,\lambda)\end{aligned}$$

Proof. (i) For $z\in\Omega^-$, we apply Stokes' theorem to $d_{\zeta,\lambda}(f(\zeta)\wedge D_{n,q}(\eta_{013})(\zeta,z,\lambda))$ on $S_{12}\times\Delta_{013}$. We then have

$$\begin{aligned}-\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{01}} f(\zeta)\wedge D_{n,q}(\eta_{01})(\zeta,z,\lambda) &= -\overline{\partial}_z\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{013}} f(\zeta)\wedge D_{n,q-1}(\eta_{013})(\zeta,z,\lambda)\\ &\quad -\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{013}} \overline{\partial}f(\zeta)\wedge D_{n,q}(\eta_{013})(\zeta,z,\lambda)\end{aligned}$$

$$-\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{03}} f(\zeta)\wedge D_{n,q}(\eta_{03})(\zeta,z,\lambda)$$
$$+\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{13}} f(\zeta)\wedge D_{n,q}(\eta_{13})(\zeta,z,\lambda).$$

Since η_1, η_3 are holomorphic in z, $q \geq 1$, we have

$$\int_{\Delta_{13}} D_{n,q}(\eta_{13})(\zeta,z,\lambda) = 0$$

for $\zeta \in S_{12}$, $z \in \Omega^-$. Therefore we can obtain (i).

(ii) For $z \in \Omega^+$, we apply Stokes' theorem to $d_{\zeta,\lambda}(f(\zeta)\wedge D_{n,q}(\eta_{023})(\zeta,z,\lambda))$ on $S_{12}\times\Delta_{023}$. We then have

$$-\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{02}} f(\zeta)\wedge D_{n,q}(\eta_{01})(\zeta,z,\lambda) = -\overline{\partial}_z\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{023}} f(\zeta)\wedge D_{n,q-1}(\eta_{023})(\zeta,z,\lambda)$$
$$-\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{023}} \overline{\partial} f(\zeta)\wedge D_{n,q}(\eta_{023})(\zeta,z,\lambda)$$
$$-\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{03}} f(\zeta)\wedge D_{n,q}(\eta_{03})(\zeta,z,\lambda)+$$
$$+\int_{(\zeta,\lambda)\in S_{12}\times\Delta_{23}} f(\zeta)\wedge D_{n,q}(\eta_{23})(\zeta,z,\lambda).$$

However, $\int_{\Delta_{23}} D_{n,q}(\eta_{23})$ is a finite sum of terms of the form

$$\eta_2\wedge\eta_3\wedge(\overline{\partial}_\zeta\eta_2)^a\wedge(\overline{\partial}_\zeta\eta_3)^b\wedge(\overline{\partial}_z\eta_2)^c\wedge(\overline{\partial}_z\eta_3)^d$$

where $a+b=n-2-q$, $c+d=q\geq 1$.

η_2 is holomorphic in ζ; thus if $a \geq 1$ the above term will vanish.

η_3 is holomorphic in z; thus if $d \geq 1$ the above term will also vanish.

The only terms that do not vanish immediately are of the form

$$\eta_2\wedge\eta_3\wedge(\overline{\partial}_\zeta\eta_3)^{n-2-q}\wedge(\overline{\partial}_z\eta_2)^q.$$

But η_3 depends on k variables and $1 \leq k \leq n-2-q$. Therefore this term will also vanish. Thus $\int_{\Delta_{23}} D_{n,q}(\eta_{23}) = 0$ and we can obtain (ii).

Substituting (2.3.1) into (2.2.1) (resp. (2.3.2) into (2.2.2)), we obtain:
For $z \in \Omega^-$,

$$f^-(z) = \overline{\partial}_z \left\{ \int_{(\zeta,\lambda)\in S_1\times\Delta_{01}} f(\zeta) \wedge D_{n,q-1}(\eta_{01})(\zeta,z,\lambda) - \int_{(\zeta,\lambda)\in S_{12}\times\Delta_{013}} f(\zeta) \wedge D_{n,q-1}(\eta_{013})(\zeta,z,\lambda) \right\} + \left\{ \int_{(\zeta,\lambda)\in S_1\times\Delta_{01}} \overline{\partial} f(\zeta) \wedge D_{n,q}(\eta_{01})(\zeta,z,\lambda) - \int_{(\zeta,\lambda)\in S_{12}\times\Delta_{013}} \overline{\partial} f(\zeta) \wedge D_{n,q}(\eta_{013})(\zeta,z,\lambda) \right\} - \int_{(\zeta,\lambda)\in S_{12}\times\Delta_{03}} f(\zeta) \wedge D_{n,q}(\eta_{03})(\zeta,z,\lambda). \tag{2.3.3}$$

For $z \in \Omega^+$,

$$f^+(z) = \overline{\partial}_z \left\{ \int_{(\zeta,\lambda)\in S_1\times\Delta_{02}} f(\zeta) \wedge D_{n,q-1}(\eta_{02})(\zeta,z,\lambda) - \int_{(\zeta,\lambda)\in S_{12}\times\Delta_{023}} f(\zeta) \wedge D_{n,q-1}(\eta_{023})(\zeta,z,\lambda) \right\} + \left\{ \int_{(\zeta,\lambda)\in S_1\times\Delta_{02}} \overline{\partial} f(\zeta) \wedge D_{n,q}(\eta_{02})(\zeta,z,\lambda) - \int_{(\zeta,\lambda)\in S_{12}\times\Delta_{023}} \overline{\partial} f(\zeta) \wedge D_{n,q}(\eta_{023})(\zeta,z,\lambda) \right\} - \int_{(\zeta,\lambda)\in S_{12}\times\Delta_{03}} f(\zeta) \wedge D_{n,q}(\eta_{03})(\zeta,z,\lambda). \tag{2.3.4}$$

We wish to take boundary values in the above expressions as $z \to \overline{\omega}$. Unfortunately, the above formulas become very singular in this case. To overcome the obstacle, we shall exploit an idea of Lieb-Range [6] and replace the integrals over S_1, S_{12} by integrals over submanifolds of $\mathbb{C}^n \setminus \overline{\Omega}$ for $z \in \Omega^-$, or submanifolds of $\overline{\Omega}$ for $z \in \Omega^+$ such that the integrand form vanishes to high enough order on $\overline{\omega}$. This will be achieved by repeated application of Stokes' theorem.

Extension of the forms. We introduce certain auxilliary regions. We first choose $\epsilon_0 > 0$ sufficiently small such that

$$G = \{z \in W' \mid |\rho(z)| < \epsilon_0, r(z) < \epsilon_0\}$$

is a bounded piecewise smooth domain. We consider the domains

$$R_1 = \{z \in G \mid 0 \le \rho \le \epsilon_0, r \le \rho\}, \quad \widetilde{R}_1 = \{z \in G \mid 0 \le -\rho \le \epsilon_0, r \le -\rho\},$$
$$R_{12} = \{z \in G \mid 0 \le \rho = r \le \epsilon_0\}, \quad \widetilde{R}_{12} = \{z \in G \mid 0 \le -\rho = r \le \epsilon_0\}.$$

We have $R_1 \cup \widetilde{R}_1 = \{z \in G \mid r < |\rho|\}$. We define

$$V =: G \cap \overline{G \setminus (R_1 \cup \widetilde{R}_1)}.$$

Then

$$\partial V \cap G = \{z \in G \mid |\rho| = r\} = R_{12} \cup \widetilde{R}_{12}.$$

For any ordered subset $I \subset \{1, 2\}$ we orient R_I in such a way such that the orientation is skew symmetric in the components of I and

$$(*) \qquad \partial R_I = -\sum_{j \notin I} R_{Ij} + S_I^0 - S_I$$

where $S_I^0 = \{z \in W' \cap \partial G \mid \rho_i = \epsilon_0 \text{ for } i \in I\}$, $\rho_1 =: \rho$, $\rho_2 =: r$.

We do not take into consideration in (*) the part of ∂R_I that comes from $\partial W'$ or ∂G (since the forms we shall be considering will vanish in these pieces). Similar expressions will hold for $\partial \widetilde{R}_1$, $\partial \widetilde{R}_{12}$.

We shall extend f to a slightly larger set $G = \{|\rho| < \epsilon_0, r < \epsilon_0\}$, using the extension Lemmas 1 and 2 in Michel-Shaw [7]. Lemma 2 in that paper states the following: Let $f \in C^{\mu}_{(0,q)}(\overline{\omega})$, $\overline{\partial}_b f \in C^{\mu}_{0,q+1}(\overline{\omega})$. For $s \in \mathbb{N}$ such that $0 \le s \le \frac{\mu-2}{2}$ there exist extension forms $E_s f \in C^{\mu-2s}_{(0,q)}(G)$ and forms $g_s \in C^{\mu-2s-1}_{(0,q+1)}(G)$, $h_s \in C^{\mu-2s-1}_{(0,q+1)}(G)$, $b_s \in C^{\mu-2s-2}_{(0,q)}(G)$ such that

(i) $\operatorname{supp} E_s f \cap \partial W' = \emptyset$, $\operatorname{supp} E_s f \cap \{|\rho| = \epsilon_0\} = \emptyset$, $\operatorname{supp} E_s f \cap \{r = \epsilon_0\} = \emptyset$,
(ii) $E_s|_{\overline{\omega}} = f$, $E_{s+1} f - E_s f = \rho^{s+1} b_s$ in G,
(iii) $\overline{\partial} E_s f - E_s \overline{\partial}_b f = \rho^s g_s + h_s$, in G where $h_s = 0$ on $G \cap \{r \le 0\}$.

Moreover we have

$$\|E_s f\|_{C^{\mu-2s}(G)} + \|g_s\|_{C^{\kappa-2s-1}(G)} + \|h_s\|_{C^{\kappa-2s-1}(G)} \le c_{\ell,n}(\|f\|_{C^\kappa(\overline{\omega})} + \|\overline{\partial}_b f\|_{C^\kappa(\overline{\omega})}).$$

We have constructed a form $\overline{\partial} E_s f - E_s \overline{\partial}_b f$ which by property (iii) vanishes to high order on $\partial\omega$. Our goal now is to derive formulas for f^-, f^+ in which the integral operators involved are created by integrating this form against certain kernels over submanifolds of Ω^+ or Ω^-.

For $z \in \Omega^-$,

$$f^- = \overline{\partial}_z \left\{ \int_{S_1\times\Delta_{01}} f \wedge D_{n,q-1}(\eta_{01}) - \int_{S_{12}\times\Delta_{013}} f \wedge D_{n,q-1}(\eta_{013}) \right\} + \left\{ \int_{S_1\times\Delta_{01}} \overline{\partial} f \wedge D_{n,q}(\eta_{01}) - \int_{S_{12}\times\Delta_{013}} \overline{\partial} f \wedge D_{n,q}(\eta_{013}) \right\} - \int_{S_{12}\times\Delta_{03}} f \wedge D_{n,q}(\eta_{03}). \tag{2.3.5}$$

For $z \in \Omega^+$,

$$f^+ = \overline{\partial}_z \left\{ \int_{S_1\times\Delta_{02}} f \wedge D_{n,q-1}(\eta_{02}) - \int_{S_{12}\times\Delta_{023}} f \wedge D_{n,q-1}(\eta_{023}) \right\} + \left\{ \int_{S_1\times\Delta_{02}} \overline{\partial} f \wedge D_{n,q}(\eta_{02}) - \int_{S_{12}\times\Delta_{023}} \overline{\partial} f \wedge D_{n,q}(\eta_{023}) \right\} - \int_{S_{12}\times\Delta_{03}} f \wedge D_{n,q}(\eta_{03}) \tag{2.3.6}$$

Recall that $f, \overline{\partial}_b f \in C^{3(l+2n-1)}(\overline{\omega})$. Let $s = \ell + 2n - 1$ and $E_s f$, $E_s(\overline{\partial}_b f)$ be the corresponding extensions of $f, \overline{\partial}_b f$ in G (constructed as above). For $z \in \Omega^-$, we apply Stokes' theorem to

$$d_{\zeta,\lambda}(E_s f \wedge D_{n,q-1}(\eta_{01})(\zeta, z, \lambda)), d_{\zeta,\lambda}(E_s \overline{\partial}_b f \wedge D_{n,q}(\eta_{01})(\zeta, z, \lambda)) \quad \text{on } R_1 \times \Delta_{01},$$

$$d_{\zeta,\lambda}(E_s f \wedge D_{n,q-1}(\eta_{013})(\zeta, z, \lambda)), d_{\zeta,\lambda}(E_s \overline{\partial}_b f \wedge D_{n,q}(\eta_{013})(\zeta, z, \lambda)) \quad \text{on } R_{12} \times \Delta_{013},$$

$$d_{\zeta,\lambda}(E_s f \wedge D_{n,q}(\eta_{03})(\zeta, z, \lambda)) \quad \text{on} R_{12} \times \Delta_{03}.$$

Then we can replace

$$\int_{S_1\times\Delta_{01}} f\wedge D_{n,q-1}(\eta_{01}) - \int_{S_{12}\times\Delta_{013}} f\wedge D_{n,q-1}(\eta_{013})$$

by the sum

$$\begin{aligned}
\text{(2.3.7)}\qquad & \overline{\partial}_z\left\{\int_{R_1\times\Delta_{01}} E_s f\wedge D_{n,q-2}(\eta_{01}) - \int_{R_{12}\times\Delta_{013}} E_s f\wedge D_{n,q-2}(\eta_{013})\right\} \\
& -\left\{\int_{R_1\times\Delta_{01}} \overline{\partial} E_s f\wedge D_{n,q-1}(\eta_{01}) - \int_{R_{12}\times\Delta_{013}} \overline{\partial} E_s f(\zeta)\wedge D_{n,q-1}(\eta_{013})\right\} \\
& + \int_{R_{12}\times\Delta_{13}} E_s f\wedge D_{n,q-1}(\eta_{13}) - \int_{R_{12}\times\Delta_{03}} E_s f\wedge D_{n,q-1}(\eta_{03}) \\
& + \int_{R_1\times\Delta_1} E_s f\wedge D_{n,q-1}(\eta_1) - \int_{R_1\times\Delta_0} E_s f\wedge D_{n,q-1}(\eta_0).
\end{aligned}$$

Similarly we replace

$$\int_{S_1\times\Delta_{01}} \overline{\partial}_b f\wedge D_{n,q}(\eta_{01}) - \int_{S_{12}\times\Delta_{013}} \overline{\partial}_b f\wedge D_{n,q}(\eta_{013})$$

by

$$\begin{aligned}
\text{(2.3.8)}\qquad & \overline{\partial}_z\left\{\int_{R_1\times\Delta_{01}} E_s\overline{\partial}_b f\wedge D_{n,q}(\eta_{01}) - \int_{R_{12}\times\Delta_{013}} E_s\overline{\partial}_b f\wedge D_{n,q}(\eta_{013})\right\} \\
& -\left\{\int_{R_1\times\Delta_{01}} \overline{\partial} E_s\overline{\partial}_b f\wedge D_{n,q}(\eta_{01}) - \int_{R_{12}\times\Delta_{013}} \overline{\partial} E_s\overline{\partial}_b f\wedge D_{n,q}(\eta_{013})\right\} \\
& + \int_{R_{12}\times\Delta_{13}} E_s\overline{\partial}_b f\wedge D_{n,q}(\eta_{13}) - \int_{R_{12}\times\Delta_{03}} E_s\overline{\partial}_b f\wedge D_{n,q}(\eta_{03}) \\
& + \int_{R_1\times\Delta_1} E_s\overline{\partial}_b f\wedge D_{n,q}(\eta_1) - \int_{R_1\times\Delta_0} E_s\overline{\partial}_b f\wedge D_{n,q}(\eta_0)
\end{aligned}$$

and finally we can replace

$$-\int_{S_{12}\times\Delta_{03}} f\wedge D_{n,q}(\eta_{03})$$

by

$$\overline{\partial}_z \int\limits_{R_{12}\times\Delta_{03}} E_s f \wedge D_{n,q-1}(\eta_{03}) + \int\limits_{R_{12}\times\Delta_{03}} \overline{\partial} E_s f \wedge D_{n,q}(\eta_{03}) + \int\limits_{R_{12}\times\Delta_0} E_s f \wedge D_{n,q}(\eta_0) - \int\limits_{R_{12}\times\Delta_3} E_s f \wedge D_{n,q}(\eta_3) \tag{2.3.9}$$

Since η_1, η_3 are holomorphic in z and η_3 depends only on a fixed number of variables, a lot of these integrals vanish. More precisely we have the following:

LEMMA 2.4. *For $z \in \Omega^-$ we have*

(i) $\displaystyle\int_{\tilde{M}\times\Delta_1} h \wedge D_{n,q}(\eta_1) = 0,$ *for* $q \geq 1$, $\tilde{M} = R_1, R_{12}, S_1$,

$$\overline{\partial}_z \int\limits_{R_1\times\Delta_1} E_s f \wedge D_{n,0}(\eta_1) + \int\limits_{R_1\times\Delta_1} E_s \overline{\partial}_b f \wedge D_{n,1}(\eta_1) = 0,$$

(ii) $\displaystyle\int_{\tilde{M}\times\Delta_3} h \wedge D_{n,q}(\eta_3) = 0,$ *for* $q \geq 1$, $\tilde{M} = R_{12}, S_{12}$,

$$\overline{\partial}_z \int\limits_{R_{12}\times\Delta_3} E_s f \wedge D_{n,0}(\eta_3) + \int\limits_{R_{12}\times\Delta_3} E_s \overline{\partial}_b f \wedge D_{n,1}(\eta_3) = 0,$$

(iii) $\displaystyle\int_{\tilde{M}\times\Delta_{13}} h \wedge D_{n,q}(\eta_{13}) = 0,$ *for* $q \geq 1$, $\tilde{M} = R_{12}, S_{12}$,

$$\overline{\partial}_z \int\limits_{R_{12}\times\Delta_{13}} E_s f \wedge D_{n,0}(\eta_{13}) + \int\limits_{R_{12}\times\Delta_{13}} E_s \overline{\partial}_b f(\zeta) \wedge D_{n,1}(\eta_{13}) = 0,$$

where h is some smooth form defined on $\tilde{M}$ of appropriate degree.

Proof. Follows directly from the fact that η_1, η_3 are holomorphic in z.

Using Lemma 2.4 and replacing (2.3.7), (2.3.8), (2.3.9) to (2.3.5), for $z \in \Omega^-$ we have

$$f^- = \overline{\partial}_z \left\{ \int\limits_{R_1\times\Delta_{01}} N_s f \wedge D_{n,q-1}(\eta_{01}) - \int\limits_{R_{12}\times\Delta_{013}} N_s f \wedge D_{n,q-1}(\eta_{013}) - \int\limits_{R_1\times\Delta_0} E_s f \wedge D_{n,q-1}(\eta_0) \right\} \tag{2.4.1}$$

$$+\left\{\int_{R_1\times\Delta_{01}} N_s\overline{\partial}_b f\wedge D_{n,q}(\eta_{01})-\int_{R_{12}\times\Delta_{013}} N_s\overline{\partial}_b f\wedge D_{n,q}(\eta_{013})\right.$$

$$\left.-\int_{R_1\times\Delta_0} E_s\overline{\partial}_b f\wedge D_{n,q}(\eta_0)\right\}$$

$$-\int_{R_{12}\times\Delta_{03}} N_s f\wedge D_{n,q}(\eta_{03})-\int_{R_{12}\times\Delta_0} E_s f\wedge D_{n,q}(\eta_0)$$

where $N_s f = E_s\overline{\partial}_b f - \overline{\partial}E_s f$.

Arguing in a similar manner and taking into account that η_2 is holomorphic in ζ, η_3 is holomorphic in $z, \zeta_1, \dots, \zeta_{n-k}$, $1 \le k \le n-2-q$, we can obtain a similar expression for f^+.

For $z \in \Omega^+$, we have

$$f^+ = \overline{\partial}_z\left\{-\int_{\tilde{R}_1\times\Delta_{01}} N_s f\wedge D_{n,q-1}(\eta_{02})+\int_{\tilde{R}_{12}\times\Delta_{013}} N_s f\wedge D_{n,q-1}(\eta_{023})\right.\tag{2.4.2}$$

$$\left.+\int_{\tilde{R}_1\times\Delta_0} E_s f\wedge D_{n,q-1}(\eta_0)\right\}$$

$$+\left\{-\int_{\tilde{R}_1\times\Delta_{02}} N_s\overline{\partial}_b f\wedge D_{n,q}(\eta_{02})+\int_{\tilde{R}_{12}\times\Delta_{023}} N_s\overline{\partial}_b f\wedge D_{n,q}(\eta_{023})\right.$$

$$\left.+\int_{\tilde{R}_1\times\Delta_0} E_s\overline{\partial}_b f\wedge D_{n,q}(\eta_0)\right\}$$

$$+\int_{\tilde{R}_{12}\times\Delta_{03}} N_s f\wedge D_{n,q}(\eta_{03})+\int_{\tilde{R}_{12}\times\Delta_0} E_s f\wedge D_{n,q}(\eta_0).$$

For $z \in \Omega^-$, we apply Stokes' theorem to

$$d_{\zeta,\lambda}(N_s f\wedge D_{n,q}(\eta_{03})) \quad \text{on } V\times\Delta_{03},$$
$$d_{\zeta,\lambda}(E_s f\wedge D_{n,q}(\eta_0)) \quad \text{on } V\times\Delta_0.$$

Then for $z \in \Omega^-$, we have

$$-\int_{R_{12}\times\Delta_{03}} N_s f \wedge D_{n,q}(\eta_{03}) = \int_{\tilde{R}_{12}\times\Delta_{03}} N_s f \wedge D_{n,q}(\eta_{03}) - \int_{V\times\Delta_0} N_s f \wedge D_{n,q}(\eta_0)$$
$$+ \overline{\partial}_z \int_{V\times\Delta_{03}} N_s f \wedge D_{n,q-1}(\eta_{03}) + \int_{V\times\Delta_{03}} N_s \overline{\partial}_b f \wedge D_{n,q}(\eta_{03})$$

and

$$-\int_{R_{12}\times\Delta_0} E_s f \wedge D_{n,q}(\eta_0) = \int_{\tilde{R}_{12}\times\Delta_0} E_s f \wedge D_{n,q}(\eta_0) - \overline{\partial}_z \int_{V\times\Delta_0} E_s f \wedge D_{n,q-1}(\eta_0)$$
$$-\int_{V\times\Delta_0} \overline{\partial} E_s f \wedge D_{n,q}(\eta_0).$$

By substituting the above expressions in (2.4.1), for $z \in \Omega^-$ we obtain

$$(2.4.3)\quad f^-(z) = \overline{\partial}_z \left\{ \int_{R_1\times\Delta_{01}} N_s f \wedge D_{n,q-1}(\eta_{01}) - \int_{R_{12}\times\Delta_{013}} N_s f \wedge D_{n,q-1}(\eta_{013}) \right.$$
$$\left. + \int_{V\times\Delta_{03}} N_s f \wedge D_{n,q-1}(\eta_{03}) - \int_{(R_1\cup V)\times\Delta_0} E_s f \wedge D_{n,q-1}(\eta_0) \right\}$$
$$+ \left\{ \int_{R_1\times\Delta_{01}} N_s \overline{\partial}_b f \wedge D_{n,q}(\eta_{01}) - \int_{R_{12}\times\Delta_{013}} N_s \overline{\partial}_b f \wedge D_{n,q}(\eta_{013}) \right.$$
$$\left. + \int_{V\times\Delta_{03}} N \overline{\partial}_b f \wedge D_{n,q}(\eta_{03}) - \int_{(R_1\cup V)\times\Delta_0} E_s \overline{\partial}_b f \wedge D_{n,q}(\eta_0) \right\}$$
$$+ \int_{\tilde{R}_{12}\times\Delta_{03}} N_s f \wedge D_{n,q}(\eta_{03}) + \int_{\tilde{R}_{12}\times\Delta_0} E_s f \wedge D_{n,q}(\eta_0).$$

Similarly, for $z \in \Omega^+$ we have

$$(2.4.4)\quad f^+ = \overline{\partial}_z \left\{ -\int\limits_{\tilde{R}_1 \times \Delta_{02}} N_s f \wedge D_{n,q-1}(\eta_{02}) + \int\limits_{R_{12} \times \Delta_{023}} N_s f \wedge D_{n,q-1}(\eta_{023}) + \int\limits_{\tilde{R}_1 \times \Delta_0} E_s f \wedge D_{n,q}(\eta_0) \right\}$$
$$+ \left\{ -\int\limits_{\tilde{R}_1 \times \Delta_{02}} N_s \overline{\partial}_b f \wedge D_{n,q}(\eta_{02}) + \int\limits_{\tilde{R}_{12} \times \Delta_{023}} N_s \overline{\partial}_b f \wedge D_{n,q}(\eta_{023}) + \int\limits_{\tilde{R}_1 \times \Delta_0} E_s f \wedge D_{n,q}(\eta_0) \right\}$$
$$+ \int\limits_{\tilde{R}_{12} \times \Delta_{03}} N_s f \wedge D_{n,q}(\eta_{03}) + \int\limits_{\tilde{R}_{12} \times \Delta_0} E_s f \wedge D_{n,q}(\eta_0).$$

We are only a step away from obtaining the desired homotopy formula. The growth of $N_s f$ will cancel the singularities of the kernels and thus the integral operators that appear in (2.4.3) and (2.4.4) will become continuous on $\overline{\omega}$. This will be established by the following lemma.

LEMMA 2.5. *Let $f \in C^{3(\ell+2n-1)}_{(0,q)}(\overline{\omega})$, $\overline{\partial}_b f \in C^{3(\ell+2n-1)}_{(0,q+1)}(\overline{\omega})$, $\ell \in \mathbb{N}$. Let D^I_z be a differential operator in z of order $|I| = \ell$. Then we have the following*:

(i) *For $z \in G \cap \overline{\Omega^-}$,*

$$\left| D^\ell_z \int\limits_{R_1 \times \Delta_{01}} N_s f \wedge D_{n,q-1}(\eta_{01}) \right| + \left| D^\ell_z \int\limits_{R_{12} \times \Delta_{013}} N_s f \wedge D_{n,q-1}(\eta_{013}) \right|$$
$$\leq C_\ell (\|f\|_{C^{3(\ell+2n-1)}} + \|\overline{\partial}_b f\|_{C^{3(\ell+2n-1)}}).$$

(ii) *For $z \in G \cap \overline{\Omega^+}$,*

$$\left| D^\ell_z \int\limits_{\tilde{R}_1 \times \Delta_{02}} N_s f \wedge D_{n,q-1}(\eta_{02}) \right| + \left| D^\ell_z \int\limits_{\tilde{R}_{12} \times \Delta_{023}} N_s f \wedge D_{n,q-1}(\eta_{023}) \right|$$
$$\leq C_\ell (\|f\|_{C^{3(\ell+2n-1)}} + \|\overline{\partial}_b f\|_{C^{3(\ell+2n-1)}}.$$

(iii) *For* $z \in G \cap \{r \leq 0\}$,

$$\left| D_z^\ell \int\limits_{V \times \Delta_{03}} N_s f \wedge D_{n,q-1}(\eta_{03}) \right| \leq C_\ell(\|f\|_{C^{3(\ell+2n-1)}} + \|\overline{\partial}_b f\|_{C^{3(\ell+2n-1)}}).$$

(iv)

$$\int\limits_{G \times \Delta_0} E_s f \wedge D_{n,q-1}(\eta_0) \in C^{3(\ell+2n-1),a}(G), \quad 0 < a < 1.$$

Proof. We shall give a detailed proof for (i). (ii), (iii) are proved similarly. (iv) is a well-known fact for the Bochner-Martinelli kernel. We shall need a lemma about the growth of $N_s f(\zeta)$ for $\zeta \in R_I$.

LEMMA 2.6. *Let f, $\overline{\partial}_b f$ be as in Lemma* (2.5). *Let $s = \ell + 2n - 1$ as before. For $\zeta \in R_I \cap \{r > 0\}$ (resp. $\zeta \in R_I \cap \{r \leq 0\}$) with $I = \{1\}$ or $I = \{12\}$ we have*

$$|N_s f(\zeta)| \leq C_{\ell,n}(r(\zeta)^{\ell+2n-2} + \rho(\zeta)^{\ell+2n-1})(\|f\|_{C^{3(\ell+2n-1)}} + \|\overline{\partial}_b f\|_{C^{3(\ell+2n-1)}})$$

$$(\text{resp. } |N_s f(\zeta)| \leq C_{\ell,n}\rho(\zeta)^{\ell+2n-1}(\|f\|_{C^{3(\ell+2n-1)}} + \|\overline{\partial}_b f\|_{C^{3(\ell+2n-1)}})).$$

Proof. Using property (iii) of the extension lemma for $f, \overline{\partial}_b f \in C^{3(l+2n-1)}$ with $s = l + 2n - 1$ we can write $N_s f$ as

$$N_s f = N_{\ell+2n-1} f = \rho^{\ell+2n-1} g_{\ell+2n-1} + h_{\ell+2n-1}$$

where $g_{\ell+2n-1}, h_{\ell+2n-1} \in C^{\ell+2n-2}_{(0,q+1)}(G)$ and $h_{\ell+2n-1} = 0$ on $G \cap \{r \leq 0\}$.

For $\zeta \in G$, let $\pi_\zeta \in \partial\mathcal{D}_0$ such that $|\zeta - \pi_z| = \text{dist}(\zeta, \partial\mathcal{D}_0)$. Then $|\zeta - \pi_z|$ is comparable to $|r(\zeta)|$. For $\zeta \in G \cap \{r > 0\}$ we Taylor expand $h_{\ell+2n-1}$ around π_ζ and we obtain the estimate

$$|h_{\ell+2n-1}(\zeta)| \leq c_{\ell,n} r(\zeta)^{\ell+2n-2} (\|f\|_{C^{3(\ell+2n-1)}} + \|\overline{\partial}_b f\|_{C^{3(\ell+2n-1)}}).$$

Thus for $\zeta \in R_1 \cap \{r > 0\}$ or $\zeta \in R_{12}$ we have

$$|N_s f(\zeta)| \leq C_{\ell,n}(r(\zeta)^{\ell+2n-2} + \rho(\zeta)^{\ell+2n-1})(\|f\|_{C^{3(\ell+2n-1)}} + \|\overline{\partial}_b f\|_{C^{3(\ell+2n-1)}}).$$

For $\zeta \in R_1 \cap \{r \leq 0\}$, $N_{\ell+2n-1} f(\zeta) = \rho(\zeta)^{\ell+2n-1} g_{\ell+2n-1}(\zeta)$. Hence we obtain the second inequality in the lemma.

We are now in a position to start proving (i). With $s = \ell + 2n - 1$ as before, let

$$A =: \int\limits_{R_1 \times \Delta_{01}} N_s f \wedge D_{n,q-1}(\eta_{01}) = \int_{R_1} N_s f \wedge \int_{\Delta_{01}} D_{n,q-1}(\eta_{01}).$$

Following [8] we can rewrite A as

$$A=\sum_{j_0+j_1=n-q-1}\int_{R_1} N_s f\wedge \frac{\mathcal{E}^1}{\Phi_0^{q+j_0}\Phi^{1+j_1}}.$$

We apply the operator D_z^I on A. Then

$$D_z^\ell A=\sum_{\ell_1+\ell_2=\ell}\ \sum_{j_0+j_1=n-q-1}\int_{R_1} N_s f\wedge D_z^{\ell_1}\frac{\mathcal{E}^1}{\Phi_0^{q+j_0}}\wedge D_z^{\ell_2}\frac{1}{\Phi^{1+j_1}}.$$

$D_z^\ell A$ can be written as a finite sum of operators of the form

$$\int_{R_1} N_s f\wedge \frac{\mathcal{E}^m}{\Phi_0^{a_0}\Phi^{a_1}}$$

where

$$\begin{aligned}
&2a_0-m\le 2(q+j_0)-1+\ell_1,\\
&a_1\le 1+j_1+\ell_2,\\
&j_0+j_1=n-q-1,\\
&\ell_1+\ell_2=\ell,\\
&a_1\ge 1,\\
&a_0\ge 0.
\end{aligned}$$

Taking into account that for $\zeta\in R_1 z\in\Omega^-$ we have

$$\begin{aligned}
&|\zeta-z|\ge\rho(\zeta)\ge r(\zeta),\\
&|\Phi(\zeta,z)|\ge c\quad(|\operatorname{Re}\Phi(\zeta,z)|+|\operatorname{Im}\Phi(\zeta,z)|)\ge c(|\rho(\zeta)|+|\rho(z)|)\ge c\,r(\zeta)
\end{aligned}$$

where c is some positive constant, for $z\in\Omega^-$ we can obtain

$$\begin{aligned}
|D_z^\ell A| &\le \left|\int_{R_1} N_s f\wedge\frac{\mathcal{E}^m}{\Phi_0^{a_0}\Phi^{a_1}}\right| \lesssim \int_{R_1}\frac{|N_s f|\quad |\zeta-z|^m dV}{|\zeta-z|^{m+2(q+j_0)-1+\ell_1}\quad |\rho(\zeta)|^{1+j_1+\ell_2}}\\
&\lesssim \int_{R_1\cap\{r>0\}}\frac{(r^{\ell+2n-2}+\rho^{\ell+2n-1})\,d\mathcal{V}}{\rho^{q+j_0+n-1+\ell}}+\int_{R_1\cap\{r\le 0\}}\frac{\rho^{\ell+2n-1}d\mathcal{V}}{\rho^{q+j_0+n-1+\ell}}\\
&\lesssim \int_{R_1\cap\{r>0\}}\frac{(r^{\ell+2n-2}+\rho^{\ell+2n-1})d\mathcal{V}}{\rho^{\ell+2n-2}}+\int_{R_1\cap\{r\le 0\}}\frac{\rho^{\ell+2n-1}\,dV}{\rho^{\ell+2(n-1)}}\lesssim C'_\ell.
\end{aligned}$$

In the above inequality we used the fact that $\ell+n-1+j_0\le\ell+2(n-1)$. C'_ℓ is a positive constant that depends on ℓ, n, $\|f\|_{C^{3(\ell+2n-1)}}$, $\|\overline{\partial}_b f\|_{C^{3(\ell+2n-1)}}$, vol (R_1) and by $A\lesssim B$ we shall denote $A\le CB$ where C is a positive constant that depends on the quantities mentioned above.

Similarly, let

$$B =: \int\limits_{R_{12}\times\Delta_{013}} N_s f \wedge D_{n,q-1}(\eta_{013}) = \int_{R_{12}} N_s f \wedge \int_{\Delta_{013}} D_{n,q-1}(\eta_{013}).$$

Since η_1 is holomorphic in z, η_3 is holomorphic in $\zeta_1, .., \zeta_{n-k}$ we have

$$B = \sum_{j_2=0}^{k-1} \sum_{j_0+j_1=n-2-q-j_2} \int_{R_{12}} N_s f \wedge \frac{\mathcal{E}^1}{\Phi_0^{q+j_0}\Phi^{1+j_1}X^{1+j_2}}.$$

$D_z^\ell B$ is a finite sum of operators of the form

$$\int_{R_{12}} N_s f \wedge \frac{\mathcal{E}^m}{\Phi_0^{a_0}\Phi^{a_1}X^{a_2}}$$

where

$$\begin{aligned}
&2a_0 - m \le 2(q + j_0) - 1 + \ell_1,\\
&a_1 + a_2 \le 2 + j_1 + j_2 + \ell_2,\\
&j_0 + j_1 = n - 2 - q - j_2,\\
&0 \le j_2 \le k - 1,\\
&a_1, \quad a_2 \ge 1.
\end{aligned}$$

Taking into consideration that for $\zeta \in R_{12}, z \in \Omega^-$ we have

$$\begin{aligned}
&|\zeta - z| \ge \rho(\zeta) = r(\zeta),\\
&|\Phi(\zeta, z)| \ge c(|\rho(\zeta)| + |\rho(z)|) \ge c\,|\rho(\zeta)| \ge c\,\rho(\zeta),\\
&|X(\zeta, z)| \ge c(|\mathrm{Re}\,X(\zeta, z)| + |\mathrm{Im}\,X(\zeta, z)|) \ge c(|r(\zeta)| + |r(z)|) \ge c\,r(\zeta),\\
&0 \le \rho(\zeta) = r(\zeta) \le \epsilon_0.
\end{aligned}$$

We have

$$\begin{aligned}
|D_z^\ell B| &\le \left| \int_{R_{12}} N_s f \wedge \frac{\mathcal{E}^m}{\Phi_0^{a_0}\Phi^{a_1}X^{a_2}} \right| \lesssim \int_{R_{12}} \frac{(r^{\ell+2n-2} + \rho^{\ell+2n-1})|\zeta - z|^m \, d\mathcal{V}}{|\zeta - z|^{2(q+j_0)+m-1+\ell_1}\rho^{a_1+a_2}}\\
&\lesssim \int_{R_{12}} \frac{(r^{\ell+2n-2} + r^{\ell+2n-1})\, d\mathcal{V}}{r^{2(q+j_0)-1+\ell+j_1+j_2}} \lesssim \int_{R_{12}} \frac{r^{\ell+2n-2}\, d\mathcal{V}}{r^{q+\ell+j_0+n-1}}.
\end{aligned}$$

But

$$q + j_0 \le n - 2.$$

Hence we have

$$|D_z^\ell B| \lesssim \int_{R_{12}} \frac{r^{\ell+2n-2}\, d\mathcal{V}}{r^{\ell+2n-3}} \lesssim C''_\ell$$

where $C''_\ell > 0$ is a constant that depends on $n, \ell, \mathrm{vol}(R_{12}), \|f\|_{C^{3(\ell+2n-1)}}, \|\overline{\partial}_b f\|_{C^{3(\ell+2n-1)}}$. Lemma 2.5 tells us that the operators appearing on the right hand side of (2.4.3) and (2.4.4) are continuous up to $\overline{\omega}$. Therefore for $z \in \overline{\omega}$ we have

$$f = \tau(f^- - f^+) = \overline{\partial}_b K_q^\ell f + K_{q+1}^\ell \overline{\partial}_b f$$

where τ is the projection operator from the space of $(0, q)$ forms defined on $\overline{\omega}$ with coefficients in $C^\infty(\overline{\omega})$ to the space $\mathcal{B}_{(0,q)}(\overline{\omega})$ which consists of all smooth $(0, q)$ forms in $\mathbb{C}^n$ restricted to $\omega \subset M$ which are pointwise orthogonal to the ideal generated by $\overline{\partial}\rho$ and where

$$\begin{aligned} K_q^\ell f = & \int_{R_1\times\Delta_{01}} N_s f \wedge D_{n,q-1}(\eta_{01}) + \int_{\tilde{R}_1\times\Delta_{02}} N_s f \wedge D_{n,q-1}(\eta_{02}) \\ & + \int_{V\times\Delta_{03}} N_s f \wedge D_{n,q-1}(\eta_{03}) - \int_{R_{12}\times\Delta_{013}} N_s f \wedge D_{n,q-1}(\eta_{013}) \\ & - \int_{\tilde{R}_{12}\times\Delta_{023}} N_s f \wedge D_{n,q-1}(\eta_{023}) - \int_{G\times\Delta_0} E_s f \wedge D_{n,q-1}(\eta_0) \end{aligned}$$

where $s = \ell + 2n - 1$. Using Lemma 2.6 we obtain

$$\|K_q^\ell f\|_{C^\ell} \lesssim c_{\ell,n}(\|f\|_{C^{3(\ell+2n-1)}} + \|\overline{\partial}_b f\|_{C^{3(\ell+2n-1)}}).$$

And this concludes the proof of the main theorem.

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