UNIVERSAL LOCALLY CONNECTED REFINEMENTS

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We shall prove the existence of a locally connected space associated with an arbitrary topological space which has a universal property with respect to maps of locally connected spaces. We shall also obtain a similar result for uniform spaces.

The proofs of these results are much easier when the notation emphasizes that a topological space is a set together with a definite family of subsets. However, for applications it is convenient to have the theorems in the usual language; hence we shall now state our results in such terms.

THEOREM A. *Let* S *be a topological space. There exist a locally connected topological space* S^* *and a continuous one-to-one mapping* φ *of* S^* *onto* S *such that*

If $\mathbf{f}$ *is any continuous mapping of a locally connected space* $\mathbf{A}$ *into* S, *then* $\mathbf{f}$ *can be factored in the form* $\mathbf{f} = \varphi \circ \mathbf{f}^*$ *where* $\mathbf{f}^*$ *is a continuous mapping of* $\mathbf{A}$ *into* S^*.

The pair $\langle \mathsf{S}^*, \varphi \rangle$ *is unique within isomorphism.*

A special case of this theorem, used in [1, pp. 54–55], provided the motivation for this work. A similar theorem for locally arcwise connected spaces is proved in [2].

The author is indebted to the referee for calling his attention to a paper of G. S. Young [5] in which many similar theorems are established. In particular the c-topology of Young is our $\mathfrak{I}'$, the a-topology is the topology of [2], and the lc-topology appears to be in general intermediate between the $\mathfrak{I}^*$-topology and the a-topology. The two latter coincide for complete metric spaces.

We shall refer to the pair $\langle \mathsf{S}^*, \varphi \rangle$ of Theorem A as a universal locally connected refinement of S.

THEOREM B. *Let* S *be a topological space, and let* $\langle \mathsf{S}^*, \varphi \rangle$ *be a universal locally connected refinement of* S. *If* S *satisfies the first axiom of countability or any of the separation axioms* T_0, T_1, T_2, *or* T_3, *then so does* S^*. *If the topology of* S *can be derived from a uniform structure or a metric, then the same is true of* S^*; *hence* S^* *will be completely regular whenever* S *is.*

On the other hand, if S is separable, compact, locally compact, paracompact, or connected, then S^* need not have these properties. Similarly the separation axiom T_4 need not transfer from S to S^*, but we do not answer the question concerning T_5.

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THEOREM C. *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be topological spaces, and let* $\langle \mathbf{S}_1^*, \varphi_1\rangle$ *and* $\langle \mathbf{S}_2^*, \varphi_2\rangle$ *be universal locally connected refinements of* $\mathbf{S}_1$ *and* $\mathbf{S}_2$, *respectively. If* φ *is the product mapping of* $\mathbf{S}_1^* \times \mathbf{S}_2^*$ *into* $\mathbf{S}_1 \times \mathbf{S}_2$ *induced by* φ_1 *and* φ_2, *then* $\langle \mathbf{S}_1^* \times \mathbf{S}_2^*, \varphi\rangle$ *is a universal locally connected refinement of* $\mathbf{S}_1 \times \mathbf{S}_2$.

THEOREM D. *Let* $\mathbf{G}$ *be a topological group, and let* $\langle \mathbf{G}^*, \varphi\rangle$ *be a universal locally connected refinement of the space* $\mathbf{G}$. *Then group structure can be introduced into* $\mathbf{G}^*$ *so that it becomes a topological group and* φ *becomes an algebraic isomorphism.*

THEOREM E. *Let* $\mathcal{S}$ *be a uniform space. There exist a uniformly locally connected space* $\mathcal{S}^*$ *and a uniformly continuous one-to-one mapping* φ *of* $\mathcal{S}^*$ *onto* $\mathcal{S}$ *such that*

> *If* $\mathbf{f}$ *is any uniformly continuous mapping of a uniformly locally connected space* $\mathcal{A}$ *into* $\mathcal{S}$, *then* $\mathbf{f}$ *can be factored in the form* $\mathbf{f} = \varphi \circ \mathbf{f}^*$ *where* $\mathbf{f}^*$ *is a uniformly continuous mapping of* $\mathcal{A}$ *into* $\mathcal{S}^*$.

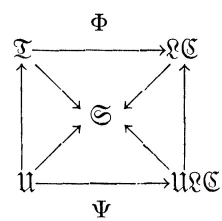
The pair $\langle \mathcal{S}^*, \varphi\rangle$ *is unique within isomorphism. If* $\mathbf{S}$ *and* $\mathbf{S}^*$ *are the topological spaces corresponding to* $\mathcal{S}$ *and* $\mathcal{S}^*$, *respectively, then* $\langle \mathbf{S}^*, \varphi\rangle$ *is a universal locally connected refinement of* $\mathbf{S}$.

Much of the content of these theorems can be summarized in the language of categories in the following statements:

The injection functor of the category $\mathfrak{L}\mathfrak{C}$ of locally connected spaces and continuous maps into the category $\mathfrak{T}$ of topological spaces and continuous maps has an adjoint Φ.

Similarly the injection of the category $\mathfrak{U}\mathfrak{L}\mathfrak{C}$ of uniformly locally connected spaces and uniformly continuous maps into the category $\mathfrak{U}$ of uniform spaces and uniformly continuous maps has an adjoint Ψ.

These adjoints can be uniquely chosen so that the diagram



is commutative, where $\mathfrak{S}$ is the category of sets, the diagonal arrows are stripping functors, and the vertical arrows represent the passage from uniform spaces to the corresponding topological spaces.

1. Lattices

By a complete lattice we mean a partially ordered set L such that every subset of L has a least upper bound (sup) and a greatest lower bound (inf). No restriction to nonvoid subsets need be made since $\sup \emptyset = \inf L$ and

$\inf \emptyset = \sup L$. We recall that if every subset of L has a greatest lower bound, then every subset of L has a least upper bound.

1.1 THEOREM. *Let L be a complete lattice with weak order relation denoted by $\subseteq$. Let $x \to x'$ be a map of L into itself which is*
(a) *increasing: for all x, $x \subseteq x'$, and*
(b) *order-preserving: if $x \subseteq y$, then $x' \subseteq y'$.*
Let F be the set of fixed points of this map. There is a unique increasing, order-preserving map $x \to x^*$ which retracts L onto F.

Proof. The set F is not empty since $\sup L \in F$ by virtue of (a). Suppose $y_\alpha \in F$ where α runs over an arbitrary index class A. Now for any $\beta \in A$ we have $\inf_\alpha y_\alpha \subseteq y_\beta$; hence $(\inf_\alpha y_\alpha)' \subseteq y'_\beta = y_\beta$, and therefore

$$(\inf_\alpha y_\alpha)' \subseteq \inf_\beta y_\beta = \inf_\alpha y_\alpha .$$

Since the opposite inequality is given by (a), we conclude $\inf_\alpha y_\alpha \in F$.

Now for any $x \in L$, put $x^* = \inf \{y \in F \mid y \supseteq x\}$. It is immediately clear that $x \to x^*$ is an increasing, order-preserving retraction of L onto F.

Suppose $x \to x^\circ$ is an increasing, order-preserving map which retracts L onto F. Pick a definite $x \in L$. If $y \in F$ and $y \supseteq x$, then $y = y^\circ \supseteq x^\circ$; thus, x° is a lower bound for the set $G = \{y \in F \mid y \supseteq x\}$. Now $x^\circ \in F$ and $x \subseteq x^\circ$, so $x^\circ \in G$. Hence $x^\circ = \inf G = x^*$. This prove the uniqueness.

We shall refer to the map $x \to x^*$ as the idempotent map associated with $x \to x'$.

1.2 COROLLARY. *If every element of L is the join of members of F, then $x \to x^*$ is the only order-preserving retraction of L onto F.*

Proof. For in this case, the map must also be increasing.

2. The lattice of topologies

Let S be a set. By a topology for S we mean a collection $\mathfrak{I}$ of subsets of S such that

(a) If $\mathcal{G} \subseteq \mathfrak{I}$, then $\bigcup_{G \in \mathcal{G}} G \in \mathfrak{I}$.
(b) If $G_1, G_2 \in \mathfrak{I}$, then $G_1 \cap G_2 \in \mathfrak{I}$.
(c) $\emptyset \in \mathfrak{I}$ and $S \in \mathfrak{I}$.

When a particular topology $\mathfrak{I}$ for S is fixed, the members of $\mathfrak{I}$ are usually called open sets. We note that the requirement $\emptyset \in \mathfrak{I}$ is redundant because we can take $\mathcal{G} = \emptyset$ in $\mathfrak{a}$).

It is obvious that the set of all topologies for S is partially ordered by set inclusion. We remark that there is a largest topology for S, namely, the set of all subsets of S; with this topology S is a discrete space. There is also a least topology for S, the set $\{\emptyset, S\}$. Note that no separation axioms are demanded for a topology.

If $\{\mathfrak{I}_\alpha\}$ is a family of topologies for S, then $\bigcap_\alpha \mathfrak{I}_\alpha$ is also a topology. Therefore, the partially ordered set of topologies is a complete lattice, in which $\inf_\alpha \mathfrak{I}_\alpha = \bigcap_\alpha \mathfrak{I}_\alpha$. It need not be true, however, that $\sup_\alpha \mathfrak{I}_\alpha = \bigcup_\alpha \mathfrak{I}_\alpha$.

If $\mathfrak{X}$ is any collection of subsets of S, then there is a least topology $\mathfrak{I}$ of S such that $\mathfrak{I} \supseteq \mathfrak{X}$. This topology may be described nonconstructively as the greatest lower bound of all topologies containing $\mathfrak{X}$, or constructively as the set consisting of S, and all sets which can be represented as arbitrary unions of finite intersections of members of $\mathfrak{X}$. We shall say that $\mathfrak{X}$ generates $\mathfrak{I}$.

Since we shall be considering several topologies for a given set at the same time, it is necessary to call attention to which is meant whenever a topological concept is used. Suppose for example that f is a map of a set A into a set S, and that $\mathcal{E}$ and $\mathfrak{I}$ are topologies for A and S, respectively. We shall say that f is $\mathcal{E}$-$\mathfrak{I}$-continuous if and only if, for all $G \in \mathfrak{I}$, $f^{-1}(G) \in \mathcal{E}$. This is, of course, one of the standard definitions for continuity. The identity map of S is $\mathfrak{I}_1$-$\mathfrak{I}_2$-continuous if and only if $\mathfrak{I}_2 \subseteq \mathfrak{I}_1$. We have the following lemma:

2.1 LEMMA. *Let f be a map of A into S. Let $\mathcal{E}$ be a fixed topology for A. There is a topology $\mathfrak{U}$ for S such that, if $\mathfrak{I}$ is a topology for S, then f is $\mathcal{E}$-$\mathfrak{I}$-continuous if and only if $\mathfrak{I} \subseteq \mathfrak{U}$.*

Proof. This is obvious as soon as we verify that

$$\mathfrak{U} = \{G \subseteq S \mid f^{-1}(G) \in \mathcal{E}\}$$

is a topology for S.

If X is subset of S and $\mathfrak{I}$ is a topology for S, then we shall say that X is $\mathfrak{I}$-connected if it is impossible to find sets G_1 and G_2 in $\mathfrak{I}$ such that

$$X \subseteq G_1 \cup G_2, \qquad G_1 \cap X \neq \emptyset, \qquad G_2 \cap X \neq \emptyset, \quad \text{and} \quad G_1 \cap G_2 \cap X = \emptyset.$$

This is the usual definition of connected, and, without using any separation axioms, it is easy to prove that every $\mathfrak{I}$-connected subset of X is contained in a maximal $\mathfrak{I}$-connected subset of X, and that X is the disjoint union of these maximal sets which we call the $\mathfrak{I}$-components of X. Moreover, if f is an $\mathcal{E}$-$\mathfrak{I}$-continuous map of A into S, and B is an $\mathcal{E}$-connected subset of A, then $f(B)$ is a $\mathfrak{I}$-connected subset of S.

A topological space is said to be locally connected if and only if every point has arbitrarily small connected neighborhoods. It is well known that this is equivalent to the assertion that every component of every open set is open. Hence we shall say that $\mathfrak{I}$ is a locally connected topology for S if and only if $\mathfrak{I}$ contains the $\mathfrak{I}$-components of every member of $\mathfrak{I}$.

Consider a topology $\mathfrak{I}$ for S, and let $\mathfrak{I}'$ be the topology generated by the $\mathfrak{I}$-components of members of $\mathfrak{I}$. Obviously, $\mathfrak{I}' \supseteq \mathfrak{I}$, and $\mathfrak{I}$ is a locally connected topology for S if and only if $\mathfrak{I}' = \mathfrak{I}$.

2.2 LEMMA. *Let f be a map of A into S. Let $\mathcal{E}$ and $\mathfrak{I}$ be topologies for A and S respectively, such that f is $\mathcal{E}$-$\mathfrak{I}$-continuous. Then f is $\mathcal{E}'$-$\mathfrak{I}'$-continuous.*

Proof. Let $\mathfrak{U} = \{X \subseteq S \mid f^{-1}(X) \in \mathcal{E}'\}$. To prove the lemma, we must prove that $\mathfrak{I}' \subseteq \mathfrak{U}$, and, since $\mathfrak{U}$ is a topology for S, it is sufficient to show that the generators of $\mathfrak{I}'$ are in $\mathfrak{U}$.

Let H be a $\mathfrak{I}$-component of a set $G \in \mathfrak{I}$. Then $f^{-1}(G) \in \mathcal{E}$. Let J be any $\mathcal{E}$-component of $f^{-1}(G)$ which touches $f^{-1}(H)$; then $f(J) \subseteq G$ and $f(J) \cap H \neq \emptyset$. Since J is $\mathcal{E}$-connected and f is $\mathcal{E}$-$\mathfrak{I}$-continuous, $f(J)$ is $\mathfrak{I}$-connected. It follows that $f(J) \subseteq H$ and $J \subseteq f^{-1}(H)$. This shows that $f^{-1}(H)$ is a union of $\mathcal{E}$-components of a member of $\mathcal{E}$. Hence, $f^{-1}(H) \in \mathcal{E}'$, or $H \in \mathfrak{U}$. This proves the lemma.

2.3 THEOREM. *Let $\mathfrak{I}$ be a topology for the set S. Among the locally connected topologies for S which are larger than $\mathfrak{I}$ there is a least, $\mathfrak{I}^*$. Suppose furthermore, that f maps a set A into S, and that $\mathcal{E}$ is a locally connected topology for A such that f is $\mathcal{E}$-$\mathfrak{I}$-continuous. Then f is $\mathcal{E}$-$\mathfrak{I}^*$-continuous.*

Proof. Consider the map $\mathfrak{I} \to \mathfrak{I}'$ in the lattice of all topologies for S. We have already noted that this map is increasing; that it is order-preserving follows from Lemma 2.2 applied to the identity map of S. Since the fixed points of the map $\mathfrak{I} \to \mathfrak{I}'$ are precisely the locally connected topologies for S, the first statement of the theorem follows immediately from Theorem 1.1.

Now consider the largest topology $\mathfrak{U}$ for S such that the map f is $\mathcal{E}$-$\mathfrak{U}$-continuous (recall Lemma 2.1). Then f is also $\mathcal{E}'$-$\mathfrak{U}'$-continuous, by Lemma 2.2. Since $\mathcal{E}$ is a locally connected topology for A, this means f is $\mathcal{E}$-$\mathfrak{U}'$-continuous. But this implies that $\mathfrak{U}' \subseteq \mathfrak{U}$, whence $\mathfrak{U}' = \mathfrak{U}$. Now $\mathfrak{I} \subseteq \mathfrak{U}$ because f is $\mathcal{E}$-$\mathfrak{I}$-continuous, and therefore, from the definition of $\mathfrak{I}^*$, $\mathfrak{I}^* \subseteq \mathfrak{U}$; this implies that f is also $\mathcal{E}$-$\mathfrak{I}^*$-continuous.

Translated into the usual language of topology, this theorem becomes Theorem A stated in the introduction except for the uniqueness statement. The uniqueness argument is the standard one for structures having a "universal" property. Suppose $\langle \mathbf{T}, \psi \rangle$ is another pair having the properties of $\langle \mathbf{S}^*, \varphi \rangle$. Applying the factorization properties to φ and ψ we have $\varphi = \psi \circ \varphi^*$ and $\psi = \varphi \circ \psi^*$ where φ^* maps $\mathbf{S}^*$ continuously into $\mathbf{T}$, and ψ^* maps $\mathbf{T}$ continuously into $\mathbf{S}^*$. Then φ^* and ψ^* are inverse maps of one another, so that $\mathbf{S}^*$ and $\mathbf{T}$ are homeomorphic.

2.4 COROLLARY. *Under the hypothesis of Lemma 2.2, f is $\mathcal{E}^*$-$\mathfrak{I}^*$-continuous.*

Proof. Since $\mathcal{E}^* \supseteq \mathcal{E}$, it is clear that f is $\mathcal{E}^*$-$\mathfrak{I}$-continuous. Now the theorem applies to show that it is also $\mathcal{E}^*$-$\mathfrak{I}^*$-continuous.

To prove the uniqueness of the adjoint functor assuming it commutes with the stripping functor we need the following proposition.

2.5 PROPOSITION. *The only order-preserving retraction of the lattice of all topologies for S onto the set of locally connected topologies is given by $\mathfrak{I} \to \mathfrak{I}^*$.*

Proof. For any subset X of S, $\{\emptyset, X, S\}$ is a locally connected topology; hence every topology of S is a join of locally connected topologies. Hence the proposition follows from 1.2.

In the next section we shall need the following result.

2.6 LEMMA. *Every member of $\mathfrak{I}^*$ is a union of $\mathfrak{I}^*$-components of members of $\mathfrak{I}$. In other words, the $\mathfrak{I}^*$-components of members of $\mathfrak{I}$ are a basis of $\mathfrak{I}^*$.*

Proof. Let $\mathfrak{I}^\circ$ be the set of all sets which can be expressed as the union of $\mathfrak{I}^*$-components of members of $\mathfrak{I}$. We shall show first that $\mathfrak{I}^\circ$ is a topology for S. Since $S \in \mathfrak{I}^\circ$, and $\mathfrak{I}^\circ$ is closed under arbitrary unions, the only point at issue here is whether it is closed under finite intersections. Suppose that H_1 and H_2 are $\mathfrak{I}^*$-components of G_1 and G_2 respectively, where G_1 and $G_2 \in \mathfrak{I}$. Let J be any $\mathfrak{I}^*$-component of $G_1 \cap G_2$ which touches $H_1 \cap H_2$. As a $\mathfrak{I}^*$-connected subset of G_1 which touches H_1, $J \subseteq H_1$. Similarly, $J \subseteq H_2$, so $J \subseteq H_1 \cap H_2$. This proves that $H_1 \cap H_2$ is the union of those components of $G_1 \cap G_2$ which it touches; hence $H_1 \cap H_2 \in \mathfrak{I}^\circ$. The intersection of two members of $\mathfrak{I}^\circ$ has the form $(\bigcup_\alpha H_{1,\alpha}) \cap (\bigcup_\beta H_{2,\beta}) = \bigcup_{\alpha,\beta}(H_{1,\alpha} \cap H_{2,\beta})$ and is therefore a member of $\mathfrak{I}^\circ$.

Since $\mathfrak{I} \subseteq \mathfrak{I}^*$ and $\mathfrak{I}^*$ is locally connected, every $\mathfrak{I}^*$-component of a member of $\mathfrak{I}$ is in $\mathfrak{I}^*$; therefore, $\mathfrak{I}^\circ \subseteq \mathfrak{I}^*$. On the other hand it is obvious that $\mathfrak{I} \subseteq \mathfrak{I}^\circ$.

Now I claim that $\mathfrak{I}^\circ$ is a locally connected topology, for every point of S has arbitrarily small $\mathfrak{I}^\circ$-neighborhoods which are $\mathfrak{I}^*$-connected. But since $\mathfrak{I}^* \supseteq \mathfrak{I}^\circ$, this implies that they are also $\mathfrak{I}^\circ$-connected. But this is the usual criterion for local connectivity.

Now because $\mathfrak{I}^*$ is the least locally connected topology larger than $\mathfrak{I}$, it follows that $\mathfrak{I}^* \subseteq \mathfrak{I}^\circ$. This proves that $\mathfrak{I}^* = \mathfrak{I}^\circ$; the lemma is proved.

3. Uniform spaces

The proof of Theorem E could be given by a procedure strictly analogous to that of Theorem A, by using the lattice of all uniform structures for S. However, it seems to be somewhat easier to use the facts proved in Section 2.

We begin by recalling the definitions to explain our notation. A uniform structure for S is a collection $\mathfrak{U}$ of subsets of $S \times S$ satisfying the conditions

(a) For all $\alpha \in \mathfrak{U}$, and all $p \in S$, $\langle p, p\rangle \in \alpha$.
(b) If $\alpha \in \mathfrak{U}$, there exists $\beta \in \mathfrak{U}$ such that $\beta^{-1} \subseteq \alpha$.
(c) If $\alpha \in \mathfrak{U}$, there exists $\beta \in \mathfrak{U}$ such that $\beta\beta \subseteq \alpha$.
(d) If $\alpha \in \mathfrak{U}$ and $\beta \in \mathfrak{U}$, there exists $\gamma \in \mathfrak{U}$ such that $\gamma \subseteq \alpha \cap \beta$.
(e) If $\alpha \in \mathfrak{U}$ and $\alpha \subseteq X \subseteq S \times S$, then $X \in \mathfrak{U}$.

The formal multiplication in (c) and the inverse in (b) refer to the operations of relation algebra:

$$\alpha\beta = \{\langle p, q\rangle \mid (\exists r)\ \langle p, r\rangle \in \alpha \text{ and } \langle r, q\rangle \in \beta\};$$

$$\beta^{-1} = \{\langle p, q\rangle \mid \langle q, p\rangle \in \beta\}.$$

We shall also write $\alpha(p)$ for $\{q \mid \langle q, p\rangle \in \alpha\}$.

If we drop the saturation requirement (e), the set $\mathfrak{U}$ is called a uniform basis for S. A uniform basis becomes a uniform structure if we adjoin all sets X satisfying the hypothesis of (e).

We shall refer to a set X as an α-set if and only if $X \times X \subseteq \alpha$.

Associated with a uniform structure $\mathfrak{U}$ for S is a topology $\mathfrak{T}$ defined by

$$\mathfrak{T} = \{G \subseteq S \mid (\forall p \in G)(\exists \alpha \in \mathfrak{U})\ \alpha(p) \subseteq G\}.$$

Let f be a mapping of a set A into the set S. Let $\mathfrak{F}$ be a uniform structure for A, and let $\mathfrak{E}$ be the corresponding topology. Let $\mathfrak{U}$ be a uniform structure for S with corresponding topology $\mathfrak{T}$. Then f is $\mathfrak{F}$-$\mathfrak{U}$-uniformly continuous if and only if, for all $\alpha \in \mathfrak{U}$, there exists $\theta \in \mathfrak{F}$ such that, if $\langle a, b\rangle \in \theta$, then $\langle f(a), f(b)\rangle \in \alpha$. Such a map is $\mathfrak{E}$-$\mathfrak{T}$-continuous, but, of course, the converse need not hold.

We shall say that $\mathfrak{U}$ is uniformly locally connected if and only if, for all $\alpha \in \mathfrak{U}$, there exists $\beta \in \mathfrak{U}$ such that, for all $p \in S$, $\beta(p)$ is in a $\mathfrak{T}$-connected subset of $\alpha(p)$. When this is true, the corresponding topology $\mathfrak{T}$ is locally connected.

3.1 LEMMA. *Let S be a set, $\mathfrak{U}$ a uniform structure for S, and $\mathfrak{T}$ the corresponding topology. For each $\alpha \in \mathfrak{U}$ let α^* be the set of all ordered pairs $\langle p, q\rangle \in S \times S$ such that p and q are in a $\mathfrak{T}^*$-connected α-set. The set of all such pairs is a basis for a uniform structure $\mathfrak{U}^*$ of S. This uniform structure is uniformly locally connected, $\mathfrak{U} \subseteq \mathfrak{U}^*$, and the topology associated with $\mathfrak{U}^*$ is $\mathfrak{T}^*$.*

Proof. We shall show first that the set of all α^* is a uniform basis for S. Requirement (a) is obvious, while (b) is valid because every α^* is symmetric. Suppose $\alpha \in \mathfrak{U}$. Choose $\beta \in \mathfrak{U}$ so that $\beta\beta \subseteq \alpha$. Then $\beta^*\beta^* \subseteq \alpha^*$; for if p and q are in a $\mathfrak{T}^*$-connected β-set X, and q and r are in a $\mathfrak{T}^*$-connected β-set Y, then p and r are in the $\mathfrak{T}^*$-connected α-set $X \cup Y$. This proves (c). Finally, if $\gamma \subseteq \alpha \cap \beta$, then $\gamma^* \subseteq \alpha^* \cap \beta^*$, so (d) holds.

Let $\mathfrak{U}^*$ be the corresponding uniform structure for S. If $\alpha \in \mathfrak{U}$, then $\alpha^* \subseteq \alpha$, so $\alpha \in \mathfrak{U}^*$. Thus $\mathfrak{U} \subseteq \mathfrak{U}^*$.

Let $\mathfrak{T}^\circ$ be the topology for S defined by $\mathfrak{U}^*$. Suppose $G \in \mathfrak{T}^\circ$. If $p \in G$, we can pick $\alpha \in \mathfrak{U}$ so that $\alpha^*(p) \subseteq G$. Now pick $\beta \in \mathfrak{U}$ so that $\beta\beta^{-1} \subseteq \alpha$. Then $p \in \mathfrak{T}\text{-Int}\,\beta(p) \in \mathfrak{T} \subseteq \mathfrak{T}^*$; let H be the $\mathfrak{T}^*$-component of $\mathfrak{T}\text{-Int}\,\beta(p)$ which contains p. Then H is a $\mathfrak{T}^*$-connected α-set, so $H \subseteq \alpha^*(p) \subseteq G$. On the other hand, $H \in \mathfrak{T}^*$ because $\mathfrak{T}^*$ is locally connected. Thus G is a $\mathfrak{T}^*$-neighborhood of any of its points, so $G \in \mathfrak{T}^*$. This proves that $\mathfrak{T}^\circ \subseteq \mathfrak{T}^*$.

For any $\alpha \in \mathfrak{U}$, the set $\alpha^*(p)$ is the union of all $\mathfrak{T}^*$-connected α-sets which contain p, and this set is $\mathfrak{T}^*$-connected. Since the identity is $\mathfrak{T}^*$-$\mathfrak{T}^\circ$-continuous, $\alpha^*(p)$ is $\mathfrak{T}^\circ$-connected. This shows that $\mathfrak{U}^*$ is uniformly locally connected, and, in particular, $\mathfrak{T}^\circ$ is locally connected. The inclusion $\mathfrak{T} \subseteq \mathfrak{T}^\circ$ is trivial, so we conclude that $\mathfrak{T}^* \subseteq \mathfrak{T}^\circ$ because $\mathfrak{T}^*$ is the least locally connected topology larger than $\mathfrak{T}$. Hence $\mathfrak{T}^\circ = \mathfrak{T}^*$; this finishes the proof of the lemma.

3.2 LEMMA. *Let S, $\mathfrak{U}$, $\mathfrak{T}$, $\mathfrak{U}^*$, $\mathfrak{T}^*$ be as in the previous lemma. Let A be any set, and let $\mathfrak{F}$ be a uniform structure for A which defines the topology $\mathfrak{E}$. Assume* *that $\mathfrak{F}$ is uniformly locally connected. Suppose that f is an $\mathfrak{F}$-$\mathfrak{U}$-uniformly continuous map of A into S. Then f is $\mathfrak{F}$-$\mathfrak{U}^*$-uniformly continuous.*

Proof. Let $\xi \in \mathfrak{U}^*$. Pick $\alpha \in \mathfrak{U}$ so that $\alpha^* \subseteq \xi$. Select $\theta \in \mathfrak{F}$ so that $\langle a, b\rangle \in \theta$ implies $\langle f(a), f(b)\rangle \in \alpha$. Choose $\theta_1 \in \mathfrak{F}$ so that $\theta_1 \theta_1^{-1} \subseteq \theta$, and finally choose $\theta_2 \in \mathfrak{F}$ so that, for any $a \in A$, $\theta_2(a)$ is contained in an $\mathcal{E}$-connected subset of $\theta_1(a)$.

Now, if $\langle a, b\rangle \in \theta_2$, a and b are in an $\mathcal{E}$-connected subset X of $\theta_1(b)$. Here X is a θ-set. We know that f is $\mathcal{E}$-$\mathfrak{J}$-continuous; by applying Theorem 2.3 it is $\mathcal{E}$-$\mathfrak{J}^*$-continuous. Therefore $f(X)$ is a $\mathfrak{J}^*$-connected α-set in S. This implies that $\langle f(a), f(b)\rangle \in \alpha^* \subseteq \xi$. This proves that f is $\mathfrak{F}$-$\mathfrak{U}^*$-uniformly continuous.

Except for the uniqueness statement, which follows from the factorization property as usual, Theorem E is a translation into customary terms of the content of Lemmas 3.2 and 3.3.

We reprove a theorem that is standard in the metric case at least.

3.3 THEOREM. *Let X be a uniformly locally connected subset of a uniform space $\mathcal{A}$. Then $\mathrm{Cl}(X)$ is uniformly locally connected.*

Proof. Let $\mathfrak{D}$ be the uniform structure of $\mathcal{A}$, and let $\alpha \in \mathfrak{D}$. We must prove that there is a $\beta \in \mathfrak{D}$ such that, for all $p \in \mathrm{Cl}(X)$, $\beta(p) \cap \mathrm{Cl}(X)$ is contained in a connected subset of $\alpha(p) \cap \mathrm{Cl}(X)$.

Choose first $\gamma \in \mathfrak{D}$ so that $\gamma\gamma\gamma \subseteq \alpha$; then, for all $p \in \mathcal{A}$, $\mathrm{Cl}(\gamma\gamma(p)) \subseteq \alpha(p)$. Next choose a symmetric δ in $\mathfrak{D}$ so that $\delta \subseteq \gamma$ and, for all $q \in X$, $\delta\delta(q) \cap X$ is contained in a connected subset of $\gamma(q) \cap X$. Finally choose β so that $\beta\beta \subseteq \delta$.

Now suppose $p \in \mathrm{Cl}(X)$. Pick q in $\delta(p) \cap X$; then

$$\delta(p) \cap X \subseteq \delta\delta(q) \cap X \subseteq K \subseteq \gamma(q) \cap X \subseteq \gamma\gamma(p) \cap X$$

where K is a connected set. Hence

$$\begin{aligned}\beta(p) \cap \mathrm{Cl}(X) &\subseteq \mathrm{Int}\,\delta(p) \cap \mathrm{Cl}(X) \subseteq \mathrm{Cl}(\mathrm{Int}\,\delta(p) \cap X) \subseteq \mathrm{Cl}(K)\\ &\subseteq \mathrm{Cl}(\gamma\gamma(p) \cap X) \subseteq \mathrm{Cl}(\gamma\gamma(p)) \cap \mathrm{Cl}(X) \subseteq \alpha(p) \cap \mathrm{Cl}(X).\end{aligned}$$

Since $\mathrm{Cl}(K)$ is connected, the theorem is proved.

3.4 THEOREM. *If $\mathcal{S}$ is a complete uniform space, then the space $\mathcal{S}^*$ of Theorem E is also complete.*

Proof. There are a complete uniform space $\mathcal{A}$ and a uniform isomorphism π of $\mathcal{S}^*$ onto a dense subspace $\mathcal{S}'$ of $\mathcal{A}$. Now the map $\varphi \circ \pi^{-1}$ is a uniformly continuous map of $\mathcal{S}'$ into $\mathcal{S}$; since the latter is complete, it can be extended to a continuous map f of $\mathcal{A}$ into $\mathcal{S}$.

As the closure of the uniformly locally connected set $\mathcal{S}'$, $\mathcal{A}$ is locally connected. Hence f can be factored: $f = \varphi \circ f^*$ where f^* maps $\mathcal{A}$ continuously into $\mathcal{S}^*$. If $x \in \mathcal{S}'$, then $\varphi(f^*(x)) = f(x) = \varphi(\pi^{-1}(x))$; since φ is one-to-one, $f^*(x) = \pi^{-1}(x)$ and $\pi(f^*(x)) = x$. Thus $\pi \circ f^*$ agrees with the identity map on a dense subset of $\mathfrak{A}$. By continuity, $\pi \circ f^*$ is the identity map of $\mathfrak{A}$. This shows that $\pi(\mathfrak{S}^*) = \mathfrak{A}$. Hence $\mathfrak{S}^*$ is complete.

If the uniform structure of $\mathfrak{S}$ can be defined by a metric, then the structure has a countable basis. It follows from the proof of Lemma 3.2 that the uniform structure of $\mathfrak{S}^*$ has a countable basis and can, therefore, be derived from a metric. However, we can do much better.

Suppose ρ is the metric which defines the uniform structure $\mathfrak{U}$ of $\mathfrak{S}$. In the notation of Lemma 3.2, put

$$\sigma(p, q) = \inf \{t \mid p \text{ and } q \text{ are in a } \mathfrak{I}^*\text{-connected set of } \rho \text{ diameter less than } t\}.$$

(This takes the value ∞ if the set on the right is void, but this is no disadvantage if we make the obvious interpretations of the axioms for a metric space. If a proper metric is required, replace σ by $\sigma' = \sigma/(1 + \sigma)$, where $\infty/(1 + \infty) = 1$.) It can be checked immediately that σ is a metric which defines the uniform structure $\mathfrak{U}^*$. It has the convenient additional property that open σ-balls are all connected in the topology $\mathfrak{I}^*$ defied by σ; moreover the $\mathfrak{I}^*$-component of S containing p is just $\{q \mid \sigma(p, q) < \infty\}$. By Theorem 3.4, if S is complete with respect to ρ it is complete with respect to σ. The referee pointed out that this definition appears in [4, p. 154 ff.] where it is used to introduce uniform local connectivity without changing the topology of a locally connected metric space.

4. Additional properties of the space S*

We discuss first the separation axioms.

4.1 Theorem. *Suppose that the topology $\mathfrak{I}$ for the set S satisfies axioms T_0, T_1, T_2, or T_3. Then $\mathfrak{I}^*$ does also.*

Proof. This is obvious in the case of axioms T_0, T_1, or T_2 because, in passing from $\mathfrak{I}$ to $\mathfrak{I}^*$, no open sets are lost. However, this argument is insufficient for axiom T_3 because some closed sets are gained.

Assume that $\mathfrak{I}$ satisfies axiom T_3. We must prove that if $p \in G \in \mathfrak{I}^*$, there is a set H such that $p \in H \in \mathfrak{I}^*$ and $\mathfrak{I}^*\text{-Cl}(H) \subseteq G$. By Lemma 2.6 we can choose $J \in \mathfrak{I}$ so that K, the $\mathfrak{I}^*$-component of J containing p, satisfies $K \subseteq G$. Since $\mathfrak{I}$ is T_3, we can choose L so that $p \in L \in \mathfrak{I}$ and $\mathfrak{I}\text{-Cl}(L) \subseteq J$. Let H be the $\mathfrak{I}^*$-component of L which contains p. Since $L \in \mathfrak{I}^*$ and the latter is locally connected, $H \in \mathfrak{I}^*$. Now $\mathfrak{I}\text{-Cl}(H)$ is $\mathfrak{I}^*$-closed, so

$$\mathfrak{I}^*\text{-Cl}(H) \subseteq \mathfrak{I}\text{-Cl}(H) \subseteq \mathfrak{I}\text{-Cl}(L) \subseteq J;$$

but $\mathfrak{I}^*\text{-Cl}(H)$ is $\mathfrak{I}^*$-connected and meets K; hence $\mathfrak{I}^*\text{-Cl}(H) \subseteq K$. This gives $\mathfrak{I}^*\text{-Cl}(H) \subseteq G$ and completes the proof.

4.2 Theorem. *If $\mathfrak{I}$ satisfies the first axiom of countability, then so does $\mathfrak{I}^*$.*

Proof. Let p be a point of S, and let $G_1, G_2, \cdots$ be a fundamental sequence of $\mathfrak{J}$-open neighborhoods of p. If H_i is the $\mathfrak{J}^*$-component of G_i which contains p, then it follows immediately from Lemma 2.6 that $H_1, H_2, \cdots$ is a fundamental sequence of $\mathfrak{J}^*$-neighborhoods.

4.3. We omit the proof of Theorem C since it is entirely straightforward by using the universal property. However, we remark that the corresponding theorem is false for infinite direct products. With substantially abbreviated notation we find

$$(\prod \mathsf{S}_\alpha)^* = (\prod \mathsf{S}_\alpha^*)^*.$$

Moreover, if $\{\mathsf{S}_\alpha\}$ is a collection of locally connected spaces, then the components of $(\prod \mathsf{S}_\alpha)^*$ have the form $\prod F_\alpha$ where, for each α, F_α is a component of S_α.

We omit even the statement of the corresponding facts for direct products of uniform spaces.

4.4 Theorem. *Let G be an algebraic group. Let $\mathfrak{J}$ be a topology for G which is compatible with the group structure. Then $\mathfrak{J}^*$ is also compatible with the group structure.*

Proof. Let us denote by $\mathfrak{J} \times \mathfrak{J}$ the topology for $G \times G$ induced by the topology $\mathfrak{J}$. Let f be the map of $G \times G$ into G given by $f(g_1, g_2) = g_1 g_2^{-1}$. We are given that f is $(\mathfrak{J} \times \mathfrak{J})$-$\mathfrak{J}$-continuous. It is, therefore, $(\mathfrak{J}^* \times \mathfrak{J}^*)$-$\mathfrak{J}$-continuous. Since $\mathfrak{J}^* \times \mathfrak{J}^*$ is a locally connected topology (the direct product of two locally connected spaces is locally connected), f is also $(\mathfrak{J}^* \times \mathfrak{J}^*)$-$\mathfrak{J}^*$-continuous, as claimed.

In classical language this becomes Theorem D.

4.5 *Examples.* If G is the character group of the additive group of rational numbers with discrete topology, and $\mathfrak{J}$ is the usual topology of G, then G is a compact, connected, metrizable group. However, with topology $\mathfrak{J}^*$, G consists of uncountably many components each of which is homeomorphic to the real numbers R. Thus we see that in the passage from $\mathfrak{J}$ to $\mathfrak{J}^*$ separability, the second axiom of countability, and connectivity have all been lost.

Now let S be the direct product of uncountably many copies of G, and let $\mathfrak{J}_1$ be the Tychonoff topology for S induced by $\mathfrak{J}$. Now in the topology $\mathfrak{J}_1^*$, the components of S are homeomorphic to uncountable direct products of copies of R. Thus S, which was compact, a fortiori locally compact, paracompact, and T_4, with the topology $\mathfrak{J}_1$, has none of these properties with topology $\mathfrak{J}_1^*$.

5. A generalization

Let $\mathfrak{C}$ be any subcategory of the category $\mathfrak{T}$ of topological spaces and continuous maps. Suppose we have another class $\mathcal{S}$ of topological spaces and a special collection Ξ of continuous maps from these spaces to the various spaces in $\mathfrak{C}$. It is unimportant whether $\mathcal{S}$ and Ξ are in the category $\mathfrak{C}$ or not, but we do require that $f \circ \xi$ be in Ξ whenever $\xi \in \Xi$ and f is a morphism of $\mathfrak{C}$.

Let each of the spaces in $\mathfrak{C}$ be replaced by the same set endowed with the largest topology which makes all of the maps of Ξ continuous. Each morphism of $\mathfrak{C}$ will remain a continuous function when the topologies of its domain and range are thus altered. Hence the change of topology of the spaces and the retention of the morphisms as set maps constitute a covariant functor from $\mathfrak{C}$ to another subcategory $\mathfrak{C}'$ of $\mathfrak{T}$. The classes $\mathcal{S}$ and Ξ will retain the necessary relations to $\mathfrak{C}'$, so they define a new functor on $\mathfrak{C}'$ which turns out to be the identity. Thus our functor is, in a sense, idempotent.

A number of examples can be given. In each of the following we shall take $\mathfrak{C}$ to be the whole of $\mathfrak{T}$, and Ξ to consist of all possible continuous maps from the members of $\mathcal{S}$ to topological spaces.

When we take $\mathcal{S}$ to be the class of all locally connected topological spaces, we get the functor Φ described above. When we take $\mathcal{S}$ to consist of just one space, the unit interval, we get the locally-arcwise connection functor described in [2]; the effect on each space is the introduction of Young's a-topology. When we take $\mathcal{S}$ to be the class of all compact spaces, the resulting functor may appropriately be called the k-functor; some of its properties are described in [3, p. 241, Exercise K]. If $\mathcal{S}$ contains just one space, and this space consists of a single convergent sequence and its limit point, then the functor replaces each space by one in which the topology is sequential. Perhaps surprisingly, the result is no different if $\mathcal{S}$ contains all metric spaces.

The ideas of this construction apply to certain other categories such as the category $\mathfrak{U}$ of uniform structures. There is also a dual construction. It may prove valuable to study these various functors carefully to obtain more detailed information concerning their properties.

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