SEMISIMPLICIAL SPECTRA

BY

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1. Introduction

The notion of a spectrum, introduced by Lima [12], has proved useful in homotopy theory. Spanier [17] used it for the study of stable homotopy theory, while E. H. Brown, Jr. [1] and G. W. Whitehead [18] have shown that there is a very close relationship between spectra and (co-) homology theories which satisfy all Eilenberg-Steenrod axioms but the dimension axiom.

Our present purpose is to define spectra in the semisimplicial context. Although the notion of a (topological) spectrum is a rather complex one (a sequence of spaces and maps), it turns out that a semisimplicial spectrum consists of only one object which very much looks like a semisimplicial complex with base point. The main differences are (i) that simplices are also allowed to have negative dimensions, and (ii) that every simplex y has an infinite number of faces $d_0 y$, $d_1 y$, $\cdots$ (but only a finite number of them are not "at the base point") and an infinite number of degeneracies $s_0 y$, $s_1 y$, $\cdots$.

Some applications will be given in [10] and [11].

There are two chapters. Chapter I deals with semisimplicial spectra and their relation to topological spectra. We also consider group spectra and show that the category of abelian group spectra is isomorphic with the category of abelian chain complexes.

In Chapter II a homotopy relation is introduced in the category of semisimplicial spectra. As for semisimplicial complexes this relation is, in general, not an equivalence relation. However on a suitable subcategory (that of spectra which "satisfy the extension condition") the homotopy relation is an equivalence relation. Consequently one has in this category the notions of homotopy equivalence and homotopy type. We end with considering minimal spectra and homotopy groups.

1.1 *Notation and terminology.* We shall freely use the results of [7] and [8] with the following changes in notation and terminology:

(i) the face and degeneracy operators will be denoted by d_i and s_i and will be written on the *left*;

(ii) c.s.s. complexes will be called *set complexes*, or for short, *complexes*; c.s.s. groups will be called *group complexes*.

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Chapter I. Semisimplicial Spectra

2. Suspension

We start with some well known facts on the topological suspension, the semisimplicial suspension, and their relationship.

2.1 Definition. Let $\mathcal{W}_*$ be the category of topological spaces with base point which have the homotopy type of a CW-complex with base point. The base point will be denoted by $*$. Then the *suspension* functor $S : \mathcal{W}_* \to \mathcal{W}_*$ is the functor which assigns to a space $Y \in \mathcal{W}_*$ the space with base point SY obtained from $Y \times I$ (I = unit interval) by shrinking the subset $Y \times 0 \cup Y \times 1 \cup * \times I$ to a point, the base point, and which maps maps accordingly.

2.2 Definition. Let $\mathcal{S}_*$ be the category of set complexes (1.1) with base point. A base point or any of its degeneracies will be denoted by $*$. Let P be a set complex which has exactly one n-simplex, ϕ_n, for every integer $n \geqq 0$. Then for $L \in \mathcal{S}_*$ its *suspension* is the set complex with base point of which the n-simplices are the appropriate degeneracy of the base point and all pairs (σ, ϕ) such that $\sigma \in L$, $\sigma \neq *$, $\phi \in P$, and $\dim \sigma + \dim \phi = n - 1$; the face and degeneracy operators are given by

$$\begin{aligned} d_i(\sigma, \phi) &= (d_i\, \sigma, \phi), & s_i(\sigma, \phi) &= (s_i\, \sigma, \phi), & 0 &\leqq i \leqq p, \\ &= (\sigma, d_{i-p-1}\, \phi), & &= (\sigma, s_{i-p-1}\, \phi), & p &< i \leqq n \end{aligned}$$

(where $p = \dim \sigma$) whenever this has a meaning, and $d_i(\sigma, \phi) = *$ otherwise. Similarly, the suspension of a map $\lambda : L \to L^1 \in \mathcal{S}_*$ is the map $S\lambda : SL \to SL^1$ given by $(\sigma, \phi) \to (\lambda\sigma, \phi)$ whenever this has a meaning, and $(\sigma, \phi) \to *$ otherwise. The function S so defined is a functor $S : \mathcal{S}_* \to \mathcal{S}_*$, the *suspension* functor.

The topological and the semisimplicial suspension functors are closely related.

Let Δ^n be the Euclidean n-simplex with vertices $A_0, \cdots, A_n$, and for $Y \in \mathcal{W}_*$ and a map $w : \Delta^n \to Y$ denote by $\Sigma w : \Delta^{n+1} \to SY$ the map given by

$$\Sigma w(\alpha_0 A_0 + \cdots + \alpha_{n+1} A_{n+1}) = (w(\beta_0 A_0 + \cdots + \beta_n A_n), \alpha_{n+1})$$

where $\beta_i = \alpha_i/(\alpha_0 + \cdots + \alpha_n)$ for $0 \leqq i \leqq n$. Let $R : \mathcal{S}_* \to \mathcal{W}_*$ be the *geometric realization* functor [13], and for $L \in \mathcal{S}_*$ and every nondegenerate n-simplex $\sigma \in L$ let $c(\sigma) : \Delta^n \to RL$ be the corresponding characteristic map. Then a rather lengthy but completely straightforward computation yields

2.3 Proposition. *Let $L \in \mathcal{S}_*$. Then there is a unique map*

$$j : RSL \to SRL \in \mathcal{W}_*$$

such that $jc(\sigma, \phi_0) = \Sigma c(\sigma)$ for every nondegenerate $\sigma \in L$. Moreover this map is natural and is a homeomorphism.

A similar result holds for the singular functor $\mathrm{Sin} : \mathcal{W}_* \to \mathcal{S}_*$ [6].

2.4 PROPOSITION. *Let $Y \in \mathcal{W}_*$. Then there is a unique map*

$$j : S \,\mathrm{Sin}\, Y \to \mathrm{Sin}\, SY \in \mathcal{S}_*$$

such that $j(\sigma, \phi_0) = \Sigma\sigma$ for all $\sigma \in \mathrm{Sin}\, Y$. Moreover this map is natural and is a weak homotopy equivalence, i.e., its geometric realization is a homotopy equivalence.

This is proved by observing that the map $j : S \,\mathrm{Sin}\, Y \to \mathrm{Sin}\, SY$ is the composition

$$S \,\mathrm{Sin}\, Y \xrightarrow{j'} \mathrm{Sin}\, SR \,\mathrm{Sin}\, Y \xrightarrow{\mathrm{Sin}\, Sh} \mathrm{Sin}\, SY$$

where j' is adjoint to the map of 2.3 ($L = \mathrm{Sin}\, Y$) and $h : R \,\mathrm{Sin}\, Y \to Y$ is adjoint to the identity map of $\mathrm{Sin}\, Y$ [9]. The proposition then follows readily from the main result of [13].

3. Topological spectra and their semisimplification

In this section we consider topological spectra and their semisimplicial analogues (called prespectra) and prove that, in some precise sense, the homotopy theories of topological spectra and of prespectra are equivalent.

3.1 DEFINITION.[2] Call a map $w : Y \to Z \in \mathcal{W}_*$ a *proper inclusion* if (i) w maps Y homeomorphically onto its image, and (ii) the pair (Z, image w) has the homotopy type of a pair of CW-complexes. A *topological spectrum* Y then consists of

(i) a sequence of spaces $Y_i \in \mathcal{W}_*$, $i = 0, 1, \cdots$,
(ii) a sequence of proper inclusions $a_i : SY_i \to Y_{i+1}$, $i = 0, 1, \cdots$.

Notation: $Y = \{Y_i, a_i\}$ or $Y = \{Y_i\}$. For two spectra $Y = \{Y_i, a_i\}$ and $Z = \{Z_i, b_i\}$ a *map* $w : Y \to Z$ will be a sequence of maps

$$w_i : Y_i \to Z_i \in \mathcal{W}_* \qquad (i = 0, 1, \cdots)$$

such that $w_{i+1} a_i = b_i(Sw_i)$ for all i. Notation: $w = \{w_i\}$. The category of topological spectra and their maps will be denoted by $\mathcal{W}s$.

3.2 DEFINITION. A *prespectrum* L consists of

(i) a sequence of set complexes $L_i \in \mathcal{S}_*$, $i = 0, 1, \cdots$,
(ii) a sequence of maps $\lambda_i : SL_i \to L_{i+1} \in \mathcal{S}_*$ which are 1-1 (into), $i = 0, 1, \cdots$.

Notation: $L = \{L_i, \lambda_i\}$ or $L = \{L_i\}$. For two prespectra $L = \{L_i, \lambda_i\}$ and $M = \{M_i, \mu_i\}$ a *map* $\psi : L \to M$ is a sequence of maps

$$\psi_i : L_i \to M_i \in \mathcal{S}_* \qquad (i = 0, 1, \cdots)$$

[2] This definition is more restrictive than the one used by G. W. Whitehead [18]. However, his results can be readily adapted to this definition by iterated use of mapping cylinders and the homotopy extension theorem.

such that $\psi_{i+1}\lambda_i = \mu_i(S\psi_i)$ for all i. Notation: $\psi = \{\psi_i\}$. The category of prespectra and their maps will be denoted by $\mathcal{P}s$.

3.3 DEFINITION. A map $w : \{Y_i\} \to \{Z_i\} \in \mathcal{W}s$ is called a *weak homotopy equivalence* if for every integer q (also negative) the induced homomorphism [8]

$$\pi_q(w) : \lim_{i\to\infty} \pi_{i+q}(Y_i) \to \lim_{i\to\infty} \pi_{i+q}(Z_i)$$

is an isomorphism. Similarly a map $\psi : \{L_i\} \to \{M_i\} \in \mathcal{P}s$ is called a *weak homotopy equivalence* if for every integer q the induced homomorphism

$$\pi_q(\psi) : \lim_{i\to\infty} \pi_{i+q}(L_i) \to \lim_{i\to\infty} \pi_{i+q}(M_i)$$

is an isomorphism.

The geometric realization R and the singular functor Sin induce functors $R : \mathcal{P}s \to \mathcal{W}s$ and $\mathrm{Sin} : \mathcal{W}s \to \mathcal{P}s$. They are given by the formulas

$$R\{L_i, \lambda_i\} = \{RL_i, (R\lambda_i)j^{-1}\}, \qquad R\{\psi_i\} = \{R\psi_i\},$$

$$\mathrm{Sin}\,\{Y_i, a_i\} = \{\mathrm{Sin}\,Y_i, (\mathrm{Sin}\,a_i)j\}, \qquad \mathrm{Sin}\,\{w_i\} = \{\mathrm{Sin}\,w_i\},$$

where j is as in 2.3 or as in 2.4. Clearly, we have

3.4 PROPOSITION. *The functors $R : \mathcal{P}s \to \mathcal{W}s$ and* Sin $: \mathcal{W}s \to \mathcal{P}s$ *map weak homotopy equivalences into weak homotopy equivalences.*

For $Y_i \in \mathcal{W}_*$ let $hY_i : R\,\mathrm{Sin}\,Y_i \to Y_i \in \mathcal{W}_*$ be the natural map of [13, Theorem 4]. It is a homotopy equivalence. The function h induces for every $Y = \{Y_i\} \in \mathcal{W}s$ a natural map $hY : R\,\mathrm{Sin}\,Y \to Y \in \mathcal{W}s$ given by $hY = \{hY_i\}$. Similarly for $L_i \in \mathcal{S}_*$ let $hL_i : L_i \to \mathrm{Sin}\,RL_i \in \mathcal{S}_*$ be the natural map of [13, Lemma 5]. It is a weak homotopy equivalence. This function h induces for every $L = \{L_i\} \in \mathcal{P}s$ a natural map $hL = \{hL_i\} : L \to \mathrm{Sin}\,RL \in \mathcal{P}s$. Clearly we have

3.5 PROPOSITION. *Let $Y \in \mathcal{W}s$ and $L \in \mathcal{P}s$. Then the natural maps*

$$hY : R\,\mathrm{Sin}\,Y \to Y \quad \textit{and} \quad hL : L \to \mathrm{Sin}\,RL$$

are weak homotopy equivalences.

3.6 DEFINITION. Let $\mathcal{C}$ be a category in which certain maps are called weak homotopy equivalences. A functor $\mathcal{C} \to \mathcal{D}$ we then call a *homotopy functor* on the category $\mathcal{C}$ with values in the category $\mathcal{D}$ if it maps every weak homotopy equivalence into an equivalence.

We can now make clear what we mean by the *equivalence of* the *homotopy theories* of topological spectra and of prespectra, namely that the following proposition holds.

3.7 PROPOSITION. *The functors R and* Sin *induce a* 1-1 *correspondence between the homotopy functors on $\mathcal{W}s$ and those on $\mathcal{P}s$.*

Proof. This follows at once from Propositions 3.4 and 3.5.

4. Semisimplicial spectra

The prespectra introduced in the previous section do not seem to have any advantage over topological spectra. In fact they are just as complicated, and it is rather cumbersome to apply to them the usual semisimplicial techniques. However,

(i) It is possible to associate with a prespectrum L *one* rather simple mathematical object which strongly resembles a set complex with base point; the main differences are that simplices are also allowed to have negative dimensions, and that every simplex has an infinite number of faces (although only a finite number of them are not "at the base point") and an infinite number of degeneracies.

(ii) Although in general the prespectrum L cannot be recovered from it, this new object contains all the homotopy information of the prespectrum L. In fact the homotopy theory of these new objects turns out to be equivalent to that of prespectra.

In view of this and the fact that the similarity of these new objects to set complexes enables one to apply to them many of the semisimplicial techniques, these objects deserve to be called spectra. Their definition is as follows.

4.1 DEFINITION. A *semisimplicial spectrum* (or *set spectrum*, or for short, *spectrum*) X consists of

(i) for *every* integer q a set $X_{(q)}$ with a distinguished element $*$ (called *base point*); the elements of $X_{(q)}$ will be called *simplices* of *degree* q,

(ii) for every integer q and every integer $i \geqq 0$ a function

$$d_i : X_{(q)} \to X_{(q-1)}$$

such that $d_i * = *$ (the *i-face* operator), and a function

$$s_i : X_{(q)} \to X_{(q+1)}$$

such that $s_i * = *$ (the *i-degeneracy* operator). These operators are required to satisfy the axioms

I. The following identities hold:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{for } i < j, \\ d_i s_j &= s_{j-1} d_i && \text{for } i < j, \\ &= \text{identity} && \text{for } i = j, j+1, \\ &= s_j d_{i-1} && \text{for } i > j+1, \\ s_i s_j &= s_j s_{i-1} && \text{for } i > j. \end{aligned}$$

II. For every simplex $\alpha \in X$ all but a finite number of its faces are the base point, i.e., there is an integer n (depending on α) such that $d_i \alpha = *$ for $i > n$.

A *subspectrum* of a spectrum X is a subset of X which is closed under all face and degeneracy operators.

For two spectra X and Y a *map* $w : X \to Y$ is a degree-preserving function which commutes with all face and degeneracy operators. An *isomorphism* is a map which is 1-1 and onto.

Clearly the set spectra and their maps form a category which will be denoted by $\mathcal{S}p$.

We proceed by relating the categories $\mathcal{P}s$ and $\mathcal{S}p$ by means of two functors $\mathrm{Sp} : \mathcal{P}s \to \mathcal{S}p$ and $\mathrm{Ps} : \mathcal{S}p \to \mathcal{P}s$.

4.2 Definition. Let $L = \{L_i, \lambda_i\} \in \mathcal{P}s$. For $\sigma \in L_i$ denote by $\lambda_i' \sigma$ the simplex $\lambda_i' \sigma = \lambda_i(\sigma, \phi_0) \in L_{i+1}$. Clearly $\dim \lambda_i' \sigma = 1 + \dim \sigma$. Because the λ_i are 1-1 (into), an equivalence relation can be introduced on the set of all simplices of all the L_i by calling two simplices $\sigma \in L_i$, $\tau \in L_{i+n}$ $(n \geqq 0)$ equivalent if $\lambda_{i+n-1}' \cdots \lambda_i' \sigma = \tau$. The resulting quotient set will be denoted by $\mathrm{Sp}\, L$. For $\sigma \in L_i$ we write $[\sigma]$ for its equivalence class. In order to turn $\mathrm{Sp}\, L$ into a spectrum, we define for every integer $j \geqq 0$ and for every $\sigma \in L_i$

$$\begin{aligned} \text{degree } [\sigma] &= \dim \sigma - i, \\ d_j[\sigma] &= [d_j \lambda_{i+j}' \cdots \lambda_i' \sigma], \\ s_j[\sigma] &= [s_j \lambda_{i+j}' \cdots \lambda_i' \sigma]. \end{aligned}$$

The elements $[*]$ will be the base points. A simple computation shows that these definitions are independent of the choice of σ in $[\sigma]$, and that indeed they turn $\mathrm{Sp}\, L$ into a spectrum.

Similarly it is easily seen that for a map $\psi = \{\psi_i\} : \{L_i\} \to \{M_i\} \in \mathcal{P}s$ the function $\mathrm{Sp}\, \psi : \mathrm{Sp}\{L_i\} \to \mathrm{Sp}\{M_i\}$ given by $[\sigma] \to [\psi_i \sigma]$ for all $\sigma \in L_i$ is well defined and is in $\mathcal{S}p$. Clearly the function Sp so defined is a functor $\mathrm{Sp} : \mathcal{P}s \to \mathcal{S}p$.

An immediate consequence of this definition is

4.3 Proposition. *Let $\{L_i\} \in \mathcal{P}s$, and let $\sigma \in L_i$ be an n-simplex. Then* degree $[\sigma] = n - i$, $d_0 \cdots d_n[\sigma] = *$, *and* $d_j[\sigma] = *$ *for* $j > n$.

It is obvious that, in general, it is impossible to recover a prespectrum L from the spectrum $\mathrm{Sp}\, L$. However Proposition 4.3 suggests

4.4 Definition. For $X \in \mathcal{S}p$ denote by $\mathrm{Ps}\, X = \{X_i, \xi_i\}$ the following prespectrum. An n-simplex of X_i is any $\alpha \in X$ such that degree $\alpha = n - i$, $d_0 \cdots d_n \alpha = *$, and $d_j \alpha = *$ for $j > n$; $*$ will be the appropriate degeneracy of the base point; the face and degeneracy operators are those induced by the corresponding operators of X. It is not difficult to verify that X is indeed a well defined set complex, i.e., that for every n-simplex $\alpha \in X_i$, $d_j \alpha$ and $s_j \alpha$ are also simplices of X_i for $0 \leqq j \leqq n$. Clearly $\alpha \in X_i$ implies $\alpha \in X_{i+1}$. The maps $\xi_i : SX_i \to X_{i+1} \in S_*$ are therefore defined as the unique maps such that $\xi_i(\alpha, \phi_0) = \alpha \ (\epsilon\, X_{i+1})$ for every $\alpha \in X_i$. Again it is not difficult to verify that this is well defined.

Similarly a map $w : X \to Y \in \mathcal{S}p$ induces a map $\mathrm{Ps}\, w : \mathrm{Ps}\, X \to \mathrm{Ps}\, Y \in \mathcal{P}s$, and the function Ps so defined clearly is a functor $\mathrm{Ps} : \mathcal{S}p \to \mathcal{P}s$.

4.5 PROPOSITION. *Let $X \in \mathcal{S}p$. Then there is a unique map*

$$j : X \to \mathrm{Sp}\ \mathrm{Ps}\ X \in \mathcal{S}p$$

such that $j\alpha = [\alpha]$ for all $\alpha \in X$. Moreover this map is natural and is an isomorphism.

This is an immediate consequence of the definitions of the functors Sp and Ps.

4.6 PROPOSITION. *Let $\{L_i\} \in \mathcal{P}s$. Then there is a unique map*

$$\{j_i\} : \{L_i\} \to \mathrm{Ps}\ \mathrm{Sp}\{L_i\} \in \mathcal{P}s$$

such that $j_i\, \sigma = [\sigma]$ for all $\sigma \in L_i$. Moreover this map is natural and is a weak homotopy equivalence.

Proof. Existence, uniqueness, and naturality are easily verified. In order to prove that $\{j_i\}$ is a weak homotopy equivalence, it suffices, in view of 3.4 and 3.5, to show that $\{Rj_i\}$ is so. Let $\mathrm{Ps}\ \mathrm{Sp}\ L = \{M_i\}$, let q be an integer, and let $\beta \in \lim_{i\to\infty} \pi_{i+q}(RM_i)$. For suitably large i one can represent β by a map $b : S^{i+q} \to RM_i \in \mathcal{W}_*$ (S^{i+q} = the $(i + q)$-sphere). As S^{i+q} is compact, the image of b is contained in a finite subcomplex of RM_i. Hence there are an integer k and a map $a : S^{i+q+k} \to RL_{i+k}$ such that $(Rj_{i+k})a$ also represents β. If $\alpha \in \lim_{i\to\infty} \pi_{i+q}(RL_i)$ is the element represented by a, then clearly $\pi_q\{Rj_i\}\alpha = \beta$, i.e., $\pi_q\{Rj_i\}$ is onto. The proof that *kernel* $\pi_q\{Rj_i\} = 0$ is similar.

In view of these propositions we can state

4.7 DEFINITION. A map $w : X \to Y \in \mathcal{S}p$ is called a *weak homotopy equivalence* if $\mathrm{Ps}\ w$ is so.

Then clearly we have

4.8 PROPOSITION. *The functors* $\mathrm{Sp} : \mathcal{P}s \to \mathcal{S}p$ *and* $\mathrm{Ps} : \mathcal{S}p \to \mathcal{P}s$ *map weak homotopy equivalences into weak homotopy equivalences. Moreover they induce* (3.6) *an equivalence between the homotopy theories of prespectra and of set spectra.*

5. Group spectra

A very useful class of spectra is formed by the group spectra. Their usefulness lies in the facts that

(i) the homotopy theory of group spectra is equivalent to that of set spectra, and

(ii) one can apply to them semisimplicial as well as group theoretical techniques, a combination which has already been applied so successfully to group complexes (see [2], [3], [6], [14]).

5.1 DEFINITION. A spectrum X is called an (*abelian*) *group spectrum* if $X_{(q)}$ is an (abelian) group for all q and all operators are homomorphisms. For two group spectra X and Y, a map $w : X \to Y \in \mathcal{S}p$ is called a *homomorphism* if the restriction $w \mid X_{(q)} : X_{(q)} \to Y_{(q)}$ is a homomorphism for all q. The category of group spectra and homomorphisms will be denoted by $\mathcal{S}p_G$, and its full subcategory of abelian group spectra by $\mathcal{S}p_A$. A map in $\mathcal{S}p_G$ is called a *weak homotopy equivalence* if it is so when considered as a map in $\mathcal{S}p$.

In order to relate the categories $\mathcal{S}p$ and $\mathcal{S}p_G$ we define a functor $F : \mathcal{S}p \to \mathcal{S}pe$ as follows.

5.2 DEFINITION. For $X \in \mathcal{S}p$ let $(FX)_{(q)}$ be the (free) group with a generator $F\alpha$ for every $\alpha \in X_{(q)}$ and one relation $F* = *$; the face and degeneracy homomorphisms are given by

$$d_i F\alpha = Fd_i \alpha, \qquad s_i F\alpha = Fs_i \alpha \qquad\qquad \text{for all } i \geqq 0.$$

Similarly for a map $w : X \to Y \in \mathcal{S}p$, let $Fw : FX \to FY$ be the homomorphism given by $F\alpha \to Fw\alpha$ for all $\alpha \in X$. Clearly the function F so defined is a functor $F : \mathcal{S}p \to \mathcal{S}p_G$.

5.3 PROPOSITION. *Let $X \in \mathcal{S}p$, and let $fX : X \to FX$ be the map given by $\alpha \to F\alpha$ for all $\alpha \in X$. Then fX is natural and is a weak homotopy equivalence.*

5.4 COROLLARY. *A map $w : X \to Y \in \mathcal{S}p$ is a weak homotopy equivalence if and only if $Fw : FX \to FY$ is so.*

Proof of Proposition 5.3. For $L \in \mathcal{S}_*$ let $E_n L$ denote the n-Eilenberg subcomplex, i.e., the largest subcomplex of L which has no nondegenerate simplices in dimension $< n$ except the base point. Define a commutative diagram

$$\begin{array}{ccc} \{X'_i, \xi'_i\} & \xrightarrow{\quad \{f'_i\} \quad} & \{Y'_i, \eta'_i\} \\ \downarrow{\scriptstyle \{a_i\}} & & \downarrow{\scriptstyle \{b_i\}} \\ \operatorname{Ps} X = \{X_i, \xi_i\} & \xrightarrow{\operatorname{Ps} fX = \{f_i\}} & \operatorname{Ps} FX = \{Y_i, \eta_i\} \end{array}$$

as follows: $X'_i = E_{d(i)} X_i$ where $d(i)$ is the largest integer $\leqq \frac{2}{3}i$, $\xi'_i = \xi_i \mid X'_i$, Y'_i is the subgroup complex (1.1) of Y_i generated by the image of X'_i under f_i, $\eta'_i = \eta_i \mid Y'_i$, $f'_i = f_i \mid X'_i$, and a_i and b_i are the inclusions. Then it is easily verified that Sp $\{a_i\}$ and Sp $\{b_i\}$ are isomorphisms, and hence (4.8) $\{a_i\}$ and $\{b_i\}$ are weak homotopy equivalences. Moreover $Y'_i = FX'_i$ where F denotes the "loops on the suspension" functor of Milnor [14], [7]. The maps $f'_i : X'_i \to Y'_i$ therefore induce isomorphisms of the homotopy groups in the stable range. This readily implies that $\{f_i'\}$ is a weak homotopy equivalence, and hence so is $\{f_i\}$. The naturality of f is obvious.

5.5 PROPOSITION. *Let $X \in \mathcal{S}p_G$, and let $g : FX \to X$ be the homomorphism given by $F\alpha \to \alpha$ for all $\alpha \in X$. Then g is natural and is a weak homotopy equivalence.*

Proof. Naturality is obvious. That g is a weak homotopy equivalence follows from 5.3 and the fact that the composite map $g(fX) : X \to X$ is the identity map of X.

5.6 COROLLARY. *The functor $F : \mathcal{S}p \to \mathcal{S}p_G$ and the inclusion functor $\mathcal{S}p_G \to \mathcal{S}p$ induce an equivalence between the homotopy theories of set spectra and of group spectra.*

We end this section by showing that the category of abelian group spectra $\mathcal{S}p_A$ is "isomorphic" to the category of abelian chain complexes $\partial\mathcal{G}$.

5.7 DEFINITION. For $G \in \mathcal{S}p_A$ its *Moore chain complex* $MG = \{(MG)_q, \partial_q\}$ is the chain complex defined by

$$(MG)_q = G_{(q)} \cap \bigcap_{i=1}^{\infty} \text{kernel } d_i,$$
$$\partial_q = d_0 \mid (MG)_q$$

for all q. Similarly for a homomorphism $g : G \to G'$ let $Mg : MG \to MG'$ be the induced chain map. Then clearly the function M so defined is a functor $M : \mathcal{S}p_A \to \partial\mathcal{G}$. Its main property is given by

5.8 PROPOSITION. *The functor $M : \mathcal{S}p_A \to \partial\mathcal{G}$ is an isomorphism of categories, i.e., there exists a functor $M' : \partial\mathcal{G} \to \mathcal{S}p_A$ such that the composite functors $M'M : \mathcal{S}p_A \to \mathcal{S}p_A$ and $MM' : \partial\mathcal{G} \to \partial\mathcal{G}$ are naturally equivalent to the identity functors of $\mathcal{S}p_A$ and $\partial\mathcal{G}$ respectively.*

This is proved in exactly the same manner as the corresponding result for abelian group complexes [3], [9].

CHAPTER II. THE HOMOTOPY RELATION

6. The reduced product

In order to define a homotopy relation for maps of set spectra, we need a suitable notion of (reduced) product of the standard 1-simplex I [8] and a set spectrum.

Let $\mathcal{S}$ be the category of set complexes, and let $\wedge : \mathcal{S}, \mathcal{S}_* \to \mathcal{S}_*$ be the *reduced product* functor, i.e., the functor which assigns a complex $K \in \mathcal{S}$ and $L \in \mathcal{S}_*$ the complex with base point $K \wedge L$ obtained from $K \times L$ by identifying the simplices of the form $(\sigma, *)$ with the appropriate degeneracy of the base point. If $I \in \mathcal{S}$ is the standard 1-simplex, $X \in \mathcal{S}p$ and Ps $X = \{X_i, \xi_i\}$, then one would expect the reduced product of I and X to be a set spectrum of the form Sp $\{I \wedge X_i, (i_I \wedge \xi_i)\gamma_i\}$ where the γ_i would be suitable maps $S(I \wedge X_i) \to I \wedge SX_i$. However, although clearly the spaces $RS(I \wedge X_i)$

and $R(I \wedge SX_i)$ are homeomorphic, the complexes $S(I \wedge X_i)$ and $I \wedge SX_i$ are, in general, not isomorphic, and it is not difficult to see that there is not even a natural map $S(I \wedge X_i) \to I \wedge SX_i$. However there exists a natural map in the "wrong" direction.

6.1 Proposition. *For $K \in \mathcal{S}$ and $L \in \mathcal{S}_*$ there is a unique map*

$$\gamma : K \wedge SL \to S(K \wedge L) \in \mathcal{S}_*$$

such that

$$\gamma(\sigma, (\tau, \phi)) = ((d_{j-n} \cdots d_j \sigma, \tau), \phi_n) \qquad \textit{where } j = \dim \sigma$$

for all $\sigma \in K$, $\tau \in L$, and $\phi_n \in P$. Moreover γ is natural.

Proof. Verification of the existence, uniqueness, and naturality of γ is straightforward.

By using this natural map γ it is now possible to construct another functor $\bullet : \mathcal{S}, \mathcal{S}_* \to \mathcal{S}_*$ which is better suited for our purpose. First we state

6.2 Definition. For $L \in \mathcal{S}_*$ and every integer $j \geqq 0$ let $\omega^j L \in \mathcal{S}_*$ be the complex, an n-simplex of which is any $(n + j)$-simplex $\sigma \in L$ such that $d_0 \cdots d_n \sigma = *$ and $d_i \sigma = *$ for $n < i \leqq n + j$; $*$ will be the appropriate degeneracy of the base point; and the operators on $\omega^j L$ will be those induced by the operators of L (with omission, of course, of the last j face and degeneracy operators). A map $\lambda : L \to L' \in \mathcal{S}_*$ induces a map $\omega^j L \to \omega^j L' \in \mathcal{S}_*$, and the function ω^j so obtained is clearly a functor $\omega^j : \mathcal{S}_* \to \mathcal{S}_*$. A simple calculation yields

6.3 Proposition. *Let $L \in \mathcal{S}_*$ and $j \geqq 0$. Then there is a unique map $t_j : S\omega^{j+1}L \to \omega^j L \in \mathcal{S}_*$ such that $t_j(\sigma, \phi_0) = \sigma$ for all $\sigma \in \omega^{j+1}L$. Moreover this map is* 1-1 *(into) and is natural.*

Now we state

6.4 Definition. Let $K \in \mathcal{S}$ and $L \in \mathcal{S}_*$, and consider the (infinite) commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
K \wedge S^2\omega^2 L & \xrightarrow{\gamma} & S(K \wedge S\omega^2 L) & \xrightarrow{S\gamma} & S^2(K \wedge \omega^2 L) & & \\
\downarrow{\scriptstyle i_K \wedge St_1 L} & & \downarrow{\scriptstyle S(i_K \wedge t_1 L)} & & & & \\
K \wedge S\omega^1 L & \xrightarrow{\gamma} & S(K \wedge \omega^1 L) & & & & \\
\downarrow{\scriptstyle i_K \wedge t_0 L} & & & & & & \\
K \wedge L & & & & & &
\end{array}$$

where S^j denotes the j-fold suspension. Define an equivalence relation on $K \wedge L$ by calling two simplices $\alpha_0, \alpha_1 \in K \wedge L$ equivalent if there is an integer j and simplices $\beta_0, \beta_1 \in K \wedge S^j\omega^j L$ such that

$$(i_K \wedge t_0 L) \cdots (i_K \wedge S^{j-1}t_{j-1} L)\beta_\varepsilon = \alpha_\varepsilon, \qquad \varepsilon = 0, 1,$$
$$(S^{j-1}\gamma) \cdots (S\gamma)\gamma\beta_0 = (S^{j-1}\gamma) \cdots (S\gamma)\gamma\beta_1 .$$

Denote by $K \bullet L$ the resulting quotient complex, and by $\pi : K \wedge L \to K \bullet L \in \mathcal{S}_*$ the projection. Clearly maps $\kappa : K \to K' \in \mathcal{S}$ and $\lambda : L \to L' \in \mathcal{S}_*$ induce a map $\kappa \bullet \lambda : K \bullet L \to K' \bullet L' \in \mathcal{S}_*$, and the function $\bullet$ so defined is a functor $\bullet : \mathcal{S}, \mathcal{S}_* \to \mathcal{S}_*$. Clearly, we have

6.5 Proposition. *Let $K \in \mathcal{S}$ and $L \in \mathcal{S}_*$. Then the map $\pi : K \wedge L \to K \bullet L$ is natural.*

6.6 Proposition. *Let $K \in \mathcal{S}$ and $L \in \mathcal{S}_*$. Then there is a unique map $j : K \bullet SL \to S(K \bullet L) \in \mathcal{S}_*$ such that the diagram*

$$\begin{tikzcd} K \wedge SL \arrow[r, "\gamma"] \arrow[d, "\pi"] & S(K \wedge L) \arrow[d, "S\pi"] \\ K \bullet SL \arrow[r, "j"] & S(K \bullet L) \end{tikzcd}$$

is commutative. Moreover j is natural and is an isomorphism.

This follows readily from the fact that $t_0 : S\omega^1 SL \to \omega^0 SL = SL$ is an isomorphism. In view of this proposition we finally state

6.7 Definition. For $K \in \mathcal{S}$, $X \in \mathcal{S}p$, and Ps $X = \{X_i, \xi_i\}$ a simple calculation shows that $\{K \bullet X_i, (i_K \bullet \xi_i)j^{-1}\}$ is a prespectrum, and hence we may define a set spectrum $K \bullet X$ by

$$K \bullet X = \text{Sp}\,\{K \bullet X_i, (i_K \bullet \xi_i)j^{-1}\}.$$

Similarly for maps $\lambda : K \to K' \in \mathcal{S}$ and $\psi : X \to X' \in \mathcal{S}p$ let Ps $\psi = \{\psi_i\}$. Then $\lambda \bullet \psi : K \bullet X \to K' \bullet X' \in \mathcal{S}p$ is the map given by $\lambda \bullet \psi = \text{Sp}\,\{\lambda \bullet \psi_i\}$. Clearly the function $\bullet$ so defined is a functor $\bullet : \mathcal{S}, \mathcal{S}p \to \mathcal{S}p$, the *reduced product* functor.

A useful consequence of this definition is

6.8 Proposition. *Let $K \in \mathcal{S}$, $X \in \mathcal{S}p$, and Ps $X = \{X_i, \xi_i\}$. Then the map $\{j_i\} : \{K \bullet X_i, (i_K \bullet \xi_i)j^{-1}\} \to$ Ps $(K \bullet X)$ of Proposition 4.6 is an equivalence.*

7. The homotopy relation

7.1 Definition. For $L \in \mathcal{S}_*$ identify L with $P \bullet L$ under the correspondence $\sigma \to \pi(\phi, \sigma)$. This induces an identification $X = P \bullet X$ for all $X \in \mathcal{S}p$. Furthermore let $j_0, j_1 : P \to I \in \mathcal{S}$ be the maps given by $j_\varepsilon \phi_0 = d_{1-\varepsilon} i^1$ where $i^1 \in I$ is the only nondegenerate 1-simplex and $\varepsilon = 0, 1$. Then two maps

$w_0, w_1 : X \to Y \in \mathcal{S}p$ will be called *homotopic* if there is a map $w : I \cdot X \to Y \in \mathcal{S}p$ (called *homotopy*) such that $w(j_0 \cdot i_X) = w_0$ and $w(j_1 \cdot i_X) = w_1$. Notation: $w : w_0 \sim w_1$ or $w_0 \sim w_1$.

As in [8] one readily proves

7.2 PROPOSITION. *Let $w' : W \to X$, $w_0, w_1 : X \to Y$, and $w'' : Y \to Z \in \mathcal{S}p$ be such that $w_0 \sim w_1$. Then $w'' w_0 w' \sim w'' w_1 w'$.*

As for set complexes [8] the homotopy relation is, in general, not an equivalence relation. But, as will be shown below, this can be remedied by restriction to a suitable subcategory $\mathcal{S}p_E$ of $\mathcal{S}p$.

7.3 DEFINITION. Let $X \in \mathcal{S}p$, and let $\operatorname{Ps} X = \{X_i\}$. Then X is said to satisfy the *extension condition* if X_i satisfies the extension condition [7] for all i. The full subcategory of $\mathcal{S}p$ generated by such spectra will be denoted by $\mathcal{S}p_E$.

7.4 *Example.* For every topological spectrum Y, clearly $\operatorname{Sp} \operatorname{Sin} Y \in \mathcal{S}p_E$.

7.5 *Example.* Every group spectrum satisfies the extension condition because every group complex does so [16].

Combining this with Corollary 5.6 we get

7.6 PROPOSITION. *Let $E : \mathcal{S}p_G \to \mathcal{S}p_E$ be the inclusion functor. Then the composite functor $EF : \mathcal{S}p \to \mathcal{S}p_E$ and the inclusion functor $\mathcal{S}p_E \to \mathcal{S}p$ induce an equivalence between the homotopy theories of all set spectra and of those satisfying the extension condition.*

7.7 DEFINITION. For every integer $n \geqq 0$ let $\Delta^n \in \mathcal{S}$ denote the standard n-simplex [8], and $\dot{i}^n \in \Delta^n$ the only nondegenerate n-simplex, and for every integer j with $0 \leqq j \leqq n$ let

$$\Delta(d_j) : \Delta^{n-1} \to \Delta^n \in \mathcal{S} \quad \text{and} \quad \Delta(s_j) : \Delta^{n+1} \to \Delta^n \in \mathcal{S}$$

be the maps given by $\Delta(d_j)\dot{i}^{n-1} = d_j \dot{i}^n$ and $\Delta(s_j)\dot{i}^{n+1} = s_j \dot{i}^n$. For $X, Y \in \mathcal{S}p$ the *function complex* $Y^X \in \mathcal{S}$ then is the complex an n-simplex of which is any map $\sigma : \Delta^n \cdot X \to Y \in \mathcal{S}p$; its faces $d_j \sigma$ and degeneracies $s_j \sigma$ are the compositions

$$\Delta^{n-1} \cdot X \xrightarrow{\Delta(d_j) \cdot i_X} \Delta^n \cdot X \xrightarrow{\sigma} Y, \qquad 0 \leqq j \leqq n,$$

$$\Delta^{n+1} \cdot X \xrightarrow{\Delta(s_j) \cdot i_X} \Delta^n \cdot X \xrightarrow{\sigma} Y, \qquad 0 \leqq j \leqq n.$$

7.8 PROPOSITION. *If $X \in \mathcal{S}p$ and $Y \in \mathcal{S}p_E$, then Y^X satisfies the extension condition.*

Proof. If $\Lambda_i^n \subset \Delta^n$ is as in [8], then it must be shown that any map $z : \Lambda_i^n \cdot X \to Y \in \mathcal{S}p$ can be extended over all of $\Delta^n \cdot X$.

Let $L \in \mathcal{S}_*$, let $A \subset L$ be a subcomplex, and let $M \in \mathcal{S}_*$ satisfy the extension condition. By a suitable modification of the main result of [5] one gets that every map $\Delta^n \bullet A \cup \Lambda_i^n \bullet L \to M \in \mathcal{S}_*$ can be extended over all of $\Delta^n \bullet L$. If $\text{Ps}\, X = \{X_i, \xi_i\}$, $\text{Ps}\, Y = \{Y_i, \eta_i\}$, and $\text{Ps}\, z = \{z_i\}$, then the map $z_0 : \Lambda_i^n \bullet X_0 \to Y_0$ admits an extension $w_0 : \Delta^n \bullet X_0 \to Y_0$. Now construct for $k > 0$ maps

$$v_k : \Delta^n \bullet (\text{image } \xi_{k-1}) \cup \Lambda_i^n \bullet X_k \to Y_k \quad \text{and} \quad w_k : \Delta^n \bullet X_k \to Y_k$$

by requiring that $v_k(i_{\Delta^n} \bullet \xi_{j-1}) = \eta_k(Sw_{k-1})i$ where j is as in 6.6, that $v_k \mid (\Delta_i^n \bullet X_k) = z_k$, and that w_k be an extension of v_k. Then clearly

$$\text{Sp}\{w_k\} : \Delta^n \bullet X \to Y \in \mathcal{S}p$$

is an extension of z.

Finally applying to Proposition 7.8 the argument of [8, §2] we get

7.9 Proposition. *Let $X \in \mathcal{S}p$ and $Y \in \mathcal{S}p_E$. Then the homotopy relation is an equivalence relation on the maps $X \to Y \in \mathcal{S}p$.*

8. Homotopy types and minimal spectra

In view of 7.2 and 7.9 it makes sense to introduce the notions homotopy equivalence and homotopy type.

8.1 Definition. A map $w : X \to Y \in \mathcal{S}p_E$ is called a *homotopy equivalence* if there is a map $w' : Y \to X \in \mathcal{S}p_E$ (called *homotopy inverse* of w) such that $w'w \sim i_X$ and $ww' \sim i_Y$. In view of 7.2 and 7.9 any two homotopy inverses of w are homotopic, and w' is a homotopy equivalence itself. Also the composition of two homotopy equivalences is again one.

Two spectra $X, Y \in \mathcal{S}p_E$ are said to have the same *homotopy type* if there exists a homotopy equivalence $w : X \to Y$. Clearly "having the same homotopy type" is an equivalence relation.

As for set complexes a (theoretical) homotopy-type classification of spectra may be obtained using an appropriate notion of minimal spectrum.

8.2 Definition. Let $M \in \mathcal{S}p_E$, and let $\text{Ps}\, M = \{M_i\}$. Then M will be called *minimal* if M_i is minimal [15] for all i. For $X \in \mathcal{S}p_E$ a subspectrum $M \subset X$ is called a *minimal subspectrum* if (i) M is minimal, and (ii) the inclusion map $M \to X$ is a homotopy equivalence.

As for set complexes one then has

8.3 Proposition. *Every spectrum $X \in \mathcal{S}p_E$ has a minimal subspectrum.*

8.4 Proposition. *Let $M, N \in \mathcal{S}p_E$ be minimal, and let $w : M \to N$ be a homotopy equivalence. Then w is an isomorphism.*

The proof of 8.3 is essentially the same as for set complexes [4] by using induction on the integer $n(\alpha)$ $(\alpha \in X)$ where $n(\alpha)$ denotes the smallest integer such that $d_j \alpha = *$ for $j > n(\alpha)$ and $d_0 \cdots d_{n(\alpha)} \alpha = *$; 8.4 follows at once from the corresponding result for set complexes [15].

8.5 COROLLARY. *Let $X \in \mathcal{S}p_E$, let $M, N \subset X$ be minimal subspectra, let $j : M \to X$ be the inclusion, and $p : X \to N$ a homotopy inverse of the inclusion. Then the composition $pj : M \to N$ is an isomorphism.*

8.6 COROLLARY. *Let $X, Y \in \mathcal{S}p_E$. Then X and Y have the same homotopy type if and only if they have isomorphic minimal subspectra.*

9. Weak homotopy equivalences and the homotopy relation

The main purpose of this section is to prove

9.1 PROPOSITION. *Let $\mathcal{C}$ be a category. Then a functor Q: $\mathcal{S}p_E \to \mathcal{C}$ is a homotopy functor (3.6) if and only if $w \sim w'$ implies $Qw = Qw'$.*

Proof. Let $Q : \mathcal{S}p_E \to \mathcal{C}$ be a homotopy functor, let $w_0, w_1 : X \to Y \in \mathcal{S}p_E$' and let $w : w_0 \sim w_1$. If $p : I \to P \in \mathcal{S}$ is the only such map, then $pj_0 = pj_1 = i_P$, and it is not difficult to verify that $j_0 \cdot i_X$, $j_1 \cdot i_X$, and $p \cdot i_X$ are weak homotopy equivalences. So are (5.4) the maps $F(j_0 \cdot i_X)$, $F(j_1 \cdot i_X)$, and $F(p \cdot i_X)$. Consequently $QF(j_0 \cdot i_X) = QF(p \cdot i_X)^{-1} = QF(j_1 \cdot i_X)$, and it follows that

$$\begin{aligned} Qw_0 &= (QfY)^{-1}(QfY)(Qw)(Q(j_0 \cdot i_X)) \\ &= (QfY)^{-1}(QFw)(QF(j_0 \cdot i_X))(QfX) \\ &= (QfY^{-1})(QFw)(QF(j_1 \cdot i_X))(QfX) \\ &= (QfY^{-1})(QFY)(QW)(Q(j_1 \cdot i_X)) = Qw_1 . \end{aligned}$$

The other half of the proposition follows at once from

9.2 PROPOSITION. *A map in $\mathcal{S}p_E$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Proof. Let $w : X \to Y \in \mathcal{S}p_E$, and let $\operatorname{Ps} X = \{X_i\}$, $\operatorname{Ps} Y = \{Y_i\}$, and $\operatorname{Ps} w = \{w_i\}$. Then by definition [7],

$$\pi_n(X_i) = \pi_{n+1}(X_{i+1}) \quad \text{and} \quad \pi_n(Y_i) = \pi_{n+1}(Y_{i+1})$$

for all n and i (the n-simplices of X_i which have all their faces $= *$ coincide with the $(n+1)$-simplices of X_{i+1} with this property).

Now if w is a homotopy equivalence, then so is $w_i : X_i \to Y_i$ for all i. This implies that the w_i induce isomorphisms of all homotopy groups, and hence w is a weak homotopy equivalence.

Conversely if w is a weak homotopy equivalence, we may (§8) assume that X and Y are minimal. Hence so are X_i and Y_i for all i. The maps $w_i : X_i \to Y_i$ induce isomorphisms of all homotopy groups, and hence [15] are isomorphisms. And so is therefore the map w.

10. Homotopy groups

We end by defining the homotopy groups of a spectrum and expressing (weak) homotopy equivalences in terms of them. The definition will be in two stages, first for spectra which satisfy the extension condition, and then for all spectra. Both definitions agree, of course, on the category $\mathcal{S}p_E$.

10.1 DEFINITION. Let $X \in \mathcal{S}p_E$, and let $\text{Ps } X = \{X_i\}$. Then (see 9.2) $\pi_n(X_i) = \pi_{n+1}(X_{i+1})$ for all n and i. Hence for *every* integer q we may define $\bar{\pi}_q X$, the q^{th} *homotopy group* of X, by

$$\bar{\pi}_q X = \pi_{q+i}(X_i) \qquad \text{where } q + i > 0.$$

Similarly for a map $w : X \to Y \in \mathcal{S}p_E$ with $\text{Ps } w = \{w_i\}$ we put

$$\bar{\pi}_q w = \pi_{q+i}(w_i) : \pi_q X \to \pi_q Y \qquad \text{where } q + i > 0.$$

The functions $\bar{\pi}_q$ so defined are clearly functors.

10.2 DEFINITION. For $X \in \mathcal{S}p$ and every integer q we define $\pi_q X$, the q^{th} *homotopy group* of X by

$$\pi_q X = \bar{\pi}_q FX.$$

Similarly for a map $w : X \to Y \in \mathcal{S}p$ we define $\pi_q w = \bar{\pi}_q Fw$. The functions π_q so defined are clearly functors, and an argument similar to the one used in the proof of Proposition 9.1 yields

10.3 PROPOSITION. *Let* w_0, $w_1 : X \to Y \in \mathcal{S}p$ *be homotopic. Then* $\pi_q w_0 = \pi_q w_1$ *for all* q.

Immediate consequences of 5.4 and 9.2 are

10.4 PROPOSITION. *A map* $w : X \to Y \in \mathcal{S}p$ *is a weak homotopy equivalence if and only if* w *induces isomorphisms of all homotopy groups.*

10.5 PROPOSITION. *A map* $w : X \to Y \in \mathcal{S}p_E$ *is a homotopy equivalence if and only if it induces isomorphisms of all homotopy groups.*

That the two definitions of homotopy groups agree on $\mathcal{S}p_E$ is stated in the following proposition, which readily follows from 5.3.

10.6 PROPOSITION. *Let* $X \in \mathcal{S}p_E$. *Then the natural homomorphism*

$$\bar{\pi}_q(fX) : \bar{\pi}_q X \to \bar{\pi}_q FX = \pi_q X$$

is an isomorphism for all q.

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