MARKOV PROCESSES ON AN ENTRANCE BOUNDARY[1]

BY
FRANK KNIGHT

This paper presents, essentially, an alternative approach to the second part of [7] in terms of "boundary theory." In [7, Section 2] the motivating idea was to regularize a temporally homogeneous Markov process $X(t)$, having as state space a set X, by using martingale theorems, independently of a topology on X or on the state space Y of the resulting process $Y(t)$. It is shown here that there is an alternative state space Y_ρ for $Y(t)$, and a compact metric topology on Y_ρ, such that $Y(t)$ (after a slight adjustment of the definition in [7] on certain sets of probability 0) has right continuous path functions. The necessary metric ρ is defined in a manner closely resembling that used by R. S. Martin [8] in defining a general boundary for the positive harmonic functions on Euclidean domains. In this way it is an extension of the boundaries discussed especially by Doob [2], Hunt [6], and Watanabe [10]. In terms of the metric ρ, the space Y_ρ is the completion of X, and the new process, denoted by $Y_\rho(t)$, is defined from $X(t)$ as the value of the process $X(t)$ on the "entrance boundary" corresponding to t, for each t. Equivalently, $Y_\rho(t)$ is simply given almost surely for all t by $Y_\rho(t) = \lim_{\tau \downarrow t, \tau \text{rational}} X(\tau)$, the limit being taken in the metric ρ.

In the second section we show the connection between this regularization of $X(t)$ and the general methods of Ray [9] for regularization of transition functions and processes corresponding to a given resolvent R_λ, $\lambda > 0$, operating on continuous functions. It may be pointed out that, aside from reasons of completeness, this connection is significant because, whenever $Y_\rho(t)$ is an instance of Ray's method, the theorems of [9] provide a transition function for $Y_\rho(t)$ together with a number of its properties which are not established otherwise.

Section 1

It is assumed in [7, Section 2] that a Markov process $X(t)$ relative to completed σ-fields $\mathfrak{F}(t)$ on a probability space (Ω, F, P) is given, together with a homogeneous transition function $p(t, x, E)$, $x \in X$, $E \in \mathcal{B}$, for $X(t)$, and such that (a) $\mathcal{B}$ is generated by countably many sets, (b) wide-sense conditional distributions over $\mathcal{B}$ exist, (c) $p(t, x, E)$ is measurable in (t, x) over $\mathcal{R} \times \mathcal{B}$ where $\mathcal{R}$ is the field of real Borel sets, and (d) $p(\cdot, x_1, \cdot) = p(\cdot, x_2, \cdot)$ implies that $x_1 = x_2$. These hypotheses are assumed again here. Under these hypotheses there is constructed in [7] a corresponding process $Y(t)$ which has for its range (at each t) families of joint distributions $F(t', E; t, w)$, $t' > 0$, $E \in \mathcal{B}$, the "conditional futures", such that for

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$t_1 > 0$,

$$F(t' + t_1, E; t, w) = \int_X p(t_1, x, E)F(t', dx; t, w).$$

These families are related to $X(t)$ as follows: for each stopping time T for $X(t)$ one has

$$\text{(1.1)} \qquad \lim_{\tau \downarrow T, \tau \text{ rational}} p(t - \tau, X(\tau), E) = F(t - T, E; T, w) \quad \text{a.s.}$$

(i.e. with probability 1) on $\{t - T > 0\}$. In fact, (1.1) is an immediate consequence of (2.9) in [7] (we have replaced the diadic rationals A used in [7] by the set of all rationals, to which the construction of [7] applies without change).

This relation may be extended to the product σ-fields over $t > T$. Let $F'_T(T)$, for each stopping time T, be the least σ-subfield of the underlying field F_T (the subscript T indicates restriction to $\{T < \infty\}$) containing all sets of the form $S = \bigcap_{i=1}^k \{X(t_i) \in E_i\} \cap \{T < t_1\}$ for $t_1 < \cdots < t_k$ and $E_i \in \mathcal{B}$. Repeating some further notation from [7], let $H^+(T)$ be the σ-field of all $S \in F_T$ such that $S \cap \{T < c\} \in \mathfrak{F}_T(c)$ for all c; let $T_n = \alpha(m, n) = m2^{-n}$ for $\alpha(m - 1, n) \leqq T < \alpha(m, n)$; let $G_n^*(T)$ be the least σ-field containing $\{X_T(T_k) \in E\}$ and $\{T_k - T \in R\}$ for $k \geqq n$, $R \in \mathcal{R}$, $E \in \mathcal{B}$; and let $G^{*+}(T) = \bigcap_{n=1}^\infty G_n^*(T)$. For each stopping time T, let $F(S; T, w)$ be the σ-additive extension to $F'_T(T)$ of the measure defined for the above sets S in terms of an indicator function $\chi_{\{T<t_1\}}$ by

$$\text{(1.2)} \qquad \begin{aligned} F(S; T, w) = \chi_{\{T<t_1\}}(w) \int_{E_1} \cdots \int_{E_{k-1}} & p(t_k - t_{k-1}, x_{k-1}, E_k) \\ & \cdot p(t_{k-1} - t_{k-2}, x_{k-2}, dx_{k-1}) \cdots F(t_1, dx_1; T, w). \end{aligned}$$

The following theorem was used, in fact, at three points in [7] (namely, for (2.14), to extend (2.19 a, b) just before Lemma 2.2, and after (2.33)) without being explicitly stated (it is a direct extension of (2.9)).

THEOREM 1.1. *Let $X(t)$ be a Markov process for which the above-mentioned hypotheses hold. Let T be a stopping time for $X(t)$. Then for $S \in F'_T(T)$,*

$$\begin{aligned} F(S; T, w) &= P(S \mid H^+(T)) \\ &= P(S \mid G^{*+}(T)) \quad a.s. \end{aligned}$$

Proof. The proof is analogous to that of (2.9) of [7], to which the theorem reduces for S of the form $\{T < t_1\} \cap \{X(t_1) \in E_1\}$. Suppose, more generally, that S is of the form in (1.2), and note that for all n sufficiently large so that $T_n < t_1$, w being fixed, we have

$$\text{(1.3)} \qquad \begin{aligned} F(S; T, w) = \int_X \int_{E_1} \cdots \int_{E_{k-1}} & p(t_k - t_{k-1}, x_{k-1}, E_k) \\ & \cdot p(t_{k-1} - t_{k-2}, x_{k-2}, dx_{k-1}) \\ & \cdots p(t_1 - T_n, x_0, dx_1)F(T_n - T, dx_0; T, w). \end{aligned}$$

Let $\{E_i'\}$ be a countable collection of sets such that

$$\{F(T_n - T, E_i'; T, w), 1 \leqq i\}$$

is sufficient for all w and large n to define uniformly approximating sums for (1.3) to within any $\varepsilon > 0$. Since $t_1 - T_n$ is countably-many-valued, it is clear from the Lebesgue definition of the integral in (1.3) that such a family exists, and by (1.1) we have a.s. for all E_i' and n

$$\lim_{m\to\infty} p(T_n - T_m, X(T_m), E_i') = F(T_n - T, E_i'; T, w). \tag{1.4}$$

By substituting $p(T_n - T_m, X(T_m), dx_0)$ for $F(T_n - T, dx_0; T, w)$ in (1.3), for $m > n$, and letting m become large, it is clear from (1.4) that a.s. on $\{T < t_1\}$

$$\begin{aligned} F(S; T, w) &= \lim_{m\to\infty} \int_X \int_{E_1} \cdots \int_{E_{k-1}} p(t_k - t_{k-1}, x_{k-1}, E_k) \\ &\qquad \cdots p(t_1 - T_n, x_0, dx_1) p(T_n - T_m, X(T_m), dx_0) \\ &= \lim_{m\to\infty} \int_{E_1} \cdots \int_{E_{k-1}} p(t_k - t_{k-1}, x_{k-1}, E_k) \\ &\qquad \cdots p(t_1 - T_m, X(T_m), dx_1). \end{aligned} \tag{1.5}$$

But this is a version of $P(S \mid H^+(T))$, measurable over $G^{*+}(T)$, and hence also a version of $P(S \mid G^{*+}(T))$. The theorem for arbitrary $S \in F_T'(T)$ now follows since the monotone convergence theorem applied for monotone sequences of sets implies that the class of sets S for which the theorem holds is a monotone class.

The metric on X mentioned in the introduction will now be defined. Let $\{E_i\}$ be a countable field of sets generating the σ-field $\mathcal{B}$, let $\lambda > 0$ be fixed, and let $g(i)$ be a strictly positive function such that $\sum_{i=1}^{\infty} g(i) < \infty$.

DEFINITION 1.1.[2] The metric $\rho(x, y)$ on X is given by

$$\rho(x, y) = \sum_{i=1}^{\infty} g(i) \left| \frac{\int_0^\infty e^{-\lambda t} p(t, x, E_i)\, dt}{\sum_{i=1}^{\infty} g(i) \int_0^\infty e^{-\lambda t} p(t, x, E_i)\, dt} - \frac{\int_0^\infty e^{-\lambda t} p(t, y, E_i)\, dt}{\sum_{i=1}^{\infty} g(i) \int_0^\infty e^{-\lambda t} p(t, y, E_i)\, dt} \right|. \tag{1.6}$$

The completion of X in this metric will be denoted by Y_ρ.

[2] By analogy with the definition of R. S. Martin, it would seem more natural to employ a metric

$$d(x, y) = \int_X \left| \int_0^\infty e^{-\lambda t}(p(t, x, dz) - p(t, y, dz))\, dt \right|.$$

where the absolute value indicates a total variation in dz. However, as a simple example shows, this metric does not always yield the convergence necessary to our definition of $Y_\rho(t)$.

In order to justify this definition it must be shown, of course, that $\rho(x, y)$ is a metric. Before doing this, however, we state that the metric so defined is not independent, even in the sense of equivalence of metrics, of the choice of the $\{E_i\}$, and it is not known to be independent of λ. However, as will appear below, any two of these metrics are equivalent insofar as the redefined processes $Y_\rho(t)$ are concerned, in the sense that two corresponding redefined processes become identical with probability 1 at each stopping time T after a natural identification.

To show that $\rho(x, y)$ is a metric, all that needs to be verified is that if $\rho(x, y) = 0$ then $x = y$. It then follows directly that Y_ρ is a compact, separable, metric space (of diameter less than 1). If $\rho(x, y) = 0$, then we have easily that

$$\int_0^\infty e^{-\lambda t} p(t, x, E_i)\, dt = \int_0^\infty e^{-\lambda t} p(t, y, E_i)\, dt$$

for each i, and by closure under monotone limits it follows that for each bounded $\mathfrak{B}$-measurable function $f(z)$,

$$\int_0^\infty e^{-\lambda t} \int_X f(z) p(t, x, dz)\, dt = \int_0^\infty e^{-\lambda t} \int_X f(z) p(t, y, dz)\, dt.$$

By introducing the resolvent

$$R_\mu f(x) = \int_0^\infty e^{-\mu t} \int_X f(z) p(t, x, dz)\, dt, \qquad \mu > 0,$$

it follows from the well-known equation

$$R_\mu = R_\lambda(I - (\mu - \lambda) R_\mu) \tag{1.7}$$

that the resolvent does not separate x and y. This implies that, contrary to hypothesis, $p(\cdot\,, x, \cdot) = p(\cdot\,, y, \cdot)$. In fact, we shall prove more generally that if $F_j(t, E)$, $j = 1$ or 2, $t > 0$, are two families of distributions on $\mathfrak{B}$ such that

$$F_j(t_1 + t_2, E) = \int_X F_j(t_1, dx) p(t_2, x, E),$$

then they are identical provided that

$$\int_0^\infty e^{-\mu t} F_1(t, E_i)\, dt = \int_0^\infty e^{-\mu t} F_2(t, E_i)\, dt$$

for all i and $\mu > 1$. If two such families are identical for one t, then they are identical for all larger t. Hence if they differ, then they differ for all t sufficiently small. It follows from this that they must differ at some E_i for all t in a set of positive Lebesgue measure. This would contradict the fact that they are determined except on a set of measure 0 by their Laplace transforms with $\mu > 1$ (see (1.14) below), assumed to be identical. Hence we find that $F_1(\cdot\,, \cdot) = F_2(\cdot\,, \cdot)$, and the proof that $\rho(x, y)$ is a metric is complete. It is to be noted that the proof does not use essentially the fact that $p(t, x, X) = 1$. One finds that Definition 1.1 also applies to "subprocesses" [4] with $0 \leqq p(t, x, X) \leqq 1$ provided that $x_1 = x_2$ whenever $p(\cdot, x_1, \cdot) = c\, p(\cdot, x_2, \cdot)$ for some $c > 0$. The same generalization holds, in fact, for the results of the rest of this section, as well as for those of [7], as may be seen by adjoining a further state "∞" to X, completing the paths by assignment of the value ∞ for all t following the "lifetime," and extending the transition function by setting

$$p(t, \{\infty\}, \{\infty\}) = 1 \quad \text{and} \quad p(t, x, \{\infty\}) = 1 - p(t, x, X).$$

It is only in the consideration of such subprocesses that the denominators in Definition 1.1 can have any significance.

The redefined process $Y_\rho(t)$ is given as follows:

DEFINITION 1.2. Let $Y_\rho(t) = \lim_{\tau \downarrow t, \tau \text{ rational}} X(\tau)$ provided that this limit exists in Y_ρ for all t. Otherwise, set $Y_\rho(t) = \underset{\sim}{x}$ for some fixed $\underset{\sim}{x} \in X$.

By a familiar application of martingale theorems we shall show that the first case of Definition 1.2 has probability 1, and that $Y_\rho(t)$ is continuous from the right. It is easily seen that

$$e^{-\lambda t} \int_X \left(\int_0^\infty e^{-\lambda s} p(s, y, E_i)\, ds \right) p(t, x, dy) \leqq \int_0^\infty e^{-\lambda t} p(t, x, E_i)\, dt.$$

Hence if $\underline{X}(t)$ is a subprocess of $X(t)$ with transition function $e^{-\lambda t} p(t, x, E)$, then

$$\int_0^\infty e^{-\lambda s} p(s, \underline{X}(\tau), E_i)\, ds$$

is easily extended to become a supermartingale in τ (for example, by setting $\underline{X}(t) = 0$ after the "lifetime"), and hence has limits from the right along the rationals at all t, outside of a set of probability 0.[3] Such a subprocess $\underline{X}(t)$ can be defined, as is well known, by subjecting $X(t)$ to a "death rate" with density $\lambda e^{-\lambda t}$ in t, which is independent of $X(t)$, and it follows easily that

$$\int_0^\infty e^{-\lambda t} p(t, X(\tau), E_i)\, dt$$

has limits in the same sense. The field $\{E_i\}$ being countable, these limits exist simultaneously for all E_i with probability 1. Since $\sum_i g(i) < \infty$, we see that the existence of these limits implies that of the limits in Definition 1.2, and our statement follows.

The connection between $Y_\rho(t)$ and the process $Y(t)$ of [7], whose range consists of the families $F(t_1, E; t, w)$, is established by an identification of part of this range with part of Y_ρ. To select the appropriate subset, we introduce an auxiliary topology similar to that of $\rho(x, y)$ but formally

[3] It also has limits from the left, by the same martingale theorem of Doob [1, p. 363].

stronger.[4] It depends on the existence of a countable set of nonnegative functions bounded by 1 which contains the indicator functions $\chi_{E_i}(x)$, and whose uniform closure is closed under the resolvents R_μ, $\mu > 1$. Such a set is easily constructed. Let $\{\mu_i\}$ be a countable dense set in $(1, \infty)$, and define recursively $S_0 = \{\chi_{E_i}\}$, S_{n+1} a countable set of nonnegative functions bounded by 1, closed under R_{μ_i}, $1 \leqq i \leqq n + 1$, and containing S_n. Then $\lim_{n\to\infty} S_n = \{f_k(z)\}$ is closed under $\{R_{\mu_i}\}$, and is countable. The uniform closure of $\{f_k(z)\}$ is then closed under R_μ, $\mu > 0$, as follows from the uniform continuity in μ and f of $R_\mu f$ in the uniform norm. The connection between the process $Y_\rho(t)$ and the families $F(t_1, E; t, w)$ is now a consequence of the following theorem:

THEOREM 1.2.[5] *For each stopping time T for $X(t)$, simultaneously for $\mu = \lambda$ and all $\mu > 1$,*

$$\lim_{\tau\downarrow T,\tau \text{ rational}} \sum_{i=1}^{\infty} g(i) \left| \frac{\int_0^\infty e^{-\mu t} \int_X p(t, X(\tau), dz) f_i(z)\, dt}{\sum_{i=1}^{\infty} \int_0^\infty e^{-\mu t} \int_X p(t, X(\tau), dz) f_i(z)\, dt} - \frac{\int_0^\infty e^{-\mu t} \int_X F(t, dz, T, w) f_i(z)\, dt}{\sum_{i=1}^{\infty} g(i) \int_0^\infty e^{-\mu t} \int_X F(t, dz; T, w) f_i(z)\, dt} \right| = 0 \quad a.s. \tag{1.8}$$

Whenever this convergence holds, then the analogous convergence in terms of $\rho(x, y)$ also holds. Let S be the set of all families $F(t_1, E; t, w)$ obtainable in this manner. Then the convergence (1.8) *induces a one-to-one correspondence between S and a subset S_ρ of Y_ρ.*

Proof. We prove the last statements first. Since the convergence in (1.8) is at least as strong as in ρ, it is only necessary to show that if $F(t_1, E; t', w')$ and $F(t_1, E; t'', w'')$ are two distinct families in S, then they are separated by ρ, or in other words, that

$$\sum_{i=1}^{\infty} g(i) \left| \frac{\int_0^\infty e^{-\lambda t} F(t, E_i; t', w')\, dt}{\sum_{i=1}^{\infty} g(i) \int_0^\infty e^{-\lambda t} F(t, E_i; t', w')\, dt} - \frac{\int_0^\infty e^{-\lambda t} F(t, E_i; t'', w'')\, dt}{\sum_{i=1}^{\infty} g(i) \int_0^\infty e^{-\lambda t} F(t, E_i; t'', w'')\, dt} \right| > 0.$$

[4] For the idea of this construction, I am indebted to Professor Daniel Ray.

[5] A metric of the form in (1.8) with $\mu = \lambda$ could replace $\rho(x,y)$ throughout the paper, and the resulting space and process would be independent of λ. We have retained Definition 1.1 because of its simplicity.

In the contrary case, it follows by monotone extensions that

$$\int_0^\infty e^{-\lambda t} \int_X F(t, dz; t', w')f(z)\, dt = \int_0^\infty e^{-\lambda t} \int_X F(t, dz; t'', w'')f(z)\, dt$$

for all bounded measurable functions $f(z)$. On the other hand, it was shown above that for some $\mu > 1$ and some i, one must have

$$\int_0^\infty e^{-\mu t} F(t, E_i; t', w')\, dt \neq \int_0^\infty e^{-\mu t} F(t, E_i; t'', w'')\, dt. \tag{1.9}$$

If we set $f = \chi_{E_i} - (\mu - \lambda) R_\mu \chi_{E_i}$, (1.7) implies that $R_\mu \chi_{E_i} = R_\lambda f$. Now let x_j' and x_j'' be sequences in X converging to $F(\cdot\, ,\cdot\, ; t', w')$ and $F(\cdot\, ,\cdot\, ; t'', w'')$ respectively, in the sense of (1.8). Then

$$\begin{aligned}
\int_0^\infty e^{-\mu t} F(t, E_i; t', w')\, dt &= \lim_{j\to\infty} \int_0^\infty e^{-\mu t} p(t, x_j', E_i)\, dt \\
&= \lim_{j\to\infty} \int_0^\infty e^{-\lambda t} \int_X p(t, x_j', dz) f(z)\, dt \\
&= \int_0^\infty e^{-\lambda t} \int_X F(t, dz; t', w') f(z)\, dt \\
&= \int_0^\infty e^{-\lambda t} \int_X F(t, dz; t'', w'') f(z)\, dt \\
&= \lim_{j\to\infty} \int_0^\infty e^{-\lambda t} \int_X p(t, x_j'', dz) f(z)\, dt \\
&= \lim_{j\to\infty} \int_0^\infty e^{-\mu t}\, p(t, x_j'', E_i)\, dt \\
&= \int_0^\infty e^{-\mu t}\, F(t, E_i; t'', w'')\, dt.
\end{aligned} \tag{1.10}$$

Since this contradicts (1.9), the result is established.

We return to the proof of (1.8); the *existence* of the limit a.s. for fixed μ follows exactly as in the discussion following Definition 1.2. It remains to identify the limit. Using the T_n of (1.3), and taking $\tau < \tau'$ rational with w fixed, we have for each bounded, measurable $f(z)$

$$\begin{aligned}
\int_0^\infty e^{-\mu t} \int_X p(t, X(\tau), dz) f(z)\, dt &= \int_0^{\tau'-\tau} e^{-\mu t} \int_X p(t, X(\tau), dz) f(z)\, dt \\
&+ e^{-\mu(\tau'-\tau)} \int_X \left(\int_0^\infty e^{-\mu t} \int_X p(t, x, dz) f(z)\, dt \right) p(\tau' - \tau, X(\tau), dx).
\end{aligned} \tag{1.11}$$

The last integral may be defined in the sense of Lebesgue as the limit of a sequence of integrals of uniformly approximating simple functions, and such a sequence depends only on a countable collection of sets $\{E_i'\}$. For each i, and hence for all, we have by (1.1) for all rational $\tau' > T$,

$$\lim_{\tau \downarrow T, \tau \text{ rational}} p(\tau' - \tau, X(\tau), E_i') = F(\tau - T, E_i'; T, w) \quad \text{a.s.}$$

By defining the limit of the last integral in (1.11) in terms of these limits it follows that a.s. on $\{T < \tau'\}$ for all rational τ'

$$\begin{aligned} \limsup_{\tau \downarrow T, \tau \text{ rational}} \Big| \int_0^\infty e^{-\mu t} \int_X p(t, X(\tau), dz) f(z)\, dt \\ -e^{-\mu(\tau'-\tau)} \int_0^\infty e^{-\mu t} \int_X F(t + \tau' - T, dz; T, w) f(z)\, dt \Big| \\ = \limsup_{\tau \downarrow T, \tau \text{ rational}} \int_0^{\tau'-\tau} e^{-\mu t} \int_X p(t, X(\tau), dz) f(z)\, dt \\ < (\tau' - T) \max f(z). \end{aligned} \tag{1.12}$$

Accordingly, we find that

$$\begin{aligned} \lim_{\tau \downarrow T, \tau \text{ rational}} \int_0^\infty e^{-\mu t} \int_X p(t, X(\tau), dz) f(z)\, dt \\ = \lim_{\tau' \downarrow T, \tau' \text{ rational}} e^{-\mu(\tau'-T)} \int_0^\infty e^{-\mu t} \int_X F(t + \tau' - T, dz; T, w) f(z)\, dt \\ = \int_0^\infty e^{-\mu t} \int_X F(t, dz; T, w) f(z)\, dt \quad \text{a. s.} \end{aligned} \tag{1.13}$$

From this it is seen that the limit (1.8) is 0 a.s. for each μ, since the convergence is equivalent to simultaneous convergence of the numerators. The same is therefore true for $\mu = \lambda$ and all μ in a countable dense subset of $(1, \infty)$. However, a differentiation of (1.8) with respect to μ shows that its derivative is bounded independently of τ, w, and μ (since the $f_i(z)$ are bounded by 1). Therefore convergence to 0 on a dense subset of $(1, \infty)$ implies convergence to 0 everywhere, and the proof of Theorem 1.2 is complete.

The next step in proving the equivalence of $Y_\rho(t)$ and the corresponding process $Y(t)$ of [7] is to show that the σ-field $\mathcal{B}(Y)$ in [7] "corresponds" to the topological σ-field $\mathcal{B}(Y_\rho)$ of Y_ρ. We recall that $\mathcal{B}(Y)$ may be taken as the least σ-field of probability measures P on the product space $\prod_{t>0} \mathcal{B}_t$ of $\mathcal{B}$ for which the sets $\{P : P(t, E) \in R\}$, $t > 0$, $E \in \mathcal{B}$, $R \in \mathcal{R}$, are measurable (in [7], t was restricted to a countable dense set, but the present definition, as noted there following Theorem 2, is equally satisfactory).

THEOREM 1.3. *To each $B \in \mathcal{B}(Y)$ there corresponds a $B_\rho \in \mathcal{B}(Y_\rho)$, and, conversely, to each $B_\rho \in \mathcal{B}(Y_\rho)$ there is a $B \in \mathcal{B}(Y)$, such that for each stopping time T for $Y(t)$,*

$$P_T(\{Y(T) \in B\} \triangle \{Y_\rho(T) \in B_\rho\}) = 0.[6]$$

[6] Here $\triangle$ is the symmetric difference: $E \triangle F = (E - F) \cup (F - E)$.

The correspondences depend only on X, $\mathcal{B}$, and the transition function $p(t, x, E)$.

Proof. By Theorem 1.2 it suffices to prove this theorem under the assumption that $Y(T) \in S$ and $Y_\rho(T) \in S_\rho$. We denote the elements of S by $F(t, E)$, so that

$$F(t + t_1, E) = \int_X p(t_1, x, E)F(t, dx) \qquad \text{for } t_1 > 0.$$

To prove the existence of a set B corresponding in the required manner to a given set B_ρ it suffices to show that

$$\int_0^\infty e^{-\lambda t}F(t, E)\, dt,$$

as a function on S, is measurable over the σ-field $[\mathcal{B}(Y) \cap S]$ consisting of the sets $E \cap S$ for $E \in \mathcal{B}(Y)$. For then the metric ρ is a measurable function on S_ρ with respect to the image σ-field of $[\mathcal{B}(Y) \cap S]$ in S_ρ under the correspondence of Theorem 1.2, and B may be taken to be any set in $\mathcal{B}(Y)$ such that the pre-image of $B_\rho \cap S_\rho$ is $B \cap S$. Since $p(t, x, E)$ is by hypothesis measurable in (t, x) over $\mathcal{R} \times \mathcal{B}$, it follows from Fubini's theorem that

$$\int_0^\infty e^{-\lambda(t+\delta/2)}p(t, x, E)\, dt$$

is measurable over $\mathcal{B}$ for each $\delta > 0$. Let $\{E_i'\}$ be a countable collection of disjoint sets such that for $x_i \in E_i$ and all $F(t, E)$ in S

$$\left| \int_{\delta/2}^\infty e^{-\lambda t}F(t, E)\, dt - \sum_{i=1}^\infty F(\delta/2, E_i') \int_0^\infty e^{-\lambda(t+\delta/2)}p(t, x_i, E)\, dt \right| < \delta/2.$$

By the usual approximation of an integral using simple functions this is clearly possible. We then have

$$\left| \int_0^\infty e^{-\lambda t}F(t, E)\, dt - \sum_{i=1}^\infty F(\delta/2, E_i') \int_0^\infty e^{-\lambda(t+\delta/2)}p(t, x_i, E)\, dt \right| < \delta.$$

Since the sum is measurable over $[\mathcal{B}(Y) \cap S]$, the first integral is seen to be measurable upon letting δ become small.

Conversely, to prove the existence of a set B_ρ corresponding to a given set $B \in \mathcal{B}(Y)$, it is enough to show that $B \cap S$ has as image in S_ρ a set of the form $B_\rho \cap S_\rho$. We may suppose that $B \cap S$ has the form $\{F : F(t, E_1) \in R\}$ since such sets generate $[\mathcal{B}(Y) \cap S]$. The problem is to define this set in terms of

$$\int_0^\infty e^{-\lambda t}\, F(t, E_i)\, dt, \qquad 1 \leq i < \infty,$$

which it is not hard to see are measurable over the field $[\mathcal{B}(Y_\rho) \cap S]$ consisting of sets $B_\rho \cap S$ for $B_\rho \in \mathcal{B}(Y_\rho)$. By monotonic extensions it follows that

$$\int_0^\infty e^{-\lambda t} \int_X F(t, dz)\, f(z)\, dt$$

is similarly measurable for f bounded and measurable. When f is taken to be the function $\chi_{E_i} - (\mu - \lambda)R_\mu \chi_{E_i}$, $\mu > 1$, the above expression becomes

$$\int_0^\infty e^{-\mu t} F(t, E_i)\, dt,$$

at least when $F(\cdot, \cdot) = p(\cdot, x, \cdot)$, $x \in X$. For any $F(t, E)$ in S, moreover, there is a sequence $x_n = X(\tau_n)$ for which the convergence (1.8) holds, and passing to the limit extends the earlier relation to all of S. Therefore

$$\int_0^\infty e^{-\mu t} F(t, E_i)\, dt$$

is measurable over $[\mathcal{B}(Y_\rho) \cap S]$, and hence as before

$$\int_0^\infty e^{-\mu t} \int_X F(t, dz) f(z)\, dt$$

is measurable for bounded, $\mathcal{B}$-measurable functions f. It is therefore sufficient to show that the set $\{F : F(t, E_1) \in R\}$ can be defined by measurable operations on the Laplace transforms

$$\int_0^\infty e^{-\mu t} \int_X F(t, dz) f(z)\, dt, \qquad \mu > 1.$$

For this purpose we use the inversion formula

$$\begin{aligned} &\lim_{k\to\infty} (-1)^k \frac{(k\tau^{-1})^{k+1}}{k!} \frac{d^k}{d\mu^k} \int_0^\infty e^{-\mu t'} F(t', E')\, dt' \Big]_{\mu = k\tau^{-1}} \\ (1.14) \qquad &= \lim_{k\to\infty} \frac{(k\tau^{-1})^{k+1}}{k!} \int_0^\infty e^{-(k\tau^{-1})t'} (t')^k F(t', E')\, dt' \\ &= F(\tau, E') \quad \text{a.e. in } (\tau > 0) \quad \text{[Widder, 11, p. 288].}^7 \end{aligned}$$

Since convergence in (1.14) is bounded for τ in (δ, t), $0 < \delta < t < \infty$, we have for each simple function $G(\tau, y)$ over $\mathcal{R} \times \mathcal{B}$ the identity

$$\begin{aligned} (1.15) \qquad &\int_\delta^t \int_X G(t - \tau, y) F(\tau, dy)\, d\tau \\ &= \lim_{k\to\infty} \int_\delta^t \int_X \int_0^\infty \frac{(k\tau^{-1})^{k+1}}{k!} e^{-(k\tau^{-1})t'} (t')^k F(t', dy) G(t - \tau, y)\, dt'\, d\tau. \end{aligned}$$

Approximating $p(t - \tau, y, E_1)$ uniformly by simple functions, we see that (1.15) holds with it in place of $G(t - \tau, y)$. By using the fact that

$$F(t, E_1) = \frac{1}{t - \delta} \int_\delta^t \int_X p(t - \tau, y, E_1) F(\tau, dy)\, d\tau,$$

[7] Note that a $k!$ is missing in Definition 6, p. 288 of [11].

it follows that

$$F(t, E_1) = \frac{1}{t-\delta} \lim_{k\to\infty} \int_\delta^t \int_X \int_0^\infty \frac{(k\tau^{-1})^{k+1}}{k!} e^{-(k\tau^{-1})t'} (t')^k F(t', dy) \cdot p(t-\tau, y, E_1)\, dt'\, d\tau. \tag{1.16}$$

For fixed k the first term of (1.14) is clearly measurable over $\mathcal{R} \times [\mathcal{B}(Y_\rho) \cap S]$ as a function of (τ, F). It follows upon approximating $p(t-\tau, y, E_1)$ once more by simple functions that (1.16) is measurable over $[\mathcal{B}(Y_\rho) \cap S]$, as was to be shown. This completes the proof of Theorem 1.3.

The proof also establishes a fact which may be of some independent interest, and which we therefore state as a corollary.

COROLLARY. *Let X be a space, let $\mathcal{B}$ be a σ-field of subsets of X, and let $p(t, x, E)$ be a transition function on $(X, \mathcal{B})$, measurable in (t, x). Then for each $\lambda > 0$ the σ-subfield of $\mathcal{B}$ generated by the sets $\{x : p(t, x, E) \in R\}$, $t > 0$, $E \in \mathcal{B}$, $R \in \mathcal{R}$, is the same as that generated by the sets*

$$\left\{x : \int_0^\infty e^{-\lambda t} p(t, x, E)\, dt \in R\right\}, \qquad E \in \mathcal{B}, \quad R \in \mathcal{R}.$$

In fact, the proof applies directly to show the equivalence of the former field with that generated by the resolvent R_μ, $\mu > 1$. By the resolvent equation (1.7) this field is generated by a single R_λ, and the corollary follows.[8]

In view of Theorems 1.2 and 1.3, the close connection of $Y_\rho(t)$ with the process $Y(t)$ in [7] is evident. Theorem 1.2 means that for each stopping time T the two processes are equal a.s. in the sense of identification on the set S, while Theorem 1.3 proves a similar connection between the respective state spaces and σ-fields. The conclusions of [7] may therefore be applied to $Y_\rho(t)$, and the result stated as a theorem:

THEOREM 1.4. *$Y_\rho(t)$ is a strong Markov process with right continuous path functions, relative to the fields $H^+(t)$ (Section 1), the metric state space Y_ρ, and the conditional probabilities (2.19 a, b) of [7] transferred to $(Y_\rho, \mathcal{B}(Y_\rho))$ by means of Theorem 1.3. Except for t in a countable set, $P\{X(t) = Y_\rho(t)\} = 1$.*

Section 2

To place the processes $Y_\rho(t)$ on an equal footing analytically with the processes $X(t)$ it is necessary to find a transition function on Y_ρ corresponding to $p(t, x, E)$ on X. In this connection it is frequently possible to use the results of Ray [9], which we mention very briefly from this viewpoint. In [9] a method of regularizing a process is introduced which proceeds in two stages. A resolvent family R_λ, $\lambda > 0$, acting on the bounded measurable functions

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[8] I wish to thank the referee for several very useful comments, in particular, for this application of the resolvent equation, and for a considerable shortening of the proof of Theorem 2.1 below.

on $(X, \mathcal{B})$ is given. For our purposes it may be defined by

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} \int_X p(t, x, dz) f(z)\, dt.$$

A part of the domain is then singled out, the space X is completed in the uniform structure generated by this part, and the resolvent acting on these functions is extended to this completion to derive a resolvent acting on the space of all continuous functions on the completion. For this new resolvent there is defined a unique transition function such that the corresponding semi-group of operators on the continuous functions is continuous from the right. Then, for each Markov process (random function) on the completion having this transition function, there is defined (in the same way as in Definition 1.2) a unique strong Markov process, and this process has the same transition function as does its antecedent.

The critical point in applying this method in our situation is the requirement that the range of the resolvent acting on bounded measurable functions be contained in the uniformly continuous functions in the new topology. It is this requirement which leads to the condition of the next theorem.

THEOREM 2.1. *A necessary and sufficient condition that each process $Y_\rho(t)$ of Definition 1.2 with $\lambda = 1$ be one of the processes of [9, Theorem III] with a common transition function is that for each sequence $E_1 \supset E_2 \supset \cdots$, $\bigcap_{i=1}^\infty E_i$ empty, $E_i \in \mathcal{B}$, one has*

$$\lim_{i\to\infty} \int_0^\infty e^{-t} p(t, x, E_i)\, dt = 0, \qquad \textit{uniformly in } x \in X. \tag{2.1}$$

(2.1)[9]

Proof. We first prove the necessity. Let $E_1 \supset E_2 \supset \cdots$ be a sequence for which (2.1) is contradicted. Then there are an $\varepsilon > 0$ and a sequence $x_i \in X$ such that

$$\int_0^\infty e^{-t} p(t, x_i, E_i)\, dt > \varepsilon \qquad \text{for all } i.$$

If the processes $Y_\rho(t)$ were instances of [9, Theorem III], then the functions

$$\int_0^\infty e^{-t} p(t, x, E)\, dt, \qquad E \in \mathcal{B},$$

being in the range of R_1, would have continuous extensions on Y_ρ. Since Y_ρ is compact, there is a subsequence x_k of the sequence x_i such that $\lim_{k\to\infty} x_k = y$, $y \in Y_\rho$, exists. In this case, one would have that

$$\lim_{k\to\infty} \int_0^\infty e^{-t} p(t, x_k, E)\, dt$$

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[9] Although "usually" satisfied, this condition fails, for example, for the transition function $p(t, \{x\}, \{x\}) = 1$, $t > 0$, x real, as is seen by setting $E_i = (0, 2^{-i})$.

exists for each $E \in \mathcal{B}$. It follows then from the theorem of Vitali-Hahn-Saks [3] that this limit defines a probability measure on $\mathcal{B}$. But since for each E_i the limit is at least ε, this contradicts the fact that $\bigcap_i E_i$ is empty.

Turning to the sufficiency of (2.1), we must show first, in accordance with [9, p. 48], that there exists a collection $\mathfrak{M}_1$ of positive functions f measurable over $(X, \mathcal{B})$ which (a) separates points in X, (b) satisfies the relation $\lambda R_{\lambda+1} f \leqq f$ for all $\lambda > 0$, (c) is dense in the uniform norm among the range of R_1 acting on the positive bounded measurable functions, and (d) determines a uniform structure on X whose completion is Y_ρ (i.e., is homeomorphic to Y_ρ with X held fixed). We shall show that the functions $\sum_{i=1}^{\infty} a_i R_1(\chi_{E_i})$ form such a collection $\mathfrak{M}_1$, where the a_i range over the nonnegative rationals, in each function at most finitely many a_i are nonzero, the E_i are the sets appearing in Definition 1.1 of $\rho(x, y)$, and $\chi_{E_i}(x)$ is 1 or 0 according as $x \in E_i$ or $x \notin E_i$. Since requirements (a) and (b) are immediate, only (c) and (d) need be proved. By approximating bounded measurable functions by simple functions, and using the linearity and boundedness of R_1, it follows that in order to prove (c) it is enough to show that $R_1 \chi_E$, for $E \in \mathcal{B}$, may be uniformly approximated by functions in $\mathfrak{M}_1$. Let $\mathcal{C}$ be the class of indicator functions χ_E for which this holds. Then $\mathcal{C}$ contains χ_{E_i} for each i, and we need only prove it closed under monotone limits. Let $F_1 \subset F_2 \subset \cdots$ be a sequence with $\chi_{F_k} \in \mathcal{C}$, $1 \leqq k < \infty$; then by (2.1) there is for each $\varepsilon > 0$ an i such that

$$\int_0^\infty e^{-t}\, p(t, x, \textstyle\bigcup_{k=1}^{\infty} F_k - F_i)\, dt < \frac{\varepsilon}{2} \qquad \text{for all } x.$$

Since $\chi_{F_i} \in \mathcal{C}$, there is an $f \in \mathfrak{M}_1$ for which

$$\left| \int_0^\infty e^{-t}\, p(t, x, F_i)\, dt - f(x) \right| < \frac{\varepsilon}{2} \qquad \text{for all } x.$$

It follows that

$$\left| \int_0^\infty e^{-t}\, p(t, x, \textstyle\bigcup_{k=1}^{\infty} F_k)\, dt - f(x) \right| < \varepsilon.$$

Hence if $F = \bigcup_{k=1}^{\infty} F_k$, then $\chi_F \in \mathcal{C}$. Next, take $F_1 \supset F_2 \supset \cdots$, $\chi_{F_k} \in \mathcal{C}$. Then by considering $F_i - \bigcap_{k=1}^{\infty} F_k$ for i large, we find, analogously to the previous case, that $\chi_F \in \mathcal{C}$ for $F = \bigcap_{k=1}^{\infty} F_k$. Hence (c) is established. Requirement (d) follows from the observation that each of the functions of $\mathfrak{M}_1$ is uniformly continuous for $\rho(x, y)$, together with the fact that every open sphere of Y_ρ contains an open set of the topology generated by the uniform structure. To prove the latter, one need only use finite intersections of sets of the form $\{y : R_1 \chi_{E_i}(y) - R_1 \chi_{E_i}(x) < \varepsilon_i\}$, as follows from the definition of $\rho(x, y)$ with $\lambda = 1$. The requirements (a)–(d) are thus met, and it follows that R_λ may be extended by continuity to the continuous functions on Y_ρ to produce the situation of [9, Theorem III]. Let $p_\rho(t, y, E)$, $y \in Y_\rho$, $E \in \mathcal{B}(Y_\rho)$, be the resulting transition function. Then by

[9], $p_\rho(t, y, E)$ is the unique transition function with this (extended) resolvent and for which

$$\int_{Y_\rho} f(z)p_\rho(t, y, dz)$$

is continuous from the right in t, $t > 0$.

The proof of Theorem 2.1 will be completed by showing that $p_\rho(t, y, E)$ is a transition function for each process $Y_\rho(t)$. In other words, it will be shown that for each t', $t > 0$ and $E \in \mathcal{B}(Y_\rho)$,

$$p_\rho(t, Y_\rho(t'), E) = P(\{Y_\rho(t' + t) \in E\} \mid Y_\rho(t')) \quad \text{a.s.} \tag{2.2}$$

Since $Y_\rho(t)$ is continuous from the right, it is sufficient to prove for f continuous, $t' > 0$, and A in $H^+(t')$, the following identity of Laplace transforms:

$$\begin{aligned} &\int_A \int_0^\infty e^{-\lambda t} \int_{Y_\rho} p_\rho(t, Y_\rho(t'), dz)f(z)\, dt\, dP \\ &\qquad = \int_A \int_0^\infty e^{-\lambda t} f(Y_\rho(t' + t))\, dt\, dP. \end{aligned} \tag{2.3}$$

If A is in $F(t')$ and $P\{Y_\rho(t') = X(t')\} = 1$, then this identity is a consequence of the analogous identity

$$\int_A \int_0^\infty e^{-\lambda t} \int_X p(t, X(t'), dz)f(z)\, dt\, dP = \int_0^\infty e^{-\lambda t} \int_A f(X(t' + t))\, dP\, dt,$$

in view of the fact that in the last term $X(t' + t)$ may be replaced by $Y_\rho(t' + t)$ because they are a.s. equal except at countably many t, and in the first term the resolvent for $Y_\rho(t)$ is the extension of that for $X(t)$. Without the assumption that $P\{Y_\rho(t') = X(t')\} = 1$, and for A in $H^+(t')$, we can find a decreasing sequence t_n with limit t' such that $P\{Y_\rho(t_n) = X(t_n)\} = 1$ for all n, and since $A \in \mathfrak{F}(t_n)$ for each n, (2.3) holds with t_n in place of t'. By letting $n \to \infty$, and using the right continuity in t of $Y_\rho(t)$ and in y of the resolvent

$$\int_0^\infty e^{-\lambda t} \int_{Y_\rho} f(z)p_\rho(t, y, dz)\, dt,$$

it follows that (2.3) holds without restriction, and the proof of Theorem 2.1 is finished.

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University of Minnesota
Minneapolis, Minnesota