DISJOINT PAIRS OF SETS AND INCIDENCE MATRICES

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In a recent paper [1] investigating the repeated appearance of zeros in the powers of a matrix the following purely combinatorial problem arose. Let $x_1, \cdots, x_n$ be n distinct objects, and let $S_1, \cdots, S_n$ be n subsets of these objects satisfying the following two conditions:

(a) For no s, $s = 1, \cdots, n - 1$, does the union of some s of the sets $S_1, \cdots, S_n$ contain s or fewer elements.

(b) No two of the sets S_i intersect in precisely one of the x_j.

Question: What is the maximum possible number $w(n)$ of nonintersecting pairs of sets, S_p and S_q, $1 \leqq p < q \leqq n$?

As usual we reformulate the problem in terms of the incidence matrix of the configuration: let A be an n-square $(0, 1)$-matrix whose (i, j) entry is 1 or 0 according as x_j belongs to S_i or not. The conditions (a) and (b) simply state respectively that A has no $s \times (n - s)$ submatrix of zeros, $s = 1, \cdots, n - 1$, and AA' has no entry equal to 1.

An n-square matrix is said to be *partly decomposable* if it contains an $s \times (n - s)$ zero submatrix for some s. Otherwise it is called *fully indecomposable.* Let $\Omega(n)$ denote the totality of fully indecomposable n-square $(0, 1)$-matrices such that AA' contains no entries equal to 1. Let $z(M)$ denote the number of zeros in a matrix M. Then the number of zeros above the main diagonal in AA', where A is fully indecomposable, is $z(AA')/2$. In our problem we consider the number

$$w(n) = \max_{A \in \Omega(n)} z(AA')/2,$$

i.e., the maximum number of zeros above the main diagonal in AA' as A varies over $\Omega(n)$. Marcus and May obtained in [1] the following results:

$$w(2) = w(3) = 0, \qquad w(4) = 1, \qquad w(5) = 2;$$

$$w(n) < n(n - 3)/2 \qquad \textit{for } n \geqq 4;$$

$$w(n) \geqq n(n - 6)/2 \qquad \textit{if } n \textit{ is even},$$

$$w(n) \geqq (n(n - 6) - 3)/2 \quad \textit{if } n \textit{ is odd}.$$

The main result of the present paper is Theorem 6 which states:

$$w(n) = n(n - 4)/2 \qquad \textit{if } n \textit{ is even and } n \geqq 6,$$

$$w(n) = (n(n - 4) - 3)/2 \quad \textit{if } n \textit{ is odd and } n \geqq 7.$$

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THEOREM 1. *A fully indecomposable n-square matrix can contain $n(n-2)$, but no more than $n(n-2)$, zero entries.*

Proof. Clearly a row of a fully indecomposable matrix cannot contain more than $n-2$ zeros, and hence the matrix cannot contain more than $n(n-2)$ zeros. It remains to show that this bound can be achieved. Consider the matrix $I+P$, where I is the n-square identity matrix and P is the n-square permutation matrix with ones in the superdiagonal. It contains $n(n-2)$ zeros. We assert that it is fully indecomposable. For suppose that Z is an $s \times t$ zero submatrix of $I+P$, and that the entries of Z are located in the rows of $I+P$ numbered $i_1, \cdots, i_s$, where $i_1 < \cdots < i_s$. If $i_s \neq n$, the columns numbered $i_1, i_1+1, i_2, i_2+1, \cdots, i_s, i_s+1$ cannot contain entries of Z. But $i_1 < i_1+1 \leqq i_2 < i_2+1 \leqq \cdots \leqq i_s < i_s+1$, and at least $s+1$ of these numbers are distinct. Therefore $t \leqq n-(s+1)$, i.e.,

$$s+t \leqq n-1.$$

If $i_s = n$, then the columns numbered $1, i_1, i_1+1, i_2, i_2+1, \cdots, i_s$ cannot contain entries of Z. If $i_1 \neq 1$, then the number of distinct columns is, as before, at least $s+1$. If $i_1 = 1$, then $i_{\alpha+1} \neq i_\alpha + 1$ for some α $(1 \leqq \alpha \leqq s-1)$, and thus at least $s+1$ of the numbers $i_1, i_1+1, i_2, i_2+1, \cdots, i_s$ are distinct, and, as before, $s+t \leqq n-1$.

Alternatively we can show that $I+P$ is fully indecomposable by applying the techniques used in the proof of Theorem 4 in [1].

COROLLARY. *Let B be any nonnegative n-square matrix. Then $I+P+B$ is fully indecomposable.*

THEOREM 2. *If A is a fully indecomposable nonnegative matrix, and if AA' contains an $s \times t$ zero submatrix, then $s+t$ does not exceed $n-2$.*

Proof. Assume for simplicity that the $s \times t$ zero submatrix is in the right top corner of AA'. Let the submatrix consisting of the first s rows of A have exactly k zero columns. Since A is fully indecomposable, $s+k \leqq n-1$. Now, each of the first s rows of A is orthogonal to each of the last t rows, and therefore all columns of A containing a nonzero entry in any of their first s positions must have zeros in all of their last t positions. Thus the last t rows contain a $t \times (n-k)$ zero submatrix. Since A is fully indecomposable, $t+n-k \leqq n-1$. Therefore $t \leqq k-1$ and $s+t \leqq n-2$.

COROLLARY 1. *If A is a fully indecomposable nonnegative matrix, then AA' is fully indecomposable.*

COROLLARY 2. *If A is a fully indecomposable nonnegative n-square matrix, then no row of AA' can have more than $n-3$ zero entries.*

THEOREM 3. *If A is a fully indecomposable nonnegative matrix and AA' has t $(t > 0)$ zeros in its i^{th} row, then the number of zeros in the i^{th} row of A is greater than t.*

Proof. Suppose that the i^{th} row of A contains s zeros (which can be assumed to be in the last s columns of A). Since AA' has t zeros in its i^{th} row the i^{th} row of A is orthogonal to t rows of A, and therefore the first $n - s$ entries in these t rows are all equal to 0. Hence the first $n - s$ columns of A contain a $t \times (n - s)$ zero submatrix. Since A is fully indecomposable, $t + n - s < n$, and therefore $t < s$.

THEOREM 4.

$$w(n) \geqq n(n-4)/2 \qquad \text{if } n \text{ is even},$$

$$w(n) \geqq (n(n-4)-3)/2 \quad \text{if } n \text{ is odd}.$$

Proof. The theorem is trivial for $n < 5$. It holds for $n = 5$ since $w(5) = 2$. For $n \geqq 6$ we exhibit for any even n a matrix $A \in \Omega(n)$ such that $z(AA') = n(n - 4)$ and for any odd n a matrix $B \in \Omega(n)$ such that $z(BB') = n(n - 4) - 3$.

(i) $n = 2k$. Let A be an n-square (0, 1)-matrix with ones in the following positions:

$$(i, 2i-1), \quad (i, 2i), \qquad i = 1, \cdots, k;$$

$$(i, 2(i-k)-1), \quad (i, 2(i-k)), \quad (i, 2(i-k)+1), \quad (i, 2(i-k)+2),$$
$$i = k+1, \cdots, 2k-1;$$

$$(n, 1), \quad (n, 2), \quad (n, n-1), \quad (n, n),$$

and zeros elsewhere, i.e.,

$$A = \begin{pmatrix} 1\,1 & 0\,0 & 0\,0 & \cdots & 0\,0 & 0\,0 \\ 0\,0 & 1\,1 & 0\,0 & \cdots & 0\,0 & 0\,0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0\,0 & 0\,0 & 0\,0 & \cdots & 1\,1 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & \cdots & 0\,0 & 1\,1 \\ 1\,1 & 1\,1 & 0\,0 & \cdots & 0\,0 & 0\,0 \\ 0\,0 & 1\,1 & 1\,1 & \cdots & 0\,0 & 0\,0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0\,0 & 0\,0 & 0\,0 & \cdots & 1\,1 & 1\,1 \\ 1\,1 & 0\,0 & 0\,0 & \cdots & 0\,0 & 1\,1 \end{pmatrix}.$$

Each of the top k rows is orthogonal to $n - 3$ rows, while each of the bottom k rows of A is orthogonal to $n - 5$ rows. Hence $z(AA') = k(n-3) + k(n-5) = n(n-4)$. Clearly all nonzero entries in AA' are equal to 2 or 4. We assert that A is fully indecomposable and thus $A \in \Omega(n)$. Let Q be the n-square permutation matrix with ones in the following positions:

$$(2i-1, i), \qquad (2i, k+i), \qquad i = 1, \cdots, k.$$

Then $QA = I + P + B$ where B is a (0, 1)-matrix, and, by the Corollary to Theorem 1, the matrix QA, and therefore A, is fully indecomposable.

(ii) $n = 2k + 1$. Let A be the $(2k)$-square $(0, 1)$-matrix described in (i), and let

$$B = \left(\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & 0 \\ & & A & & & \vdots \\ & & & & & 0 \\ & & & & & 1 \\ \hline 0 & 0 & \cdots & 0 & 1 \quad 1 & 1 \end{array}\right),$$

so that the only nonzero entries in the last row and in the last column of B are in the $(n-1, n)$, $(n, n-2)$, $(n, n-1)$, and (n, n) positions. We show that B is fully indecomposable. Let Z be an $s \times t$ zero submatrix of B. If Z has no entries in A, it must be fully contained in the last row or in the last column of B, and then $s + t < n$. If Z has entries in A, then A contains an $(s-1) \times t$ or an $s \times (t-1)$ zero submatrix, and since A is a fully indecomposable $(n-1)$-square matrix, $s + t - 1 < n - 1$, i.e., $s + t < n$. Hence B is fully indecomposable.

This can be also proved directly by a method similar to the one employed in case (i). Clearly BB' has no ones in it, and therefore $B \in \Omega(n)$. The off-diagonal entries in the top left principal $(n-1)$-square submatrix of BB' are those of AA' which contains $(n-1)(n-5)$ zeros. In addition to these, BB' contains $n - 4$ zeros in its last row and $n - 4$ zeros in its last column. Therefore $z(BB') = n(n-4) - 3$.

THEOREM 5. $w(6) = 6$, $w(7) = 9$.

Proof. By Theorem 4, $w(6) \geqq 6$ and $w(7) \geqq 9$. We show that the assumptions $w(6) > 6$ and $w(7) > 9$ lead to contradictions. Suppose then that there exists a matrix $S \in \Omega(6)$ such that $z(SS') \geqq 14$. Since, by Corollary 2 to Theorem 2, no row of SS' can have more than 3 zeros, at least two rows of SS' have exactly 3 zeros, and therefore at least two rows of S have exactly 2 ones. There exist therefore permutation matrices P, Q such that

$$PSQ = \left(\begin{array}{cc|cccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & & & & \\ 1 & 1 & & & & \\ 0 & 0 & & X & & \\ 0 & 0 & & & & \end{array}\right),$$

where X is a 4-square matrix whose first two columns are, with a suitable choice for P and Q, either

$$\text{(a)} \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \text{(b)} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that the 4 zeros in the first two columns of X cannot all be in the bottom two rows, for the last two rows of PSQ would then contain a 2×4 zero submatrix. Now, SS' is assumed to have at least 14 zeros. Therefore, by Theorem 3, S must have at least 20 zeros, and the last two columns of X must contain at least 4 zeros. If X is of the form (a), none of these 4 zeros can be in the last row, and only the second row of X can have both its entries in the last two columns equal to zero. But then exactly one of the last two entries in the first row of X, i.e., the third row of PSQ, is zero, and the $(3, 6)$ entry in $(PSQ)(PSQ)'$ is equal to 1. This contradicts the assumption that $S \in \Omega(6)$.

If X is of the form (b), then at least one of the last two entries in each row of X must be nonzero. Since the last two columns of X must contain at least 4 zeros, exactly one of the last two entries in each row of X must be zero. Moreover, these zeros, with a suitable choice for Q, must be the $(3, 5)$, $(4, 5)$, $(5, 6)$, $(6, 6)$ entries in PSQ; otherwise $(PSQ)(PSQ)'$ would contain entries equal to 1. But then PSQ contains a 3-square zero submatrix and is therefore partly decomposable. Contradiction.

We now prove that $w(7) = 9$. Since, by Theorem 4, $w(7) \geqq 9$, it suffices to prove that $T \in \Omega(7)$ and $z(TT') > 18$ are contradictory. Suppose then that there exists a matrix $T \in \Omega(7)$ such that $z(TT') \geqq 20$. We shall consider three cases.

(i) TT' has at least two rows with 4 zeros in each, i.e., T has at least two rows with 5 zeros in each. There exist therefore permutation matrices P, Q such that

$$PTQ = \left(\begin{array}{cc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & & & & & \\ 1 & 1 & & & & & \\ 0 & 0 & & & Y & & \\ 0 & 0 & & & & & \\ 0 & 0 & & & & & \end{array}\right),$$

where for a suitable choice of P the first two columns of Y are either

$$\text{(a)} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \text{(b)} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

If (a) is the case, let $W = Y + E_{11} + E_{12}$, where E_{ij} is the matrix of appropriate order with 1 in the (i, j) position and zeros elsewhere. Then W is fully indecomposable. For if W were partly decomposable, then PTQ, and therefore T, would also be partly decomposable. Thus $z(WW') \leqq 4$. Now let $B = PTQ + E_{33} + E_{34}$. Then $z(TT') = z(BB') + 2$ as $(PTQ)(PTQ)'$ has zeros in the $(2, 3)$ and $(3, 2)$ positions, while the corresponding entries in

BB' are equal to 2, and, it is easily seen, all other zeros in $(PTQ)(PTQ)'$ and in BB' match. But $z(BB') = 8 + 4 + z(WW') \leqq 16$, and therefore $z((PTQ)(PTQ)') = z(TT') \leqq 18$. Contradiction.

Note that the same method could have been used to prove that $z(SS')$ cannot exceed 12 if PSQ is of the form (a). We shall also use the same technique in the proof of our main theorem.

If Y is of the form (b) we can use the method employed in the previous case setting $W = Y + E_{11} + E_{12}$ or $Y + E_{21} + E_{22}$, etc. Alternatively we observe that if Y is of this form, then we can choose two permutation matrices P and Q such that

$$PTQ = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & y \\ 1 & 1 & 0 & 0 & 1 & 1 & y \\ 0 & 0 & 1 & 1 & 1 & 1 & y \\ 0 & 0 & 1 & 1 & 1 & 1 & y \\ 0 & 0 & 0 & 0 & 1 & 1 & y \end{pmatrix},$$

where at least two of the entries marked y are equal to 1. But then $z(TT') = z((PTQ)(PTQ)') = 14$.

(ii) TT' has exactly one row with 4 zeros; the corresponding row of T has 5 zeros. If T has another row with 5 zeros, then there exist permutation matrices P and Q such that

$$PTQ = \left(\begin{array}{cc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & & & & & \\ 1 & 1 & & & & & \\ 0 & 0 & & & Y & & \\ 0 & 0 & & & & & \\ 0 & 0 & & & & & \end{array}\right),$$

where the first two columns of Y are one of

$$\text{(a)} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \text{(b)} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \text{(c)} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If Y is either of the form (a) or of the form (b), then we show that $z(TT')$ cannot exceed 18 as above in the case (i). If Y is of the form (c), then it is fully indecomposable, for otherwise T itself would be partly decomposable. Therefore $z(YY') \leqq 4$, and

$$z(TT') = z((PTQ)(PTQ)') = 8 + 4 + z(YY') \leqq 16.$$

Suppose now that all rows of T but one have less than 5 zeros each. Then at least four of them must have 4 zeros. We can again find permutation matrices P and Q such that

$$PTQ = \left(\begin{array}{cc|ccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & & & & & \\ 1 & 1 & & & & & \\ 0 & 0 & & & Y & & \\ 0 & 0 & & & & & \\ 0 & 0 & & & & & \end{array}\right).$$

Clearly neither the third nor the fourth row of PTQ can have 4 zeros. For if the first or the second row of Y had a single nonzero entry, then either one entry of TT' would be equal to 1, or, if each row had a single nonzero entry in the same column as the other, PTQ would contain a 3×4 zero submatrix. Thus each of the last three rows must have exactly three ones, at least two of which must be in the first three columns of Y. Observe that the last three rows cannot be identical; otherwise they would contain a 3×4 zero submatrix. Therefore at least one of the last two columns, say the last, has exactly one nonzero entry in the last three rows. Now, neither the third nor the fourth row of PTQ can have a nonzero entry in the last column, for in that case TT' would contain an entry equal to 1. Therefore the last column of PTQ has only one nonzero entry, and T is partly decomposable. Contradiction.

(iii) TT' has no rows with 4 zeros. If T has a row with 5 zeros, then the proof is similar to the proof of the case (ii). If no row of T has 5 zeros, then at least six rows must have 4 zeros each. Now these rows must either be orthogonal to each other or have an inner product equal to 2 or 3. Moreover no three of them can be identical, for PTQ would then contain a 3×4 zero submatrix. Let P and Q be permutation matrices such that

$$PTQ = \left(\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & & T_1 & & T_2 \\ \hline & & & 0 & 0 & 0 & \\ & T_3 & & 0 & 0 & 0 & T_4 \\ \hline & & & T_5 & & & \end{array}\right).$$

Now, neither T_2 nor T_4 can be a zero matrix; otherwise PTQ would contain a 3×4 zero submatrix. But if T_2 and T_4 have nonzero entries, say in the third and in the fifth row of PTQ respectively, then the $(3, 5)$ entry of $(PTQ)(PTQ)'$ is equal to 1. Contradiction.

THEOREM 6.

$$w(n) = n(n-4)/2 \qquad \textit{if } n \textit{ is even and } n \geqq 6,$$
$$w(n) = (n(n-4)-3)/2 \quad \textit{if } n \textit{ is odd and } n \geqq 7.$$

Proof. Let

$$f(n) = n(n-4)/2 \qquad \text{if } n \text{ is even,}$$
$$f(n) = (n(n-4)-3)/2 \quad \text{if } n \text{ is odd.}$$

Since, by Theorem 4, $w(n) \geqq f(n)$ for $n \geqq 6$, it remains to prove that $w(n) > f(n)$ is impossible. We use induction on n. By Theorem 5, $w(6) = f(6)$ and $w(7) = f(7)$. Suppose now that for some $n > 7$ there exists a matrix $A \in \Omega(n)$ such that $z(AA') > 2f(n)$. Observe that if n is even, then $z(AA') \geqq n(n-4)+2$, and AA' must contain at least two rows with $n-3$ zeros in each. If n is odd, there is apparently a possibility of AA' containing less than two such rows. We shall first assume that AA' has at least two rows with exactly $n-3$ zeros and therefore that A has at least two rows each with exactly two entries equal to 1, without assuming anything about the parity of n. Let P and Q be permutation matrices such that

$$PAQ = \left(\begin{array}{cc|cccccc} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 \\ \hline 1 & 1 & & & & & \\ 1 & 1 & & & & & \\ 0 & 0 & & & X & & \\ \vdots & \vdots & & & & & \\ 0 & 0 & & & & & \end{array}\right),$$

where the first two columns of X are either

$$\text{(a)} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \text{(b)} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

If (a) is the case, let $Y = X + E_{11} + E_{12}$ and $B = PAQ + E_{33} + E_{34}$. Then Y is a fully indecomposable $(n-2)$-square matrix and $z(YY') \leqq 2f(n-2)$ by the induction hypothesis. Also $z(AA') \leqq z(BB') + 4$ as $(PAQ)(PAQ)'$ has zeros in the $(2, 3)$, $(3, 2)$, and possibly in the $(3, 5)$, $(5, 3)$ positions, while the corresponding entries BB' are equal to 2; all other zeros in $(PAQ)(PAQ)'$ match those in BB'. Now

$$\begin{aligned} z(BB') &= 2(n-3) + 2(n-5) + z(YY') \\ &\leqq 2f(n-2) + 4n - 16. \end{aligned}$$

Therefore

$$z(AA') \leqq 2f(n-2) + 4n - 12.$$

If n is even,

$$z(AA') \leqq (n-2)(n-6) + 4n - 12 = n(n-4) = 2f(n),$$

and if n is odd,

$$z(AA') \leqq (n-2)(n-6) - 3 + 4n - 12 = n(n-4) - 3 = 2f(n).$$

Thus in both cases $z(AA') \leqq 2f(n)$, contradicting our assumption that $z(AA') > 2f(n)$.

If X is of the form (b), we assert that it can always be reduced, with a suitable choice for P and Q, to the form (a). Suppose that this cannot be done and that AA' has exactly t rows with $n - 3$ zero entries in each. Clearly t cannot exceed $n/3$. We can assume therefore that the 1^{st}, $4^{\text{th}}, \cdots$, $(3t-2)^{\text{th}}$ rows of AA' contain $n - 3$ zeros each. Then there exist permutation matrices P and Q such that PAQ has ones in the following positions:

$$\begin{array}{lll} (3i-2, 2i-1), & (3i-2, 2i), & \\ (3i-1, 2i-1), & (3i-1, 2i), & \\ (3i, 2i-1), & (3i, 2i), & i = 1, \cdots, t; \end{array}$$

and all other entries in the first $2t$ columns are equal to 0. Note that the $(3i-1)^{\text{th}}$ and the $(3i)^{\text{th}}$ $(i = 1, \cdots, t)$ rows must have at least 4 ones each; otherwise A could not be fully indecomposable, or AA' would have an entry equal to 1. Thus the three rows of AA', the $(3i-2)^{\text{th}}$, the $(3i-1)^{\text{th}}$, and the $(3i)^{\text{th}}$, have between them at most $(n-3) + 2(n-5) = 3n - 13$ zeros. Therefore the first $3t$ rows of AA' contain at most $t(3n-13)$ zeros, and since $z(AA')$ is assumed to be greater than $n(n-4) - 3$, the remaining $n - 3t$ rows must contain at least

$$n(n-4) - 1 - t(3n-13) = (n-3t)(n-4) + t - 1$$

zeros. In this part of the proof we assume that $t \geqq 2$. Hence at least $t - 1$ rows among the last $n - 3t$ rows must have $n - 3$ zeros each. This contradicts our assumption that AA' has exactly t such rows.

To conclude the proof we show that if $A \in \Omega(n)$, n is odd, greater than 7, and $z(AA') \geqq n(n-4) - 1$, then AA' must have at least two rows with $n - 3$ zeros in each. Suppose that AA' has only one row with $n - 3$ zeros. Let P and Q be permutation matrices such that

$$PAQ = \left(\begin{array}{cc|c} 1 & 1 & 0 \quad 0 \quad \cdots \quad 0 \\ \hline 1 & 1 & \\ 1 & 1 & \smash{A_1} \\ \hline 0 & 0 & \\ 0 & 0 & \\ \vdots & \vdots & A_2 \\ 0 & 0 & \end{array}\right).$$

Then, as above, each row of A_1 must have at least two entries equal to 1. Therefore AA' has at most $(n-3)+2(n-5)=3n-13$ zeros in the first three rows, and the remaining $n-3$ rows must have at least $n(n-4)-1-(3n-13)=(n-3)(n-4)$ zeros. Thus, unless AA' has another row with $n-3$ zeros, the second and the third row of AA' have exactly $n-5$ zeros each, and each of the last $n-3$ rows has exactly $n-4$ zeros. It follows that each of the last $n-3$ rows of A has at most 3 ones in it and a nonzero inner product with exactly three other rows. Since $n \geqq 9$, there must be at least two rows of A orthogonal to each of the first three rows. Suppose that the last row is orthogonal to all but the last four rows. Not all these rows can contain only 2 ones since this would imply that the last four rows of A contain a $4 \times (n-2)$ zero submatrix. Let the last row have 3 ones in the last three columns. Then at least two of the nonzero entries in each of the other three rows must be confined to the last three columns. The remaining nonzero entry, if any, may be in another column, but the column must be the same for all three rows. In any case the last four rows of A contain a $4 \times (n-4)$ zero submatrix.

We now show that the alternative that AA' has no rows with $n-3$ zeros is also untenable. Suppose then that $z(AA') \geqq n(n-4)-1$ and no row of AA' contains more than $n-4$ zeros. Since the number of zeros in AA' clearly must be even, one row of AA', say the first, contains $n-5$ zeros, while each of the remaining rows has exactly $n-4$ zeros. But then the first row of A has at most 4 ones and an inner product greater than 1 with four other rows each of which contains at most 3 ones and has an inner product greater than 1 with exactly three other rows. This is impossible unless A is partly decomposable.

A matrix A is *decomposable* if there exists a permutation matrix P such that PAP' is a subdirect sum; otherwise A is *indecomposable.*

Using the methods in [1] we can conclude with the following result:

THEOREM 7. *Let $N(n)$ be the set of n-square indecomposable normal matrices with distinct eigenvalues. Let $\tau(n)$ be the largest integer m for which there exists an $A \in N(n)$ with m positions (i, j), $1 \leqq i < j \leqq n$, such that $(A^k)_{ij} = 0$ for all positive integers k. Then*

$$\tau(2) = \tau(3) = 0, \qquad \tau(4) = 1, \qquad \tau(5) = 2,$$

and

$$\tau(n) \leqq n(n-4)/2 \qquad \text{if } n \text{ is even and } n \geqq 6,$$

$$\tau(n) \leqq (n(n-4)-3)/2 \qquad \text{if } n \text{ is odd and } n \geqq 7.$$

We omit the proof which is essentially contained in [1].

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