# ON THE HOMOLOGY DECOMPOSITION OF POLYHEDRA<sup>1</sup>

BY

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#### Introduction. Summary

Let  $X_1$  be a simply connected polyhedron (i.e., a simply connected finite CW-complex). According to Eckmann-Hilton [6]—see also Brown-Copeland [3]—it is homotopy-equivalent to a polyhedron X built up by subcomplexes  $X_i$ ,  $X_2 \subset X_3 \subset \cdots \subset X_N = X$ , where  $X_r$  is constructed out of  $X_{r-1}$  in a very perspicuous way by means of the  $r^{\text{th}}$  integer homology group of X and an element in a homotopy group of  $X_{r-1}$ . Following [6] we call  $X = \{X_r\}$  a normal polyhedron, and the collection  $\{X_r\}$  of the  $X_r$  a homology decomposition of X.

It is the purpose of this note to exemplify our opinion that the concept of the homology decomposition can be used profitably to study homotopy sets  $\Pi(X, Y)$  of the maps of a space X into a space Y.

All considerations rely on Proposition 2.2 which describes the circumstances under which a map  $f: X \to Y$  of the normal polyhedra  $X = \{X_r\}, Y = \{Y_r\}$ induces a map  $f_r: X_r \to Y_r$  compatible with f. Proposition 2.2 follows from Proposition 2.1, which generalizes the Blakers-Massey theorem on relative homotopy groups [2, p. 198].

Section 3 contains the first example of an application of the homology decomposition. Proposition 3.3 is a powerful lemma of Thom [10, p. 59], for which we give a new proof. The idea of our proof is to climb up a homology decomposition, using at each step known facts about homotopy groups of spheres.

From Section 4 on, we restrict our attention to "selfmaps"  $f: X \to X$  of a simply connected polyhedron X. The composition of maps defines in the homotopy set  $\Pi(X, X)$  a multiplication turning  $\Pi(X, X)$  into a monoid. Denote by T(X) the homotopy set of all selfmaps of X which induce the trivial endomorphism of  $\bigoplus_k H^k(X; H_k(X))$ . It is a multiplicatively closed subset of  $\Pi(X, X)$ . Theorem 4.2 states that T(X) is nilpotent. The order t(X) of nilpotency of T(X) is a homotopy invariant of X which, by appealing to a theorem of Novikov [8], can be shown to assume any given value for an appropriate X (Proposition 4.5).

An endomorphism  $\Phi$  of  $\bigoplus_k H^k(X; H_k(X))$  induced by a map  $f: X \to X$ satisfies necessarily a certain relation, and such a  $\Phi$  will be called admissible (Definition 4.6, Lemma 4.7). The question for which spaces X every admissible endomorphism of  $\bigoplus_k H^k(X; H_k(X))$  can be realized by a selfmap of X is dealt with in Theorem 4.9: For 2-connected X this is the case if and

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only if X is homotopy-equivalent to a wedge of Moore-spaces (polyhedra with exactly one nonvanishing integer homology group) and

$$\operatorname{Ext}(H_r, H_{r+1}) = 0$$

for all r.

In Section 5 we consider selfmaps of a suspension  $\Sigma X$ . Its natural comultiplication defines in  $\Pi(\Sigma X, \Sigma X)$  a group operation in addition to the multiplication by composition of maps, and  $\Pi(\Sigma X, \Sigma X)$  is a near-ring (i.e., a set with two binary operations coupled by only one law of distributivity; see e.g. [1]). By using the distributor series of a near-ring (Definition 5.1, compare [7]) Theorem 5.3 measures to what extent  $\Pi(\Sigma X, \Sigma X)$  fails to be a ring. It furthermore suggests the introduction of an integer-valued homotopy invariant d(X) of X which is equal to one in case  $\Pi(\Sigma X, \Sigma X)$  is a ring (Definition 5.2).

In the sense of the Eckmann-Hilton duality [4] everything carries over under considerable simplifications to spaces with a finite number of homotopy groups.

It is a pleasure to thank P. Olum for his scrutinizing questions and suggestions on the subject of this note. Our thanks are also due to J. C. Moore it was a remark of his on the "dual" situation which enabled us to give Theorem 4.2 its present form—, to P. J. Hilton for the example at the end of Section 5, and to the referee for the proof of the present version of Proposition 2.1, replacing a much weaker proposition of an earlier draft.

### 1. Notations. Definitions

(a) All spaces considered are polyhedra and have a basepoint 0 which is respected by maps  $f, g, \cdots$  and their homotopies  $f \sim g, \cdots$ . The trivial map  $X \to 0 \ \epsilon \ Y$  is also denoted by 0. A map  $f: X \to Y$  and its homotopy class in the homotopy set  $\Pi(X, Y)$  will usually not be distinguished.  $\Sigma$ stands for the suspension operators for spaces, maps, and homotopy sets. The space obtained by attaching the cone CA over A to X by means of the map (or class) f is denoted by  $CA \cup_f X$ .

(b) For Moore-spaces (polyhedra with exactly one nontrivial integer homology group in dimension r > 1) consult e.g. [9]. We denote them by K'(G, r) or simply  $K'_r$ ,  $L'_r$ . Their groups  $G, H, \cdots$  as well as all coefficient groups considered will be supposed to be finitely generated. The "homotopy groups of X with coefficients G" are defined as  $\Pi(K'(G, r), X)$  and are related to the ordinary homotopy groups  $\pi_r(X)$  by an exact sequence [5], "the coefficient formula for homotopy groups". For  $\Pi(X, K'(G, r))$ , the cohomotopy groups with coefficients G, and their coefficient formula, see [9].

(c) The inclusion

$$i: X_r \to X = CK'(G, r) \cup_{\alpha} X_r$$

is called an elementary cofibration (with attaching class or map  $\alpha$  and projection  $\phi: X \to X/X_r = K'(G, r+1)$ ) if

- (1)  $X_r$  is simply connected;
- (2) dim  $X_r \leq r + 1$  and  $H_{r+1}(X_r) = 0;$
- (3)  $i_*: H_k(X_r) \cong H_k(X)$  for  $k \leq r$ .

It follows from (3) that

$$\phi_*: H_{r+1}(X) \cong H_{r+1}(K'(G, r+1)) \cong G.$$

A normal polyhedron  $X = \{X_r\}$  is one obtained by a sequence

$$X_2 \to X_3 \to \cdots \to X_N = X$$

of elementary cofibrations. The groups of the cofibers  $X_r/X_{r-1}$  are the integer homology groups of X; if  $H_s(X) = 0$  for some s, we insert a term  $X_s = X_{s-1}$  so that we have  $X_r/X_{r-1} = K'(H_r(X), r)$  for all r. Denote by  $i_r: X_r \to X$  the imbedding;  $i_r^*: H^k(X; G) \to H^k(X_r; G)$ , G arbitrary, is isomorphic for  $k \leq r$  and epimorphic for  $k \geq r+1$ . The r<sup>th</sup> fundamental class  $h^r \in H^r(X; H_r(X))$  of the normal polyhedron  $X = \{X_r\}$  is defined as

$$h^{r} = i_{r}^{*-1} \phi_{r}^{*} h_{1}^{r}$$
,

where  $h_1^r$  is the fundamental class of  $H^r(K'(H_r(X), r); H_r(X))$ , and  $\phi_r$  the projection  $X_r \to K'(H_r(X), r)$ ; for arbitrary G the induced

$$\phi_r^*: H^r(K'(H_r(X), r); G) \to H^r(X_r; G)$$

is monomorphic. Distinguish  $X_r$  and the r-skeleton  $X^r$ .

### 2. Compression problems in cofibrations

For  $X = S^N$  the following proposition reduces to Theorem II of [2].

PROPOSITION 2.1. Let X be an N-dimensional polyhedron, and  $j: A \to Y$ a cofibration with projection  $\psi: Y \to Y/A$  such that

Y is 
$$(N - r)$$
-connected,  
Y/A is r-connected,  
 $N - r \le r$ .

Then the sequence of homotopy sets

$$\Pi(X,A) \xrightarrow{j_*} \Pi(X,Y) \xrightarrow{\psi_*} \Pi(X,Y/A)$$

is exact: If  $\psi_* f = 0$ , then f is compressible into A.

*Proof.*<sup>2</sup> Replace  $\psi$  by a fibration  $p: E \to Z_{\psi}$ , where  $\psi$  is the mapping

<sup>&</sup>lt;sup>2</sup> A proof is also possible by methods developed by I. Namioka in *Maps of pairs in* homotopy theory, Proc. London Math. Soc. (3), vol. 12 (1962), pp. 725-738.

cylinder of  $\psi$ , and E the space of all paths in  $Z_{\psi}$  which start in

$$Y = Y \times 0 \subset Z_{\psi}.$$

The fibre F of p consists of the paths in E ending at the basepoint of  $Z_{\psi}$ . Let  $\tau: Y/A \to Z_{\psi}$  be the imbedding, and  $\sigma: Y \to E$  the map defined by  $\sigma y = y \times I \ \epsilon E$ ;  $\sigma$  induces a map  $\rho: A \to F$ . Consider the homotopy sequences of the pairs (Y, A) and (E, F):

In view of  $\pi_k(E, F) \cong \pi_k(Z_{\psi})$  and the Blakers-Massey theorem,  $(\sigma, \rho)_*$  is isomorphic for  $k \leq N$  and epimorphic for k = N + 1. By the five-lemma  $\rho_*$  is epimorphic on  $\pi_N(A)$  and isomorphic in the lower dimensions. Hence the map  $\rho$  is N-connected:

$$\to \pi_N(A) \xrightarrow{\rho_*} \pi_N(F) \to \pi_N(\rho) \to \pi_{N-1}(A) \xrightarrow{\rho_*} \pi_{N-1}(F) \to .$$

Replace  $\rho$  by the inclusion of A into the mapping cylinder  $Z_{\rho}$  to infer that under these circumstances any map  $f: X \to F$  can be factored up to homotopy through  $\rho$  if dim  $X \leq N$ :



This means that  $\rho_*$  is epimorphic on  $\Pi(X, A)$ :

The lower line of this diagram is exact;  $\sigma_*$  and  $\tau_*$  are one-to-one correspondences, and  $\rho_*$  is epimorphic. Therefore the upper line is also exact.

We now apply Proposition 2.1 to maps of normal polyhedra.

PROPOSITION 2.2. Let  $f: X \to Y$  be a map of the normal polyhedra  $X = \{X_r\}, Y = \{Y_r\}, and let h^{r+1} \in H^{r+1}(Y; H_{r+1}(Y))$  be the  $(r + 1)^{st}$  fundamental class of  $Y = \{Y_r\}$  (see Section 1(c)). There exists a map  $f_r: X_r \to Y_r$  compatible with  $f, fi_r \sim j_r f_r$ :



if and only if

$$i_r^* f^* h^{r+1} = 0 \ \epsilon \ H^{r+1}(X_r \ ; H_{r+1}(Y)) \cong \operatorname{Ext}(H_r(X), H_{r+1}(Y)).$$

*Proof.* (a) Suppose there is a map  $f_r$  with  $fi_r \sim j_r f_r$ . Write

$$L_{r+1} = K'(H_{r+1}(Y), r+1),$$

and pass to cohomology with  $H_{r+1}(Y)$  as coefficients:

$$H^{r+1}(X) \xleftarrow{f^*} H^{r+1}(Y)$$

$$\downarrow j^*_{r+1}$$

$$i^*_r \qquad H^{r+1}(Y_{r+1}) \xleftarrow{\psi_r} H^{r+1}(L'_{r+1}).$$

$$\downarrow j^*_{r,r+1}$$

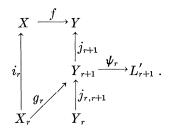
$$H^{r+1}(X_r) \xleftarrow{f^*_r} H^{r+1}(Y_r)$$

By definition  $h^{r+1} = j_{r+1}^{*-1} \psi_r^* h_1^{r+1}$ , where  $h_1^{r+1}$  denotes the fundamental class of  $H^{r+1}(L'_{r+1})$ . Together with  $i_r^* f_{j_{r+1}}^{***-1} = f_r^* j_{r,r+1}^*$  we have

$$i_r^* f^* h^{r+1} = i_r^* f^* j_{r+1}^{*-1} \psi_r^* h_1^{r+1} = f_r^* j_{r,r+1}^* \psi_r^* h_1^{r+1} = 0$$

because of  $\psi_r j_{r,r+1} = 0$ .

(b) By the cellular approximation theorem there exists a map  $g_r: X_r \to Y_{r+1}$  with  $fi_r \sim j_{r+1} g_r$ :



By hypothesis  $i_r^* f^* h^{r+1} = 0$ . Therefore, with  $h_1^{r+1}$  as in (a), we have

$$g_r^* \psi_r^* h_1^{r+1} = i_r^* f_{r+1}^{*j*-1} \psi_r^* h_1^{r+1} = 0.$$

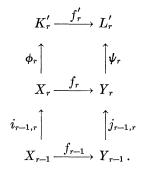
Because any map  $k: X_r \to L'_{r+1}$  is homotopic to zero if and only if  $k^* h_1^{r+1} = 0$ , we conclude  $\psi_r g_r \sim 0$ . By Proposition 2.1 it follows from  $\psi_r g_r \sim 0$  that  $g_r$ is compressible into  $Y_r$ , i.e., that there exists a map  $f_r: X_r \to Y_r$  with  $g_r \sim j_{r,r+1}f_r$ , and the proof of Proposition 2.2 is complete.

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COROLLARY 2.3. Let  $f: X \to Y$  be a map of the normal polyhedra  $X = \{X_r\}, Y = \{Y_r\}$ . Suppose that f induces the trivial homomorphism on  $\bigoplus_k H^k(Y; H_k(Y))$ . Then there exist maps  $f_r: X_r \to Y_r$  and  $f'_r: K'_r \to L'_r$  such that

 $f_r i_{r-1,r} \sim j_{r-1,r} f_{r-1}$  and  $f'_r \phi_r \sim \psi_r f_r$ 

for all elementary cofibrations  $i_{r-1,r}$  of X and  $j_{r-1,r}$  of Y:



*Proof.* Since  $i_r^* f^* h^{r+1} = 0$  for all r, there are, by Proposition 2.2, maps  $f_r: X_r \to Y_r$  for all r. By carrying out the compressions from the cofibration at the "top" on downwards, the map  $f_{r-1}$  will be compatible with  $f_r$ . In view of  $\psi_r f_r i_{r-1,r} \sim \psi_r j_{r-1,r} f_{r-1} = 0$  and the homotopy extension property of  $i_{r-1,r}$ , there exists a map  $f'_r$  with  $f'_r \phi_r \sim \psi_r f_r$ .

## 3. A lemma of Thom

The following Proposition 3.3 is a lemma of Thom [10, p. 59] for which we give a proof by induction on homology decompositions. Since the lemma deals with the homology homomorphisms induced by maps, we first consider Lemmata 3.1 and 3.2 which follow directly from the coefficient formulae for cohomology and homotopy groups (for the latter see [5]). Note that any induced homomorphism is always an element in an abelian group Hom (, ).

LEMMA 3.1. Let  $f: \Sigma X \to Y$  be a map. Suppose  $f_* | H_k(\Sigma X)$  of finite order for k = r - 1, r. Then  $f^* | H^r(Y; G)$  is also of finite order for all coefficients G.

LEMMA 3.2. Consider a map f of Moore-spaces,  $f: K'(G, r) \to K'(H, r)$ . Then we have

(a)  $f^* | H^r(K'(H, r); H) \equiv 0$  if and only if  $f_* | H_r(K'(G, r)) \equiv 0$ . (b) If  $f_* | H_r(K'(G, r)) \equiv 0$ , then there exists an integer m such that  $mf \sim 0$ .

**PROPOSITION 3.3** (Thom). Let X, Y be polyhedra, X of dimension

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 $\leq 2n-2$  and Y at least (n-1)-connected. Suppose  $f_* \mid H_*(X)$  of finite Then f itself is of finite order in the group order.

$$\Pi(X, Y) \cong \Pi(\Sigma X, \Sigma Y) \cong \cdots$$

*Proof.* (a) It is no restriction to assume Y simply connected and  $X = \Sigma^2 X'$ . Let  $\{X_r\}$  be the suspension of a homology decomposition of  $\Sigma X'$ , and  $\{Y_r\}$  any decomposition of Y. All  $X_r$  are suspensions, and all

$$\phi_r: X_r \to K'(H_r(X), r)$$

are suspended maps. If for  $f, g: X \to Y$  there exist maps

 $f_r, q_r: X_r \to Y_r$ 

then this is also the case for f + g, and we can take  $(f + g)_r = f_r + g_r$ .

(b) Let  $j: Y_{r-1} \to Y_r$  with projection  $\psi: Y_r \to L'_r$  be part of  $\{Y_r\}$ , and  $K'_s$  one of the cofibers of  $\{X_r\}$ . Then we have

- The sequence  $\Pi(K'_s, Y_{r-1}) \xrightarrow{j_*} \Pi(K'_s, Y_r) \xrightarrow{\psi_*} \Pi(K'_s, L'_r)$  is exact. The group  $\Pi(K'_s, Y_r)$  is finite for s > r. (1)
- (2)

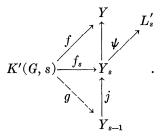
(1) follows from the Blakers-Massey theorem in [2, p. 198]; (2) is an easy consequence of (1), the fact that  $\pi_r(S^m)$  is finite for  $r \leq 2m - 2$ , and the coefficient formulae for homotopy and cohomotopy groups.

(c) Let  $f'_* | H_*(X)$  be of finite order. By Lemma 3.1 there exists an integer q such that  $(qf')^* = f^* = 0$  on  $H^k(Y; H_k(Y))$  for all k. By Corollary 2.3 there are maps  $f_r: X_r \to Y_r$  compatible with each other for all r.

To anchor the induction let X be a Moore-space, X = K'(G, s). There are two possibilities (1), (2) for the corresponding  $Y_s$ :

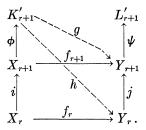
 $Y_s = K'(H, s)$ . We know that  $f_s^* \mid H^s(K'(H, s); H) = 0$ . Apply (1)Lemma 3.2(a) and (b) to conclude that  $mf \sim 0$ .

 $Y_s$  has several homology groups (write  $\psi = \psi_s$ ,  $j = j_{s-1,s}$ ): (2)



In view of  $(\psi f_s)^* | H^s(L'_s; H_s(Y)) \equiv 0$  and Lemma 3.2(a) and (b) there is an integer m such that  $m(\psi f_s) = m(\psi_* f_s) = \psi_* m f_s = 0$ . By remark (1) of (b) we infer  $mf_s = j_* g$ ,  $g \in \Pi(K'(G, s), Y_{s-1})$ , and this element g is of finite order by remark (2) of (b). Thus  $mf_s = j_*g$  and, consequently, mf are also of finite order.

(d) For the induction step we have to prove: Suppose  $f_r: X_r \to Y_r$  of finite order, and, of course,  $f_{r+1}^* | H^{r+1}(Y_{r+1}, H_{r+1}(Y)) \equiv 0$ . Then  $f_{r+1}$  is also of finite order (write  $\phi = \phi_r$ ,  $i_{r,r+1} = i$ , etc.):



We can assume  $f_r \sim 0$ . Therefore  $f_{r+1} \sim g\phi$  (see diagram) because of the homotopy extension property of *i* and the relation  $f_{r+1} i \sim jf_r \sim 0$ . Since  $\phi^*$  is monomorphic on the (r + 1)-dimensional cohomology, we conclude  $(\psi g)^* \mid H^{r+1}(L'_{r+1}; H_{r+1}(Y)) \equiv 0$ , and by Lemma 3.2 also

$$m(\psi g) = m(\psi_* g) = \psi_*(mg) = 0.$$

Again by (b) we have  $mg = j_* h$  with  $h \in \Pi(K'_{r+1}, Y_r)$  of finite order. Since  $\phi$  is a suspended map,  $\phi^*$  is homomorphic. Therefore  $mf_{r+1} = \phi^*(mg) = \phi^* j_* h$ , which means that  $mf_{r+1}$  is of finite order as the homomorphic image of h, and the proof of Proposition 3.3 is complete.

### 4. Selfmaps of polyhedra. The invariant t(X)

The composition of maps defines in  $\Pi(X, X)$  the structure of a monoid with identity and zero element. Consider the cohomology functor  $\mathfrak{K}^*$  which maps the monoid  $\Pi(X, X)$  homomorphically into the multiplicative structure of the ring of endomorphisms of  $\bigoplus_k H^k(X; H_k)$ ,  $H_k = H_k(X)$ . The set  $\mathfrak{K}^{*-1}(0)$  of all "cohomologically trivial" selfmaps is multiplicatively closed in  $\Pi(X, X)$  and will be denoted by T(X),  $T(X) \subset \Pi(X, X)$ . In the first part of this section we study T(X) and define an integer-valued homotopy invariant t(X) of X; in the second part (Definition 4.6 et seq.) we discuss the case when  $\mathfrak{K}^*$  is epimorphic.

The following proposition prepares for Theorem 4.2.

PROPOSITION 4.1. Consider a Moore-space K'(G, r). If G has no 2-torsion, then T(K'(G, n)) = 0. Otherwise T(K'(G, r)) is nilpotent of order  $\leq 2$ .

*Proof.* (a) Write K' = K'(G, r),  $G = H_r(K')$ . By Lemma 3.2(a) we know that  $f \in T(K')$  if and only if  $f_* | G \equiv 0$ . Consider the coefficient formula for homotopy groups in (b) below. Since  $\pi_{r+1}(K') \cong G \otimes Z_2$ , obviously Ext  $(G, G \otimes Z_2) = 0$  if G has no 2-torsion. Therefore, if G has no 2-torsion, any  $f : K' \to K'$  with  $f_* | G \equiv 0$  is homotopic to zero.

(b) Let G be arbitrary and  $f_* | G \equiv g_* | G \equiv 0$ . The coefficient formula is natural with respect to covariant maps:

$$\begin{array}{cccc} 0 \to \operatorname{Ext}(G, \pi_{r+1}(K')) & \xrightarrow{\alpha} & \Pi(K', K') & \xrightarrow{\beta} & \operatorname{Hom}(G, G) \to 0 \\ & & & & & \downarrow f_1 & & \downarrow f_2 \\ 0 \to \operatorname{Ext}(G, \pi_{r+1}(K')) & \xrightarrow{\alpha} & \Pi(K', K') & \xrightarrow{\beta} & \operatorname{Hom}(G, G) \to 0, \end{array}$$

where  $f_i$  are induced by  $f_0: \pi_k(K') \to \pi_k(K')$ . By hypothesis  $\beta g = 0$ . Therefore  $g = \alpha g_1$  and  $f_* g = f_* \alpha g_1 = \alpha f_1 g_1$ . In (c) below, we show that  $f_1 = 0$ , and the proof of Proposition 4.1 will be complete.

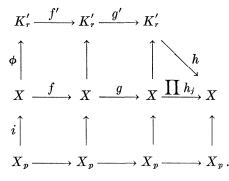
(c) To compute  $\pi_{r+1}(K')$ , inspect, according to Peterson [9], the first derived homotopy exact couple of Massey (see Fig. 1 in [9]) which is natural with respect to maps f:

$$\begin{array}{cccc} \pi_{r+1}(K') & \xleftarrow{\cong} & \Gamma & \xrightarrow{\cong} & H_r(K'; Z_2) \cong G \otimes Z_2 \\ & & & & & \downarrow f_0 & & \downarrow f_{**} \\ \pi_{r+1}(K') & \xleftarrow{\cong} & \Gamma & \xrightarrow{\cong} & H_r(K'; Z_2) \cong G \otimes Z_2 \,. \end{array}$$

In view of  $f_* | G \equiv 0$  we have  $f_{**} | G \otimes Z_2 \equiv 0$ . Therefore  $f_0 = 0$ , and  $f_1 | \operatorname{Ext}(G, \pi_{r+1}(K')) \equiv 0$  as was to be proved.

THEOREM 4.2. Let X be a simply connected polyhedron with q nontrivial homology groups,  $q_1$  of which have 2-torsion. Then any product of  $q + q_1$  selfmaps which induce the trivial endomorphism of  $\bigoplus_k H^k(X; H_k(X))$  vanishes. In other words, T(X) is nilpotent of order  $\leq q + q_1$ .

*Proof.* Apply Proposition 4.1 if X = K'(G, r). Suppose for induction that for any Y with s homology groups T(Y) is nilpotent of order t. Consider a homology decomposition  $\{X_r\}$  of an X with s + 1 homology groups, and let p be such that  $X_p$  has s homology groups. Let f, g,  $h_1, \dots, h_t$  be elements of T(X), and recall Corollary 2.3:



By hypothesis  $\prod_{j=1}^{t} h_j$  induces on  $X_p$  the trivial map. Therefore  $\prod h_j \sim h\phi$ .

If  $H_r(X)$  has 2-torsion, then  $g'f' \sim 0$  by Proposition 4.1, and  $(\prod h_i)af \sim hg'f'\phi \sim 0;$ 

in the opposite case  $g' \sim f' \sim 0$ , and already  $(\prod h_j)g \sim 0$ . Thus T(X) is nilpotent of order  $\leq t+2$  or  $\leq t+1$  according to the presence or ab-

COROLLARY 4.3. Let X be a simply connected polyhedron with q nontrivial homology groups,  $q_1$  of which have 2-torsion. Denote by  $T_0(X) \subset \Pi(X, X)$ the set of all selfmaps of X which induce the trivial endomorphism of  $H_*(X)$ . Then  $T_0(X)$  is nilpotent of order  $\leq 2q + 2q_1$ .

*Proof.* Write out three times the coefficient formula for cohomology groups to verify that  $[T_0(X)]^2 \subset T(X)$ . Therefore  $[T_0(X)]^{2(q+q_1)} = 0$ .

Examples of applications of Theorem 4.2. Let X be a simply connected polyhedron.

(1) The endomorphism of  $\pi_r(X)$  induced by a selfmap f such that  $f_* \mid H_*(X) \equiv 0$  is nilpotent for all r.

(2) An idempotent element  $f \in \Pi(X, X)$ ,  $f^2 = f$ , cannot satisfy the condition  $f_* \mid H_*(X) \equiv 0$  unless it is equal to zero.

(3) If  $f \in \Pi(X, X)$  suspends trivially,  $\Sigma f = 0$ , then f is nilpotent (consider the induced homomorphisms of  $\Pi(X, X)$  and  $\Pi(\Sigma X, \Sigma X)$ ).

Theorem 4.2 suggests Definition 4.4 which in its most general form reads as follows:

DEFINITION 4.4. Let X be a topological space, and H a homology theory. Define t(H; X) to be the order of nilpotency of the multiplicatively closed subset T(H; X) of  $\Pi(X, X)$  consisting of all maps which induce the trivial endomorphism of  $\bigoplus_k H^k(X; H_k(X))$ .

Considering the integer homology one defines analogously  $t_0(H; X)$ . For polyhedra t(H; X),  $t_0(H; X)$  depend only on the homotopy type of X. In this case we write t(H; X) = t(X),  $t_0(H; X) = t_0(X)$  and, as we have already done, T(H; X) = T(X),  $T_0(H; X) = T_0(X)$ . To determine the range of values of t(X) considered as a function of X, we invoke the following theorem of Novikov.

THEOREM OF NOVIKOV [8]. For any given integer r > 0 there exist integers  $k_1 > k_2 > \cdots > k_r$  and maps  $f_s$  of spheres,  $f_s : S^{k_s} \to S^{k_{s+1}}$ , such that the composition

 $S^{k_1} \xrightarrow{f_1} S^{k_2} \xrightarrow{f_2} \cdots \xrightarrow{f_{r-1}} S^{k_r}$ 

is not homotopic to zero.

**PROPOSITION** 4.5. 1. For any given integer r > 0 there exists a simply connected polyhedron X with r nontrivial homology groups and t(X) = r.

sence of 2-torsion in  $H_r(X)$ .

2. For any given integer r > 0 there exists a simply connected polyhedron X with r nontrivial homology groups and t(X) = 1.

**Proof.** 1. Consider  $X = S^{k_1} \vee \cdots \vee S^{k_r}$  and  $f: X \to X$  defined by  $f \mid S^{k_s} = f_s: S^{k_s} \to S^{k_{s+1}}$ , the  $k_s$  and  $f_s$  being the integers and maps of Novikov's theorem. Obviously  $f \in T_0(X)$ . Let  $i: S^{k_1} \to X$  and  $p: X \to S^{k_r}$  be the injection of the first and the projection onto the last factor of the wedge X. Since  $pff \cdots fi = pf^{r-1}i = f_{r-1}f_{r-2} \cdots f_1$  is not homotopic to zero,  $f^{r-1}$  is not trivial either. This means  $t_0(X) \ge r$ . Because all homology groups of X are infinite cyclic, we have  $t_0(X) = t(X)$ . By Theorem 4.1,  $t(X) \le r$ . Therefore t(X) = r.

2. Let  $M_{(r)}$  be the complex projective space of r complex dimensions. Since dim  $M_{(r)} = 2r$  and  $\pi_i(M_{(r)}) = 0$  for i < 2 and 3 < i < 2r, a map  $f: M_{(r)} \to M_{(r)}$  is homotopic to zero if and only if its induced homomorphism on  $H_2(M_{(r)})$  is trivial. This means that  $f_* = 0$  implies  $f \sim 0$ . Hence  $t_0(M_{(r)}) = 1$ . But again  $t_0(M_{(r)}) = t(M_{(r)})$  because all homology groups of  $M_{(r)}$  are infinite cyclic.

We now turn to the question which endomorphisms of  $\bigoplus_k H^k(X; H_k)$ ,  $H_k = H_k(X)$ , can be realized by a selfmap  $f: X \to X$ .

Consider  $f^*$  induced by  $f: X \to X$ :

where  $f_k$  is induced by  $f_* | H_*$ . The natural ring structure of Hom  $(H_r, H_r)$  defines a ring structure in  $H^r(X; H_r)/\alpha \operatorname{Ext}(H_{r-1}, H_r)$ , and  $f^*$  induces an endomorphism  $f^{\sharp}$  of the additive group, (or, as we shall say, an additive endomorphism) of the ring  $H^r(X; H_r)/\alpha \operatorname{Ext}(H_{r-1}, H_r)$ . Since  $f^{\sharp}$  is essentially  $f_2$  induced by  $f_* | H_r$ , we have  $f^{\sharp}(ab) = af^{\sharp}(b)$  in the ring

$$H^r(X; H_r) / \alpha \operatorname{Ext}(H_{r-1}, H_r)$$

We summarize these considerations in Definition 4.6 and Lemma 4.7.

DEFINITION 4.6. The endomorphism  $\Phi = \bigoplus_k \Phi_k$  of  $\bigoplus_k H^k(X; H_k)$  is called admissible if  $\Phi_k$  induces for all k an additive endomorphism  $\Phi_k^{\sharp}$  of the ring  $H^k(X; H_k)/\alpha \operatorname{Ext}(H_{k-1}, H_k)$  such that  $\Phi_k^{\sharp}(ab) = a\Phi_k^{\sharp}(b)$ .

**LEMMA 4.7.** The endomorphism  $f^*$  of  $\bigoplus_k H^k(X; H_k)$  induced by  $f: X \to X$  is admissible in the sense of Definition 4.6.

To study the problem under which circumstances every admissible endomorphism of  $\bigoplus_k H^k(X; H_k)$  can be realized by a selfmap, we first have to consider Proposition 4.8.

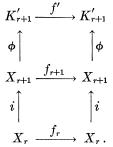
**PROPOSITION 4.8.** Let X be a simply connected polyhedron. Consider the following admissible endomorphism  $\Phi = \bigoplus_k \Phi_k$  of  $\bigoplus_k H^k(X; H_k)$ :

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$$\Phi_k^{(r)} \equiv \text{Id} \quad for \quad k \leq r,$$
$$\equiv 0 \quad for \quad k > r.$$

Suppose that  $\Phi^{(r)}$  can be realized by a map  $f: X \to X$ ,  $\Phi^{(r)} = f^*$ . Then  $\operatorname{Ext}(H_r, H_{r+1}) = 0$ . If furthermore X is 2-connected, then in any homology decomposition  $\{X_n\}$  of X the  $r^{th}$  attaching map  $\alpha_r: K'(H_{r+1}, r) \to X_r$  vanishes.

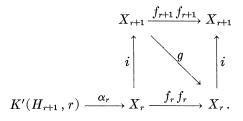
*Proof.* (a) Consider a homology decomposition  $\{X_n\}$  of X. By Proposition 2.2 and the homotopy extension property of  $i = i_{r,r+1}$ , there exist maps  $f_r$ ,  $f_{r+1}$ , f' compatible with each other (write  $\phi = \phi_{r+1}$ ,  $K'_{r+1} = K'(H_{r+1}, r+1)$ ):



It follows from the coefficient formula for cohomology groups and the fact that an epimorphic endomorphism of a finitely generated group is isomorphic that  $f_*$  is an automorphism of  $H_k$  for  $k \leq r$ . Therefore  $f_r$  is a homotopy equivalence. Consider on the other hand cohomology with  $H_{r+1}$  as coefficients:

Since  $i_r^*$  is epimorphic and  $f^* = 0$ , we have  $f_r^* = 0$ . But  $f_r$  was seen to be an equivalence. Therefore  $H^{r+1}(X_r) \cong \operatorname{Ext}(H_r, H_{r+1}) = 0$ .

(b) For the map  $f': K'_{r+1} \to K'_{r+1}$  induced by  $f_{r+1}$  and  $f_r$  (see diagram of (a)) we have  $f'^* | H^{r+1}(K'_{r+1}; H_{r+1}) \equiv 0$ . Therefore  $f'f' \sim 0$  by Proposition 4.1. This implies  $\phi f_{r+1} f_{r+1} \sim f'f' \phi \sim 0$ . If X is 2-connected, it follows from Proposition 2.1 that  $f_{r+1}f_{r+1}$  is compressible into  $X_r$ ,  $f_{r+1}f_{r+1} \sim ig$ :



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In view of  $if_r f_r \sim f_{r+1} f_{r+1} i \sim igi$ , we have  $i_* g_* i_* = i_* (f_r f_r)_*$  for the homology groups;  $i_*$  is isomorphic because i is an elementary cofibration, and  $(f_r f_r)_*$  is isomorphic because  $f_r$  is a homotopy equivalence of  $X_r$ . Therefore  $gi: X_r \to X_r$  is also an equivalence. On the other hand,  $gi\alpha_r \sim 0$  because g maps  $X_{r+1} = CK'(H_{r+1}, r) \bigcup_{\alpha_r} X_r$  into  $X_r$ . Therefore  $\alpha_r \sim 0$ , and the proof of Proposition 4.8 is complete.

THEOREM 4.9. 1. Let X be homotopy-equivalent to a wedge of Moorespaces. Suppose  $\operatorname{Ext}(H_r, H_{r+1}) = 0$  for all  $r, H_r = H_r(X)$ . Then every admissible endomorphism (see Lemma 4.7) of  $\bigoplus_k H^k(X; H_k)$  can be realized by a selfmap  $f: X \to X$ .

2. Let X be an m-connected polyhedron,  $m \ge 1$ , with the property that every admissible endomorphism of  $\bigoplus_k H^k(X; H_k)$  can be realized by a selfmap  $f: X \to X$ . Then  $\text{Ext}(H_r, H_{r+1}) = 0$  for all r. Furthermore,

(a) If  $m \ge 2$ , then X is homotopy-equivalent to a wedge of Moore-spaces. (b) If m = 1, then all attaching maps of any homology decomposition of X suspend trivially (i.e.,  $\Sigma X$  is homotopy-equivalent to a wedge of Moore-spaces).

*Proof.* 1. In view of  $\operatorname{Ext}(H_r, H_{r+1}) = 0$  for all r, it suffices to prove the following statement (the application of which to all Moore-spaces of the wedge X will prove part 1 of Theorem 4.9): Every admissible automorphism  $\Phi$  of  $H^r(K'(G, r); G)$  can be realized by a selfmap  $f: K'(G, r) \to K'(G, r)$ .

Let  $\Phi : H^r(K'(G, r); G) \to H^r(K'(G, r); G)$  be admissible. Define  $\Phi' : \operatorname{Hom}(G, G) \to \operatorname{Hom}(G, G)$  by  $\Phi' = \beta \Phi \beta^{-1}$ :

Since  $\Phi$  is admissible, so is  $\Phi' : \Phi'(ab) = a\Phi'(b)$ ; in particular,  $\Phi'(a) = a\Phi'(Id)$  for all  $a \in \text{Hom}(G, G)$ . Realize  $\Phi'(Id) : G \to G$  by a map

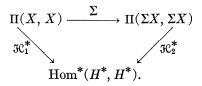
$$f: K'(G, r) \to K'(G, r)$$

with the aid of the coefficient formula for homotopy groups. Obviously  $f^* = \Phi$ , and part 1 of Theorem 4.9 is proved.

2(a). By hypothesis the admissible endomorphisms  $\Phi^{(r)}$  of Proposition 4.8 can be realized by maps for all r. It follows from the same Proposition 4.8 that  $\text{Ext}(H_r, H_{r+1}) = 0$ , and that all attaching maps of any homology decomposition of X vanish provided X is 2-connected.

2(b). Identify  $\bigoplus_k H^k(X; H_k(X)) = \bigoplus_k H^k(\Sigma X; H_k(\Sigma X)) = H^*$ , and denote by Hom<sup>\*</sup>( $H^*$ ,  $H^*$ ) the subgroup of admissible endomorphisms of  $H^*$ .

The meaning of  $\mathfrak{K}_1^*$ ,  $\mathfrak{K}_2^*$  in the following commutative diagram is obvious:



If  $\mathfrak{K}_1^*$  is epimorphic, so is  $\mathfrak{K}_2^*$ . Apply 2(a) to  $\Sigma X$  to conclude that  $\Sigma X$  is homotopy-equivalent to a wedge of Moore-spaces, and the proof of Theorem 4.9 is complete.

## 5. Selfmaps of suspensions. The invariant d(X)

The natural comultiplication of  $\Sigma X$  defines in  $\Pi(\Sigma X, \Sigma X)$  a group operation which will be written as addition though the group  $\Pi(\Sigma X, \Sigma X)^+$  may be nonabelian. The addition is connected with the monoid-multiplication by only one law of distributivity in general: f(a + b) = fa + fb. It is customary to call a set P with two binary operations + and  $\cdot$  (called addition and multiplication) a near-ring if (1) (P, +) is a group, (2)  $(P, \cdot)$  is a monoid, and (3) multiplication is left distributive with respect to addition (for nearrings see e.g. [1]). Since the composition of maps is associative,  $\Pi(\Sigma X, \Sigma X)$ is an associative near-ring with an identity. Call  $g \in \Pi(\Sigma X, \Sigma X)$  distributive if (a + b)g = ag + bg for all a, b; e.g. any suspended map  $g = \Sigma g'$  is distributive. If  $\Sigma : \Pi(X, X) \to \Pi(\Sigma X, \Sigma X)$  is epimorphic for some reason, then  $\Pi(\Sigma X, \Sigma X)$  is a ring.

To measure the deviation of  $\Pi(\Sigma X, \Sigma X)$  from being a ring we use the following definition of the distributor series  $D^{r}(P)$  of a near-ring P. It is slightly different from that given in [7].

Definition 5.1. Let P be a near-ring.

(1) Define  $D^0(P) = P$ .

(2) Call  $[a, b, f] \equiv (a + b)f - bf - af$  with  $a, b, f \in P$  a 1-distributor of P, and denote the subgroup of  $P^+$  generated by all 1-distributors by  $D^1(P)$ .

(3)  $D^{n}(P), n \geq 2$ , is defined as the subgroup of  $P^{+}$  generated by all elements of the form [a, b, f] with  $f \in P$  and  $a, b \in D^{n-1}(P)$ . The generators of  $D^{n}(P)$  are called *n*-distributors of *P*.

It is obvious that  $D^{n+k}(P) = 0$  for k > 0 if  $D^n(P) = 0$ . Thus it makes sense to write d(P) = n if  $D^{n-1}(P) \neq 0$ ,  $D^n(P) = 0$  such that d(P) = 1if P is a ring. If  $D^k(P) \neq 0$  for all k, we write  $d(P) = \infty$ . This obviously suggests the following definition.

DEFINITION 5.2. Let X be a topological space. Call  $d(X) = d(\Pi(\Sigma X, \Sigma X))$  the order of distributivity of X.

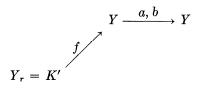
The integer d(X) is an invariant of the homotopy type of X but not of  $\Sigma X$  in general. If e.g. X is an (n - 1)-connected polyhedron of dimension  $\leq 2n - 1$ , then d(X) = 1.

THEOREM 5.3. Let X be a simply connected polyhedron with n nontrivial homology groups. Then  $d(X) \leq n$  (i.e., all n-distributors of  $\Pi(\Sigma X, \Sigma X)$  vanish).

*Proof.* (a) Write  $\Sigma X = Y$ . Consider a homology decomposition  $\{Y_r\}$  of Y obtained by suspending one of X. By induction we shall prove in (b), (c) below the following statement which obviously includes Theorem 5.3: Let  $Y_r$  have s nontrivial homology groups, and let  $a^{(s-1)}$ ,  $b^{(s-1)}$  be two (s-1)-distributors of  $\Pi(Y, Y)$ . Then for any  $f: Y_r \to Y$  we have

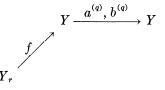
$$[a^{(s-1)}, b^{(s-1)}, f] \equiv (a^{(s-1)} + b^{(s-1)})f - b^{(s-1)}f - a^{(s-1)}f = 0.$$

(b) Let s = 1, i.e.,  $Y_r$  is a Moore-space K';  $a, b \in \Pi(Y, Y)$ .



In all possible cases considered  $\Sigma : \Pi(\Sigma^{-1}K', X) \to \Pi(K', Y), Y = \Sigma X$ , is epimorphic. Therefore  $f : K' \to Y$  is a suspended map. Consequently [a, b, f] = 0.

(c) We assume the theorem for s = q. If  $a^{(q)}$ ,  $b^{(q)}$  are q-distributors of  $\Pi(Y, Y)$  and  $f: Y_r \to Y$  is any map, we have to prove that  $[a^{(q)}, b^{(q)}, f] = 0$  if  $Y_r$  has q + 1 homology groups:

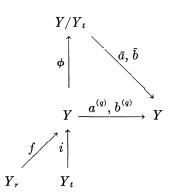


By definition  $a^{(q)} = [a_1, a_2, f']$ , where  $a_i$  are (q-1)-distributors of  $\Pi(Y, Y)$  and  $f': Y \to Y$  is any map.

Let  $Y_t$ , t < r, have q nontrivial homology groups. Since  $i: Y_t \to Y$  is a suspended map, we have

$$a^{(q)}i = [a_1, a_2, f']i = [a_1, a_2, f'i]; \quad f'i: Y_t \to Y \to Y.$$

By induction hypothesis,  $[a_1, a_2, f'i] = 0$ . Therefore we can find maps  $\bar{a}, \bar{b}$  with  $\bar{a}\phi \sim a^{(q)}, \bar{b}\phi \sim b^{(q)}$ :



Now use the fact that  $\phi$  and  $\phi f$  are suspended maps:

 $\begin{aligned} (a^{(q)} + b^{(q)})f &\sim (\bar{a}\phi + \bar{b}\phi)f \sim ((\bar{a} + \bar{b})\phi)f \sim (\bar{a} + \bar{b})(\phi f) \\ &\sim \bar{a}\phi f + \bar{b}\phi f \sim a^{(q)}f + b^{(q)}f, \end{aligned}$ 

which means  $[a^{(q)}, b^{(q)}, f] = 0.$ 

*Example.* The following example of a space X with three nontrivial homology groups and d(X) = 3 is due to P. J. Hilton.

Take

 $X = \bigvee_{k=1}^{4} S_{k}^{1} \lor S^{2} \lor S^{5}, \qquad \Sigma X = \bigvee_{k=1}^{4} S_{k}^{2} \lor S^{3} \lor S^{6}.$ 

Consider the identity maps  $i_k : S_1^2 \to S_k^2$ , the Hopf map  $\gamma : S^3 \to S_1^2$ , the generator  $\alpha : S^6 \to S^3$  of  $\pi_6(S_3)$ . Then  $[i_1, i_2, \gamma] = \delta_1 \neq 0$ ,  $[i_3, i_4, \gamma] = \delta_2 \neq 0$ ,  $[\delta_1, \delta_2, \alpha] \neq 0$ . Therefore  $d(X) \geq 3$ . But  $d(X) \leq 3$  by a simple direct argument. Hence d(X) = 3.

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