ASYMPTOTIC BEHAVIOR OF SUCCESSIVE COEFFICIENTS OF SOME POWER SERIES

BY

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Introduction

Suppose that $f_1, f_2 \ldots$ is a sequence of numbers satisfying the two conditions:

(i) $\sum_{n=1}^{\infty} f_n = 1$,
(ii) $f_n \geqq 0$, and the greatest common divisor of the indices k such that $f_k \neq 0$ is one.

We shall set

$$F(t) = \sum_{n=1}^{\infty} f_n t^n, \qquad U(t) = \sum_{n=0}^{\infty} u_n t^n,$$

where

I.1 $$U(t) = 1/(1 - F(t)).$$

We are interested in the behavior of the ratio

$$r_n = u_{n+1}/u_n$$

as $n \to \infty$. It was shown in [2] that as n tends to infinity

$$u_n \to (F'(1))^{-1},$$

with the expression on the right being interpreted as zero when $F'(1)$ is infinite. We seek conditions on the f_k which do not imply $F'(1) < \infty$ but insure[2]

$$\liminf_{n \to \infty} r_n = \limsup_{n \to \infty} r_n = 1.$$

The simplest condition we found was

$$\limsup_{n \to \infty} f_{n+1}/f_n \leqq 1.$$

This condition has the serious drawback that it does not permit $f_k = 0$ for infinitely many k. We have found more satisfactory conditions which include the above condition as a special case. Nevertheless this special case is of special interest because the arguments then are simpler and more transparent.

We shall proceed with stating the above-mentioned conditions. Let $\mu_1, \mu_2, \cdots, \mu_{N-1}, \lambda$ be real numbers, and suppose $\mu_{N-1} \neq 0$, $\lambda > 0$. We

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[2] Our results, like those of [2], have applications in probability theory; these will be discussed elsewhere.

shall say that the sequence $f_1, f_2, \cdots, f_n, \cdots$ satisfies condition

$$A_N(\mu_1, \mu_2, \cdots, \mu_{N-1}; \lambda)$$

if and only if the expression $f_n + \mu_1 f_{n+1} + \cdots + \mu_{N-1} f_{n+N-1}$ becomes greater than zero for large n, and in addition

I.2 $$\limsup_{n\to\infty} \frac{f_{n+1} + \mu_1 f_{n+2} + \cdots + \mu_{N-1} f_{n+N}}{f_n + \mu_1 f_{n+1} + \cdots + \mu_{N-1} f_{n+N-1}} \leqq \lambda.$$

We shall say that the sequence $f_1, f_2, \cdots$ satisfies condition B_N when it satisfies $A_N(1, 1, \cdots, 1; 1)$.

We can now state more precisely the results that will be presented in this paper. First of all we shall show that (i), (ii), and $A_N(\mu_1, \cdots, \mu_{N-1}, \lambda)$ imply that as $n \to \infty$, $\limsup r_n < \infty$ and $\liminf r_n > 0$. In addition it will be shown that in any case, given (i) and (ii) we have[3]

I.3 $$\limsup_{n\to\infty} u_{n+1}/u_n \leqq (\limsup f_{n+1}/f_n) \cup 1.$$

Finally, it will be established that (i), (ii), and B_N imply that

I.4 $$\limsup_{n\to\infty} u_{n+1}/u_n \leqq 1.$$

The latter inequality will be shown to imply the convergence of u_{n+1}/u_n to one.

The methods developed in this paper have been used to deduce $r_n \to 1$ using only (i), (ii), and $A_N(\mu_1, \mu_2, \cdots, \mu_{N-1}; 1)$ where $\mu_1, \mu_2, \cdots \mu_{N-1}$ are to be nonnegative. However, the proof of this extension is somewhat technical and will not be included.

In the Remark at the end of Section 2 we show that the condition $r_n \to 1$ is equivalent to each of two other conditions.

Two questions are left unanswered.

(1) Do (i), (ii), and condition $A_N(\mu_1, \mu_2, \cdots, \mu_{N-1}; \lambda)$ imply the convergence of u_{n+1}/u_n? (In particular, does $\limsup f_{n+1}/f_n < \infty$ imply $r_n \to 1$?)

(2) Does a result of the type I.3 hold in general? In other words, do (i), (ii), and $A_N(\mu_1, \mu_2, \cdots, \mu_{N-1}; \lambda)$ imply at least

$$\limsup_{n\to\infty} u_{n+1}/u_n \leqq \lambda \cup 1?$$

1.

We shall start by establishing a few identities and inequalities.

1.1. We observe that I.1 implies that $u_0 = 1$, and that

1.11 $$u_{n+1} = f_1 u_n + f_2 u_{n-1} + \cdots + f_{n+1} u_0.$$

[3] We write $a \cup b$ for the maximum of a and b.

By induction it is easy to show that

1.12 $$u_n \leqq 1.$$

On the other hand, (ii) implies that we can find $\rho + 1$ natural numbers $N_0, k_1, k_2, \cdots, k_\rho$ with the following properties:

(a) $f_{k_i} \neq 0, \quad i = 1, 2, \cdots, \rho;$
(b) every natural number $n \geqq N_0$ can be written in the form

1.13 $$n = k_1 n_1 + k_2 n_2 + \cdots + k_\rho n_\rho,$$

where the n_i are nonnegative integers.

From 1.11 when $n \geqq N_0$ it is easily deduced that 1.13 implies

1.14 $$u_n \geqq (f_{k_1})^{n_1}(f_{k_2})^{n_2} \cdots (f_{k_\rho})^{n_\rho}.$$

We thus conclude that $u_n \neq 0$ at least when $n \geqq N_0$.

1.2. Given a set of constants $\mu_1, \mu_2, \cdots, \mu_{N-1}$ we shall set

$$M(t) = \mu_1 t + \mu_2 t^2 + \cdots + \mu_{N-1} t^{N-1}.$$

From I.1 we get

1.21 $$U(t) = \frac{1 + M(t)}{1 - [(1 + M(t))F(t) - M(t)]},$$

so that setting

1.22 $$F^M(t) = \sum_{k=1}^{\infty} f_k^M t^k = (1 + M(t))F(t) - M(t),$$

we shall have (for $n > N - 2$)

1.23 $$u_{n+1} = f_1^M u_n + \sum_{k=1}^{n} f_{k+1}^M u_{n-k},$$

and also (for $n > N - 1$)

1.24 $$u_n = \sum_{k=1}^{n} f_k^M u_{n-k}.$$

For convenience of notation, $N = 1$ shall mean $M(t) = 0$, so that when $N = 1$, the f_k^M in our formulas are to represent the old f_k.

We divide 1.24 by u_n and manipulate to obtain (for $n \geqq N + N_0 + N_1$)

1.25 $$1 \geqq \sum_{k=1}^{N_1} \frac{f_k^M}{r_{n-1} r_{n-2} \cdots r_{n-k}},$$

provided only $f_k^M \geqq 0$ for $k > N_1$.

Let λ be a real number greater than one. Multiplying 1.24 by λ and subtracting from 1.23 we get

1.26 $$u_{n+1} = (\lambda + f_1^M) u_n + \sum_{k=1}^{n} (f_{k+1}^M - \lambda f_k^M) u_{n-k}.$$

Suppose now that

1.27 $$f_{k+1}^M - \lambda f_k^M \leqq 0 \qquad \text{for} \quad k \geqq N_2.$$

Using this inequality in 1.26, dividing by u_n, and manipulating, we obtain

1.28 $$r_n \leqq \lambda + f_1^M + \sum_{k=1}^{N_1} \frac{f_{k+1}^M - \lambda f_k^M}{r_{n-1} r_{n-2} \cdots r_{n-k}}$$

when $n \geqq N + N_0 + N_1$ and $N_1 \geqq N_2$.

1.3. To simplify the exposition we shall introduce a new term. Let

$$n_1 < n_2 < \cdots < n_k < \cdots$$

be integers. The set of numbers $n_1, n_2, \cdots, n_k, \cdots$ will be called a "determining subsequence" if and only if, for $\alpha = 0, \pm 1, \pm 2, \cdots$, etc., the variable $r_{n_k+\alpha}$ converges as $k \to \infty$.

We set

1.31 $$\lim_{k\to\infty} r_{n_k+\alpha} = R_\alpha .$$

If $n_1, n_2, \cdots, n_k, \cdots$ is a determining subsequence and 1.31 holds, passing to the limit for $n = n_k + \alpha$ in 1.25 and (if 1.27 is valid) in 1.28, we obtain

1.32 $$1 \geqq \sum_{k=1}^{N_1} \frac{f_k^M}{R_{\alpha-1} R_{\alpha-2} \cdots R_{\alpha-k}},$$

1.33 $$R_\alpha \leqq \lambda + f_1^M + \sum_{k=1}^{N_1} \frac{f_{k+1}^M - \lambda f_k^M}{R_{\alpha-1} R_{\alpha-2} \cdots R_{\alpha-k}}.$$

Notice now that since the left-hand side of 1.32 is independent of N_1, we must also have

1.34 $$1 \geqq \sum_{k=1}^{\infty} \frac{f_k^M}{R_{\alpha-1} R_{\alpha-2} \cdots R_{\alpha-k}}.$$

On the other hand, if 1.27 is valid, the tail of the sum in 1.33 can be estimated by means of the tail of the series in 1.34. So we can also write

1.35 $$R_\alpha \leqq \lambda + f_1^M + \sum_{k=1}^{\infty} \frac{f_{k+1}^M}{R_{\alpha-1} \cdots R_{\alpha-k}} - \lambda \sum_{k=1}^{\infty} \frac{f_k^M}{R_{\alpha-1} \cdots R_{\alpha-k}}.$$

When $N = 1$ ($M(t) = 0$), this relation will be used in the form

1.36 $$R_\alpha - \lambda \leqq \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-1} \cdots R_{\alpha-k}} (R_{\alpha-k} - \lambda).$$

1.4. We can now deduce a few consequences of the inequalities that we have established. For convenience, here and in the following we shall set

$$\liminf_{n\to\infty} r_n = m, \qquad \limsup_{n\to\infty} r_n = M.$$

LEMMA 1.41. *If* $M < \infty$, *then* $M \geqq 1$, *and*

1.41 $$m \geqq F(1/M) M.$$

Proof. The inequality in 1.34 for $\alpha + 1$ can be written in the form

1.42 $$R_\alpha \geqq f_1^M + \sum_{k=1}^{\infty} \frac{f_{k+1}^M}{R_{\alpha-1} \cdots R_{\alpha-k}}.$$

For $M(t) = 0$ this relation yields

1.43 $$R_\alpha \geqq f_1 + \sum_{k=1}^{\infty} f_{k+1}/M^k.$$

Since for any given α there are determining sequences such that $R_\alpha = m$, 1.43 yields[4] 1.41.

LEMMA 1.42. *If* $M \leqq 1$, *then* r_n *converges, and* $\lim_{n\to\infty} r_n = 1$.

Proof. The assertion follows immediately from 1.41 and assumption (i).

LEMMA 1.43. *If* $\liminf_{n\to\infty} u_{n+1}/u_n > 0$, *then*

$$\limsup_{n\to\infty} u_{n+1}/u_n \leqq (\limsup f_{n+1}/f_n) \cup 1.$$

Proof. If $\limsup_{n\to\infty} f_{n+1}/f_n = \infty$, there is nothing to prove. Suppose then that

1.44 $$\lambda' = (\limsup_{n\to\infty} f_{n+1}/f_n) \cup 1 < \infty.$$

Since 1.27 has to hold for each $\lambda > \lambda'$ and suitable N_2, 1.28 is valid for $M(t) = 0$, and passing to the limit we obtain

$$M \leqq \lambda + f_1 + \sum_{k=1}^{N_1} \frac{|f_{k+1} - \lambda f_k|}{m^k}.$$

We can thus pick a determining sequence such that $R_0 = M$.

We now observe that 1.36 will necessarily hold for each $\lambda > \lambda'$; therefore we shall have also

$$R_\alpha - \lambda' \leqq \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-1} R_{\alpha-2} \cdots R_{\alpha-k}} (R_{\alpha-k} - \lambda').$$

For $\alpha = 0$ we obtain

$$\sum_{k=1}^{\infty} \frac{f_k}{R_{-1} R_{-2} \cdots R_{-k}} (M - R_{-k}) \leqq (\lambda' - M)\left(1 - \sum_{k=1}^{\infty} \frac{f_k}{R_{-1} R_{-2} \cdots R_{-k}}\right).$$

Since $M \geqq R_{-k}$, the assumption that $\lambda' \leqq M$ (in view of 1.34 written for $M(t) = 0$) implies that $R_{-k} = M$ for all k such that $f_k \neq 0$, say for $k \geqq \alpha_0$. Using this fact for $\alpha = -\alpha_0$ we get

$$0 \leqq (\lambda' - M)(1 - \textstyle\sum_{k=1}^{\infty} f_k/M^k).$$

Observe now that if $M > 1$, we necessarily have $\sum_{k=1}^{\infty} f_k/M^k < 1$, and thus we must conclude that

$$\lambda' \geqq M.$$

[4] The use of such a sequence was suggested by a new proof of the renewal theorem due to W. Feller [3]. See also Choquet and Deny [1].

2.

We shall proceed to show that condition $A_n(\mu_1, \mu_2, \cdots, \mu_{N-1}; \lambda)$ is sufficient to guarantee that $m > 0$ and $M < \infty$.

2.1. We begin by establishing

LEMMA 2.1. *Under the assumption* (ii) (*and therefore also* (a), (b) *of Section* 1) *for every $k \geqq N_0$ there exists a constant $C(k) > 0$ such that for $n \geqq N_0 + k$*

2.11 $$r_{n-1} r_{n-2} \cdots r_{n-k} \geqq C(k).$$

Proof. We observe first that by (b) there exist ρ and $k_1, \cdots k_\rho$ such that every $k \geqq N_0$ can be written in the form

2.12 $$k = k_1 n_1 + k_2 n_2 + \cdots + k_\rho n_\rho \qquad (n_i \geqq 0).$$

On the other hand, from 1.24 ($M(t) = 0$) we obtain for each $k < n$

2.13 $$u_n \geqq f_k u_{n-k},$$

and for $k = k_i$

2.14 $$r_{n-1} r_{n-2} \cdots r_{n-k_i} \geqq f_{k_i} \qquad \text{for } n \geqq N_0 + k_i.$$

Using 2.12, 2.13, and 2.14 we get (by grouping terms)

$$r_{n-1} r_{n-2} \cdots r_{n-k} \geqq (f_{k_1})^{n_1}(f_{k_2})^{n_2} \cdots (f_{k_\rho})^{n_\rho},$$

for $k \geqq N_0$ and $n > k$. Thus we get 2.11 with

$$C(k) = (f_{k_1})^{n_1}(f_{k_2})^{n_2} \cdots (f_{k_\rho})^{n_\rho}.$$

2.2. To obtain a lower bound for $r_{n-1} r_{n-2} \cdots r_{n-k}$ for small k it is necessary to assume some condition in addition to (ii). In fact, it can be shown by examples that if $f_1 = 0$, then m need not be greater than zero. We shall also show that it is sufficient to assume condition $A_N(\mu_1, \mu_2, \cdots, \mu_{N-1}; \lambda')$.

LEMMA 2.21. *If for each $k > N_0$ there exist $n(k)$ and $C(k)$ such that*

2.21 $$r_{n-1} r_{n-2} \cdots r_{n-k} \geqq C(k) > 0 \qquad \textit{for } n \geqq n(k)$$

and condition $A_N(\mu_1, \mu_2, \cdots, \mu_{N-1}; \lambda')$ holds, then there exist constants $C(k)$ and $n(k)$ such that 2.21 *holds also for* $1 \leqq k \leqq N_0$. *In addition we have that*

2.22 $$M = \limsup_{n\to\infty} u_{n+1}/u_n < \infty.$$

Proof. We shall proceed by reverse induction. We assume that for each $k \geqq k_0 > 1$ there exist constants $C(k)$ and $n(k)$ ($n(k) > k$) so that 2.21 is satisfied and shall deduce the same result for $k = k_0 - 1$.

Let λ be a given number greater than λ'. In view of the hypothesis, 1.27 will hold for a sufficiently large N_2. Thus 1.28 holds, and we get

2.23 $$r_n \leqq \lambda + |f_1^M| + \sum_{k=1}^{N_2} \frac{|f_{k+1}^M - \lambda f_k^M|}{r_{n-1} r_{n-2} \cdots r_{n-k}}.$$

Dividing this inequality by $r_n r_{n+1} \cdots r_{n+k_0-1}$, in view of the induction hypothesis, we obtain that

$$\text{2.24} \qquad \frac{1}{r_{n+1} \cdots r_{n+k_0-1}} \leqq \frac{\lambda + |f_1^M|}{C(k_0)} + \sum_{k=1}^{N_2} \frac{|f_{k+1}^M - \lambda f_k^M|}{C(k_0 + k)},$$

at least when

$$\text{2.25} \qquad n \geqq \max\{N_0, n(k_0), \cdots, n(k_0 + N_2)\}.$$

We then define $[C(k_0 - 1)]^{-1}$ to be equal to the right-hand side of 2.24, and $n(k_0 - 1)$ to be equal to the right-hand side of 2.25 plus k_0.

To prove the last assertion of the lemma, we observe that from 2.23 we obtain

$$r_n \leqq \lambda + |f_1^M| + \sum_{k=1}^{N_2} \frac{|f_{k+1}^M - \lambda f_k^M|}{C(k)}.$$

2.3. We can now combine the results in Lemmas 1.41, 1.42, 1.43, and 2.21 to obtain the following:

THEOREM 2.3. *If* (i) *and* (ii) *hold and*

$$1 \smile \limsup_{n\to\infty} f_{n+1}/f_n = \lambda < \infty,$$

we have

$$\lambda F(1/\lambda) \leqq \liminf_{n\to\infty} u_{n+1}/u_n \leqq \limsup_{n\to\infty} u_{n+1}/u_n \leqq \lambda.$$

If, in addition $\lambda = 1$, *then* u_{n+1}/u_n *is convergent, and*

$$\lim_{n\to\infty} u_{n+1}/u_n = 1.$$

Remark. The work of the last section makes evident that $r_n \to 1$ if and only if equality always holds in 1.34. Each of these conditions is equivalent to

$$\lim_{N\to\infty} \left[\sup_{n \geqq N} \frac{\sum_{k=N}^{n} f_k u_{n-k}}{u_n} \right] = 0.$$

Clearly $r_n \to 1$ implies the truth of this condition. Conversely if the condition holds, we can pass to the limit in 1.24 along $\{n_k + \alpha\}$ (with $M(t) = 0$) and obtain equality in 1.34.

3.

In this section we shall be concerned with the proof of the following:

THEOREM 3. *If the sequence* $f_1, f_2, \cdots, f_n, \cdots$ *satisfies* (i), (ii), *and condition* B_N, *then*

$$\lim_{n\to\infty} u_{n+1}/u_n = 1.$$

3.1. Before proceeding with our arguments we need to establish an auxiliary result which is of some intrinsic interest.

THEOREM 3.1. *Under the assumptions* (i), (ii), *and* $M < \infty$, *we shall have*[5]

3.11 $$\limsup_{n\to\infty} u_{n+N}/u_n \leqq 1$$

for some $N \geqq 1$ *only if*

3.12 $$\lim_{n\to\infty} u_{n+1}/u_n = 1.$$

Proof. For $N = 1$ the theorem follows from Lemma 1.41. Suppose $N \geqq 2$, and let n_k be a determining subsequence such that $R_0 = M$. The assumption in 3.11 implies that

3.13_α $$R_{\alpha-1} R_{\alpha-2} \cdots R_{\alpha-N} \leqq 1, \qquad \alpha = 0, \pm 1, \pm 2, \cdots .$$

Suppose we set

3.14 $$\Gamma_\alpha = R_{\alpha-2} \cdots R_{\alpha-N} + R_{\alpha-3} \cdots R_{\alpha-N} + \cdots + R_{\alpha-N} + 1.$$

The inequality $3.13_{\alpha-1}$ can also be written in the form

3.15_α $$\Gamma_\alpha R_{\alpha-N-1} \leqq \Gamma_{\alpha-1}.$$

A repeated application of 3.15 yields

3.16 $$\Gamma_\alpha R_{\alpha-N-1} R_{\alpha-N-2} \cdots R_{\alpha-N-k} \leqq \Gamma_{\alpha-k}.$$

We thus have that

3.17 $$\sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-N-1} \cdots R_{\alpha-N-k}} \frac{\Gamma_{\alpha-k}}{\Gamma_\alpha} \geqq \sum_{k=1}^{\infty} f_k = 1.$$

On the other hand, in view of 1.34 written for $M(t) = 0$ and with $\alpha - j + 1$ in place of α we shall have

3.18 $$\sum_{k=1}^{\infty} \frac{f_k R_{\alpha-k-j} \cdots R_{\alpha-k-N}}{R_{\alpha-j} \cdots R_{\alpha-N} R_{\alpha-N-1} \cdots R_{\alpha-N-k}} = \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-j} \cdots R_{\alpha-j+1-k}} \leqq 1,$$

and this implies that

$$\sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-N-1} \cdots R_{\alpha-N-k}} \Gamma_{\alpha-k} \leqq \Gamma_\alpha.$$

Therefore equality must hold in 3.17 and 3.18. Since α is arbitrary, we shall have

$$\sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-1} \cdots R_{\alpha-k}} = 1 \qquad \text{for } \alpha = 0, \pm 1, \pm 2, \cdots;$$

thus also

3.19 $$R_\alpha = \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-1} \cdots R_{\alpha-k}} R_{\alpha-k}.$$

By assumption $R_0 = M$, and of course $R_\alpha \leqq M$ for all other α. We deduce that $R_{-k} = M$ for each k such that $f_k \neq 0$. And by (ii) for a suitable α_0,

[5] The theorem remains true even if 3.11 is weakened to $\limsup u_{(n+1)N}/u_{nN} \leqq 1$ as $n \to \infty$.

we shall have

$$R_\alpha = M \qquad \text{for all } \alpha \leqq \alpha_0 .$$

On the other hand, 3.19 written for $\alpha = \alpha_0$ gives

$$M = \textstyle\sum_{k=1}^{\infty} f_k/M^{k-1},$$

but because of (i) this equality can only hold for $M = 1$.

3.2. We proceed with the proof of Theorem 3. Since condition B_N guarantees (by Lemmas 2.1 and 2.21) that $M < \infty$, in view of Theorem 3.1 we need only establish that

3.21 $$\limsup_{n\to\infty} u_{n+N}/u_n \leqq 1.$$

We shall thus pick a determining subsequence $n_1, n_2, \cdots, n_k, \cdots$ such that

3.22 $$R_0 R_1 \cdots R_{N-1} = \limsup_{n\to\infty} u_{n+N}/u_n = M^*,$$

and suppose $M^* > 1$. We let $M(t) = t + t^2 + \cdots + t^{N-1}$. Condition B_N guarantees that 1.35 will be satisfied for any $\lambda > 1$. We shall therefore have also

$$R_\alpha \leqq 1 + f_1^M + \sum_{k=1}^{\infty} \frac{f_{k+1}^M - f_k^M}{R_{\alpha-1} \cdots R_{\alpha-k}} .$$

This inequality can be written for $\alpha - 1$ in the form

3.23 $$1 \leqq \frac{1 + f_1^M}{R_{\alpha-1}} + \sum_{k=1}^{\infty} \frac{f_{k+1}^M - f_k^M}{R_{\alpha-1} \cdots R_{\alpha-k-1}} .$$

From 1.22 an easy calculation yields

$$f_1^M + 1 = f_1, \quad f_2^M - f_1^M = f_2, \quad \cdots, \quad f_{N-1}^M - f_{N-2}^M = f_{N-1} ;$$
$$f_N^M - f_{N-1}^M = f_N + 1,$$

and for $k \geqq 1$

$$f_{N+k}^M - f_{N+k-1}^M = f_{N+k} - f_k .$$

Substituting in 3.23 we obtain

3.24 $$1 \leqq \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-1} \cdots R_{\alpha-k}} + \frac{1}{R_{\alpha-1} \cdots R_{\alpha-N}} - \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-1} \cdots R_{\alpha-(N+k)}} .$$

Observe now that for each k we can write

$$\frac{1}{R_{\alpha-1} \cdots R_{\alpha-k}} = \frac{1}{R_{\alpha-1} \cdots R_{\alpha-N}} \frac{R_{\alpha-k-1} \cdots R_{\alpha-k-N}}{R_{\alpha-N-1} \cdots R_{\alpha-N-k}},$$

so that 3.24 can be given the more suggestive form

$$R_{\alpha-1} \cdots R_{\alpha-N} - 1 \leqq \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-N-1} \cdots R_{\alpha-N-k}} (R_{\alpha-k-1} \cdots R_{\alpha-k-N} - 1).$$

For convenience we shall set $R_{\alpha-1} R_{\alpha-2} \cdots R_{\alpha-N} = P_\alpha$, so that we get

3.25 $$P_\alpha - 1 \leqq \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-N-1} \cdots R_{\alpha-N-k}} (P_{\alpha-k} - 1).$$

The assumptions imply that $P_\alpha \leqq M^*$ and $P_N = M^*$. From 3.25 we obtain that if, for some α_0, $P_{\alpha_0} = M^*$, then necessarily $P_{\alpha_0 - k} = M^*$ for all k such that $f_k \neq 0$. In addition we must have

$$\sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha_0-N-1} \cdots R_{\alpha_0-N-k}} = 1.$$

In view of (ii) we deduce that there is an α_0 such that for all $\alpha \leqq \alpha_0$

$$\text{(e)} \quad P_\alpha = M^*, \qquad \text{and} \qquad \text{(ee)} \quad \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-1} \cdots R_{\alpha-k}} = 1.$$

From (e) we deduce that $R_\alpha = R_{\alpha-N}$ for all $\alpha < \alpha_0$. On the other hand (ee) for $\alpha + 1$ in place of α can be written in the form

$$R_\alpha = \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-1} \cdots R_{\alpha-k}} R_{\alpha-k}.$$

Let $R = \max (R_{\alpha_0-1}, R_{\alpha_0-2}, \cdots, R_{\alpha_0-N})$. For $\alpha < \alpha_0$ we have also

$$R - R_\alpha = \sum_{k=1}^{\infty} \frac{f_k}{R_{\alpha-1} \cdots R_{\alpha-k}} (R - R_{\alpha-k}).$$

Consequently if $R_{\alpha_1} = R$ for some α_1, we have $R_{\alpha_1-k} = R$ for all k such that $f_k \neq 0$. This implies (in view of (ii)) that

$$R_\alpha = R \qquad \text{for all } \alpha < \alpha_0.$$

Writing (e) and (ee) for such an α we obtain

$$R^N = M^*, \qquad \textstyle\sum_{k=1}^{\infty} f_k/R^k = 1,$$

and this, in view of (i), gives the desired contradiction.

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