ON FINITE GROUPS WITH DIHEDRAL SYLOW 2-SUBGROUPS

BY

DANIEL GORENSTEIN AND JOHN H. WALTER[1]

PART I. INTRODUCTION AND PRELIMINARIES

1. Introduction

Much attention has recently been given to the characterization of classes of simple groups in terms of conditions which specify the centralizers of their involutions[2] or their Sylow 2-subgroups. (Cf. R. Brauer [2]; R. Brauer, M. Suzuki, and G. E. Wall [7]; W. Feit [11], M. Suzuki [15], [16], [17], [18], [19]; and J. H. Walter [21], [22].) This paper presents such a characterization for the simple groups[3] $PSL(2, q)$, where q is an odd prime power, and improves the results obtained in [7] and [16].

It is easy to show that in a group with a dihedral[4] Sylow 2-subgroup S the centralizer of an involution τ in the center of S has a normal 2-complement U, and our characterization is given in terms of the structure of U.

THEOREM I. *Let G be a finite group with a dihedral Sylow 2-subgroup S, and let τ be an involution in the center of S. Suppose that the centralizer of τ possesses an abelian 2-complement U. Then G contains a normal subgroup K of odd order and one of the following holds:*

(i) *G has no normal subgroups of index 2, and G/K is isomorphic to $PSL(2, q)$ with q odd or to the alternating group A_7;*

(ii) *G contains a normal subgroup of index 2 but no normal subgroup of index 4, and G/K is isomorphic[3] to $PGL(2, q)$ with q odd;*

(iii) *G contains a normal subgroup of index 4, and G/K is isomorphic to a Sylow 2-subgroup S of G.*

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[2] An involution is understood to be an element of order 2.

[3] These are the groups of unimodular projectivities of a projective line defined with a finite coordinate field F_q of q elements. The groups $PGL(2, q)$ are the groups of projectivities of the projective line defined over F_q. Also $PSL(2, q)$ and $PGL(2, q)$ are the homomorphic images of the unimodular group $SL(2, q)$ and the general linear group $GL(2, q)$ by the homomorphism of $GL(2, q)$ onto $GL(2, q)/Z_2$ where Z_2 is the subgroup of scalar transformations.

[4] We also understand that an elementary abelian group of type (2, 2) is a dihedral group. Such groups will be called *four-groups* in this paper.

The dihedral group S is generated by elements α and β which satisfy the relations $\alpha^{2^a} = \beta^2 = 1$ and $\beta\alpha\beta^{-1} = \alpha^{-1}$. In [7], Brauer, Suzuki, and Wall[5] treat the special case in which β induces an automorphism which inverts every element of U, the centralizer in G of any element $x \neq 1$ of U is contained in the centralizer of τ, and G possesses no normal subgroups of index 2; while in [16], Suzuki treats the special case in which S has order 8, α induces an automorphism which inverts every element of U, and G possesses no normal subgroups of index 2. (In both cases the action of α and β forces U to be abelian.)

The hypotheses of our theorem are satisfied when S has order 4 and is its own centralizer in G. Therefore we obtain the following corollary, which verifies a conjecture of Brauer [3].

COROLLARY. *Let G be a finite group of order $4g'$, g' odd, and assume that a Sylow 2-subgroup of G is its own centralizer. Then G contains a normal subgroup K of odd order such that G/K is isomorphic either to a Sylow 2-subgroup of G or to $PSL(2, q)$, where $q \equiv 3, 5 \pmod 8$.*

The hypotheses of Theorem I are also satisfied when the centralizer of τ is itself a dihedral group. Furthermore, Theorem I can be combined with [14] and [16] to obtain additional results on groups which possess a subgroup of order 4 which is its own centralizer including, as a special case, a classification of groups which admit an automorphism of order 2 with exactly two fixed points. These results are established in §15.

The proof of Theorem I is carried out by induction on the order of G. Because of this, Case (ii) is shown to be a consequence of Case (i); Case (iii) is established from theorems of Burnside and Grün. Therefore, the proof of Theorem I reduces to Case (i). In this regard, most of our attention is paid to investigating p-groups in G which admit a four-group as a group of automorphisms. Utilizing a recent theorem of Brauer and Suzuki concerning this class of groups, which has now been presented in a very general and elegant context by H. Wielandt [23], we are able to construct p-subgroups whose orders may be compared with certain factors in the formulas for the order of G derived from character theory. These groups are constructed for those primes p for which there exists a p-element of U whose centralizer in G is not contained in U. Using the formulas for the order of G together with the group-theoretic information, we are able to show that G either satisfies conditions in terms of which Suzuki [16] characterized A_7, or that G satisfies conditions in terms of which Brauer, Suzuki, and Wall [7] characterized the groups $PSL(2, q)$, q odd.[5]

Part of the character theory which we develop holds for any finite group containing a dihedral Sylow 2-subgroup and no normal subgroups of index 2.

[5] Actually they assume, in addition, that the group U is cyclic. The generalization to the abelian case, which we need, has been carried out and, in fact, can be obtained by making minor adjustments in their arguments.

We obtain formulas for the order of these groups which are of independent interest. The assumptions of Theorem I are employed to obtain an additional formula for the order of G and congruences for the degrees of certain characters.

For convenience, we shall call a group G which satisfies the hypothesis of Theorem I an *L-group*. We shall denote by $O(H)$ the maximal normal subgroup of odd order in a group H. Also $C_H^*(\sigma)$ denotes the set of elements λ in H such that $\lambda^{-1}\sigma\lambda = \sigma^{\pm 1}$. In other respects our notation is standard.

2. The action of certain automorphism groups

The results of a recent paper of Wielandt [23] are very important to us. We restate without proof two of his preliminary results and also a particular case of his main theorem.

LEMMA 1. *Let T be a solvable group*[6] *of automorphisms of a group K such that $(|T|, |K|) = 1$. Then for each p dividing $|K|$, K possesses a T-invariant*[7] *Sylow p-subgroup, and each maximal T-invariant p-subgroup of K is a Sylow subgroup.*

LEMMA 2. *Let T be a group of automorphisms of a group K such that $(|T|, |K|) = 1$. Then two T-invariant Sylow p-subgroups of K are conjugate by an element of K left fixed by T.*

If P is a T-invariant Sylow subgroup of K, and P' is the subgroup of P left fixed[7] *by a subgroup T' of T, then P' is a Sylow subgroup of the subgroup K' of K left fixed by T'.*

LEMMA 3. *Let T be a four-group*[4] *of automorphisms of a group K of odd order. Let τ_i, $i = 1, 2, 3$, denote the three involutions of T, and K_i the fixed subgroup of τ_i. Then if K_0 is the fixed subgroup of T,*

$$|K|\,|K_0|^2 = |K_1|\,|K_2|\,|K_3|. \tag{1}$$

An important structural consequence of Lemma 3 is given in part (ii) of the following lemma. This result was also discovered independently by Steven Bauman.

LEMMA 4. *Let T and K be defined as in Lemma 3.*

(i) *$K_i = K_i' K_0 = K_0 K_i'$ where K_i' is the subset of K_i consisting of the elements inverted*[8] *by τ_j, $j \neq i$. If $K_0 = 1$, $K_i = K_i'$ is abelian.*

(ii) *K admits the factorizations*

$$K = K_1 K_2 K_3 = K_0 K_1' K_2' K_3'. \tag{2}$$

[6] This assumption of solvability is necessary to secure the well-known result on the existence of invariant Sylow p-groups which we include in the statement of this lemma.

[7] A subset H of a group K is said to be left *invariant* by a group T of automorphisms of K if $H^T = H$; if $x^\sigma = x$ for all x in H and σ in T, then H is said to be left *fixed* by T.

[8] An element x of G is said to be *inverted* by an element σ if $x^\sigma = x^{-1}$. A subset is said to be *inverted* by σ if each of its elements is inverted by σ.

An element of K has a unique representation as a product $a_0 a_1 a_2 a_3$, where $a_0 \in K_0$ and $a_i \in K_i'$, $i = 1, 2, 3$. Also $K_i \cap K_j = K_0$ and $K_i' \cap K_j' = 1$, $i \neq j$.

Proof. (i) If $j \neq i$, τ_j induces an automorphism of K_i of order 2 with fixed subgroup K_0 and inverted set K_i'. The first assertion of the lemma now follows from [12, Lemma 1]. If $K_0 = 1$, then $K_i = K_i'$ and, consequently, is abelian.

(ii) It suffices to show that $K = K_0 K_1' K_2' K_3'$. Because of Lemma 3, this will follow when we prove the uniqueness of the representation of an element x as the product $x = a_0 a_1 a_2 a_3$ where $a_0 \in K_0$ and $a_i \in K_i'$, $i = 1, 2, 3$. Hence suppose that

$$a_0 a_1 a_2 a_3 = b_0 b_1 b_2 b_3, \tag{3}$$

where $b_0 \in K_0$ and $b_i \in K_i'$, $i = 1, 2, 3$. Using (i), we have that

$$b_1^{-1} b_0^{-1} a_0 a_1 = c_0 c_1 \quad \text{and} \quad b_3 a_3^{-1} = d_3 d_0$$

where c_0, $d_0 \in K_0$, $c_1 \in K_1'$, and $d_3 \in K_3'$. Hence

$$c_0 c_1 a_2 = b_2 d_3 d_0. \tag{4}$$

Applying τ_1, τ_2, and τ_3 in succession gives

$$c_0 c_1 a_2^{-1} = b_2^{-1} d_3^{-1} d_0, \qquad c_0 c_1^{-1} a_2 = b_2 d_3^{-1} d_0, \qquad c_0 c_1^{-1} a_2^{-1} = b_2^{-1} d_3 d_0. \tag{5}$$

From (4) and (5), we obtain

$$a_2^2 = d_0^{-1} d_3 b_2^2 d_3 d_0 = d_0^{-1} d_3^{-1} b_2^2 d_3^{-1} d_0. \tag{6}$$

Hence

$$b_2^{-2} d_3^2 b_2^2 = d_3^{-2}. \tag{7}$$

As both b_2 and d_3 have odd order, (7) implies $d_3 = 1$; hence $b_3 = d_0 a_3$. Since each coset of K_0 in K_3 contains a unique element of K_3', $d_0 = 1$. Consequently $b_3 = a_3$. A similar calculation shows that $c_1 = 1$, and hence from (4), $c_0 = 1$ and $a_2 = b_2$. Thus $a_0 a_1 = b_0 b_1$; whence $a_0 = b_0$ and $a_1 = b_1$.

Let K be a group of odd order which admits a four-group $T = \{\tau_1, \tau_2, \tau_3\}$ as a group of automorphisms. Then the decompositions (2) will be called *T-decompositions of K*. In the following lemma, G will be a group containing a normal subgroup M. Designate by $\bar{H}$ and $\bar{\sigma}$ the images of a subgroup and an element σ of G in $\bar{G} = G/M$.

LEMMA 5. *Let G be a finite group containing a four-group T and a normal subgroup M of odd order. Then if τ is an involution of T, $\overline{C_G(\tau)} = C_{\bar{G}}(\bar{\tau})$. Also $\overline{C_G(T)} = C_{\bar{G}}(\bar{T})$. If K is a T-invariant subgroup of G of odd order with T-decompositions (2), then $\bar{K}$ has the $\bar{T}$-decomposition*

$$\bar{K} = \bar{K}_1 \bar{K}_2 \bar{K}_3 = \bar{K}_0 \bar{K}_1' \bar{K}_2' \bar{K}_3'. \tag{8}$$

Proof. Let $T = \{\tau_1, \tau_2, \tau_3\}$. Clearly $\overline{C_G(\tau_i)} \leqq C_{\bar{G}}(\bar{\tau}_i)$ and $\overline{C_G(T)} \leqq C_{\bar{G}}(\bar{T})$. Let $\sigma \in G$, and suppose that for given $i = 1, 2, 3$, $\sigma^{\tau_i} = \sigma\lambda_i$ where $\lambda_i \in M$. Then $\lambda_i \lambda_i^{\tau_i} = 1$. Since $|M|$ is odd, $\lambda_i = \mu_i^2$ where $\mu_i \in M$. Then $(\sigma\mu_i)^{\tau_i} = \sigma\mu_i$. Thus $\overline{C_G(\tau_i)} = C_{\bar{G}}(\bar{\tau}_i)$.

We may now suppose that σ has been chosen in σM so that $\sigma^{\tau_1} = \sigma$ and $\sigma^{\tau_2} = \sigma\lambda_2$. Then $(\sigma\lambda_2)^{\tau_1} = \sigma\lambda_2$ and $\lambda_2^{\tau_1} = \lambda_2$. As μ_2 is a power of λ_2, $\mu_2^{\tau_1} = \mu_2$. Consequently, $(\sigma\mu_2)^{\tau_i} = \sigma\mu_2$, $i = 1, 2, 3$; hence $\overline{C_G(T)} = C_{\bar{G}}(\bar{T})$.

The remaining statement is a direct consequence of the foregoing.

LEMMA 6. *Let $T = \{\tau_1, \tau_2, \tau_3\}$ be a four-group of automorphisms of a p-group P, p odd, with T-decomposition $P = P_1 P_2 P_3$, and assume that each P_i is abelian. Let $P_0 = C_P(T)$, and let P_i', $i = 1, 2, 3$, be the subset of P_i inverted by τ_j, $j \neq i$. Then the complex $P_1' P_2' P_3'$ is a group, and we have*

$$P = P_0 \times P_1' P_2' P_3'. \tag{9}$$

Proof. The lemma is proved by induction on the order of P. We first note that $P_i = P_0 \times P_i'$ because of Lemma 4(i) and the fact that P_i is abelian of odd order. Then $P_0 \leqq Z(P_i)$, $i = 1, 2, 3$; hence $P_0 \leqq Z(P)$.

Suppose first that $P_0 < Z(P)$. Then T acts on $Z(P)$ which is abelian. Hence $Z(P)$ admits the T-decomposition by virtue of Lemmas 2 and 4

$$Z(P) = (P_1 \cap Z(P))(P_2 \cap Z(P))(P_3 \cap Z(P)).$$

Since $P_0 \leqq Z(P)$, we must have that $P_i \cap Z(P) = P_0 \times P_i' \cap Z(P)$. Set $P_i'' = P_i' \cap Z(P)$. Considering the way the involutions of T act on P_0 and P_i'', $i = 1, 2, 3$, we see that

$$Z(P) = P_0 \times P_1'' \times P_2'' \times P_3''.$$

This proves the lemma in case $P = Z(P)$. Otherwise set

$$F = P_1'' \times P_2'' \times P_3'';$$

since by assumption $P_0 < Z(P)$, $F \neq 1$. We may then apply the lemma by induction to $\bar{P} = P/F$. Adopting an analogous notation for $\bar{P}$, we have $\bar{P} = \bar{P}_0 \times \bar{P}'$ where $\bar{P}' = \bar{P}_1' \bar{P}_2' \bar{P}_3'$. It is an easy consequence of Lemma 5 that $\bar{P}_i'$ is the image of P_i' under the natural mapping of P onto $\bar{P}$. Since $F \cap P_0 = 1$, the inverse image P' of $\bar{P}'$ is disjoint from P_0, and hence $P = P_0 \times P'$. Since P' contains each P_i' and is invariant under T, $P' = P_1' P_2' P_3'$.

Finally assume that $P_0 = Z(P) < P$. Consider the second center $Z_2(P)$, which has the T-decomposition

$$Z_2(P) = (P_1 \cap Z_2(P))(P_2 \cap Z_2(P))(P_3 \cap Z_2(P)).$$

As $Z_2(P) > Z(P) = P_0$, $P_i \cap Z_2(P) > P_0$ for some $i = 1, 2, 3$. Since P_i is abelian, $P_i \cap Z_2(P) = P_0 \times (P_i' \cap Z_2(P))$. Choose $x_i \neq 1$ in $P_i' \cap Z_2(P)$.

If $x_j \in P_j$, $j \neq i$, then

$$x_j^{-1} x_i x_j = x_i x_0, \tag{10}$$

where $x_0 \in Z(P) = P_0$. Applying τ_j to (10), we obtain

$$x_j^{-1} x_i^{-1} x_j = x_i^{-1} x_0. \tag{11}$$

Combining (11) with (10) gives $x_i x_0 = x_0^{-1} x_i$, and hence $x_0 = 1$. Thus x_i centralizes P_j, $j \neq i$. Since P_i is abelian, $x_i \in Z(P)$, which is a contradiction. This excludes the last possibility and proves the lemma.

LEMMA 7. *Let the hypotheses and notation be as in Lemma 6, and assume $P > P_1$. Then there exists an element σ in P_2' or P_3' which centralizes P_1.*

Proof. Let $Z_r(P)$ be the first term of the ascending central series of P which is not contained in P_1. Since $Z_r(P)$ is T-invariant, it follows from Lemma 4 that

$$Z_r(P) = (Z_r(P) \cap P_1)(Z_r(P) \cap P_2')(Z_r(P) \cap P_3').$$

One of $Z_r(P) \cap P_2'$ or $Z_r(P) \cap P_3'$ contains an element $\sigma \neq 1$, and σ is inverted by τ_1. Hence for $x \in P_1$, $\sigma x \sigma^{-1} = x\mu$, where $\mu \in Z_{r-1}(P) \leqq P_1$. Conjugating by τ_1 gives $\sigma^{-1} x \sigma = x\mu$, and it follows that σ^2 commutes with x. Since σ has odd order, σ commutes with x. Since x was arbitrary, σ centralizes P_1.

3. The Sylow 2-subgroups and the involutions in G

Let G be a group with dihedral Sylow 2-subgroups, and let S be a fixed Sylow 2-subgroup of G. S is generated by elements α and τ_2, where α has order 2^a, $a \geqq 1$, τ_2 has order 2, and $\tau_2 \alpha \tau_2^{-1} = \alpha^{-1}$. Set $\tau_1 = \alpha^{2^{a-1}}$, $\beta = \alpha^2$, and if $a \geqq 2$, $\gamma = \alpha^{2^{a-2}}$. Also define $\tau_3 = \tau_1 \tau_2$. When $a \geqq 2$, $A = \{\alpha\}$ is the unique cyclic subgroup of index 2 in S and thus is characteristic in S; $B = \{\beta\}$ is the commutator subgroup of S.

There are three classes of involutions in S; namely, one consisting of the central involution τ_1 above, a second consisting of the elements $\tau_2 \alpha^{2i}$, $i = 1, 2, \cdots, 2^{a-1}$, and a third consisting of the elements $\tau_2 \alpha^{2i+1}$, $i = 1, 2, \cdots, 2^{a-1}$. Consequently when $a > 1$, there are two conjugate classes of four-groups which are represented by $S_0 = \{\tau_1, \tau_2\}$ and $S_1 = \{\tau_1, \tau_2 \alpha\}$. When $a > 1$, $\tau_2^\gamma = \tau_3$, $\tau_3^\gamma = \tau_2$, $(\tau_2 \alpha)^\gamma = \tau_1(\tau_2 \alpha)$, and $(\tau_1(\tau_2 \alpha))^\gamma = \tau_2 \alpha$.

In particular, γ normalizes any four-subgroup T of S. When $a > 1$, let $C_{G,S}'(T)$ be the subgroup of G generated by γ and $C_G(T)$. Generally, there will be no confusion if we set $C_G'(T) = C_{G,S}'(T)$, although there may be other Sylow subgroups of G containing T and an element $\gamma \in N_G(T) - T$. When $a = 1$, set $C_G'(T) = C_G(T)$. Then $N_G(T)/C_G(T)$ is isomorphic to a subgroup of the symmetric group on 3 letters. Hence $|N_G(T):C_G'(T)| = 1$ or 3.

When $N_G(T) > C_G'(T)$, there exists a 3-element ρ in $N_G(T) - C_G'(T)$ such that $\tau_1^\rho = \tau_2$, $\tau_2^\rho = \tau_3$, and $\tau_3^\rho = \tau_1$.

LEMMA 8. *Let G be a group with a dihedral Sylow 2-subgroup S. Then one of the following holds:*

(i) *G contains no normal subgroups of index 2. Then all involutions in G are conjugate, $N_G(S_0) > C'_G(S_0)$, and when $S > S_0$, $N_G(S_1) > C'_G(S_1)$.*

(ii) *G contains a normal subgroup of index 2, but no normal subgroup of index 4. Then $S > S_0$, and for exactly one value of $i = 0, 1, N_G(S_i) > C'_G(S_i)$.*

(iii) *G contains a normal subgroup of index 4. Then G possesses a normal 2-complement.*

In particular, $C_G(\tau_1)$ has a normal 2-complement.

Proof. Suppose first that $S = S_0$. If G contains no normal subgroups of index 2, then by Burnside's theorem [13, p. 203], $N_G(S_0) > C_G(S_0)$. Then all involutions in S and consequently in G are conjugate. Conversely, if G contains a normal subgroup of index 2, $N_G(S_0) = C_G(S_0)$, and Burnside's theorem implies that G has a normal 2-complement.

Now assume that $S > S_0$. The automorphism group of S is a 2-group, and hence $N_G(S) = SC_G(S)$. It follows then from Grün's theorem [13, p. 214] that the maximal abelian 2-factor group of G is isomorphic to S/S_2, where S_2 is generated by the subgroups $S \cap S'^{\sigma}$, $\sigma \in G$. Since $S' = B$, A is not contained in $S \cap S'^{\sigma}$ for any σ, and it follows that $S = S_2$ if and only if τ_2 and $\tau_2\alpha$ lie in some $S \cap S'^{\sigma}$. This is equivalent to having τ_1 conjugate to τ_2 and to $\tau_2\alpha$ in G. Thus if G has no normal subgroups of index 2, all the involutions of S and hence of G are conjugate. Furthermore, there exists then an element σ such that $\tau_1^{\sigma} = \tau_2$. Replacing γ^{σ} by $\gamma^{\sigma\lambda}$ where $\lambda \in C(\tau_2)$, if necessary, we may suppose that γ^{σ} is in $C'_{G,S^{\sigma}}(S_0)$. Then γ^{σ} centralizes τ_2 and interchanges τ_1 and τ_3. A simple calculation shows that $\rho = \gamma\gamma^{\sigma}$ conjugates τ_1 into τ_3, τ_2 into τ_1, and τ_3 into τ_2. Thus $\rho \in N_G(S_0) - C'_G(S_0)$. A similar calculation shows that $N_G(S_1) > C'_G(S_1)$.

The same argument shows that G has a normal subgroup of index 2, but no normal subgroup of index 4 if and only if $S_2 = \{B, \tau_2\}$ or $\{B, \tau_2\alpha\}$; and it is clear that in this case $N_G(S_i) > C'_G(S_i)$ for exactly one value of $i = 0, 1$.

If G has a normal subgroup G_0 of index 4, then $S_2 = B$, and B is a Sylow 2-subgroup of G_0. Since B is cyclic, G_0 and hence G has a normal 2-complement.

Finally if $H = C_G(\tau_1)$, then $H \geqq S$, and τ_1 is not conjugate to τ_2 or $\tau_2\alpha$ in H. Hence by the preceding argument applied to H, H has a normal 2-complement.

4. Properties of $PSL(2, q)$, $PGL(2, q)$, and A_7

We will list the properties of these groups which are to be used in this paper. For further reference, the reader is referred to L. Dickson [9] or J. Dieudonné [10].

(A) *The groups $PSL(2, q)$, $PGL(2, q)$, q odd, and A_7 are L-groups of orders $\frac{1}{2}q(q^2 - 1)$, $q(q^2 - 1)$, and 2520, respectively; $PSL(2, q)$ and A_7 are* *simple groups except for* $PSL(2, 3)$, *which is isomorphic to the alternating group* A_4. *Also* $PSL(2, 5)$ *is isomorphic to* A_5.

(B) *Let* τ *and* τ' *be involutions in* $PSL(2, q)$ *and* $PGL(2, q) - PSL(2, q)$, *respectively. Then* $D = C_{PGL(2,q)}(\tau)$ *and* $D' = C_{PGL(2,q)}(\tau')$ *are dihedral groups of orders* $2(q - \delta)$ *and* $2(q + \delta)$, *respectively, where* $\delta = \pm 1$ *and* $\delta \equiv q \pmod 4$. $D \cap PSL(2, q)$ *and* $D' \cap PSL(2, q)$ *are dihedral groups of orders* $q - \delta$ *and* $q + \delta$, *respectively.* D *contains a Sylow* 2*-subgroup of* $PGL(2, q)$.

The elements of D and D' may be represented, respectively, by matrices in either (12) or (13), or in (13) or (12) according as $q \equiv 1$ or $q \equiv -1 \pmod 4$.

$$(12) \qquad \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \begin{pmatrix} ac & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$(13) \qquad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \begin{pmatrix} ac & bc \\ -b & a \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $a^2 + b^2 = 1$ and c is a nonsquare. We shall refer to these matrices as (12a), (12b), (12c), etc.

Two matrices of the form (12a), (12b), (13a), or (13b) represent the same coset in $PGL(2, q)$ if and only if they are negatives of each other. Thus the matrices (12a) and (13a) represent elements of cyclic groups C_0 and C_0' of orders $\frac{1}{2}(q - 1)$ and $\frac{1}{2}(q + 1)$ respectively, in $PSL(2, q)$. The matrices (12b) and (13b) represent, respectively, generators of cyclic groups C and C' in $PGL(2, q)$ which contain C_0 and C_0' as subgroups of index 2. The matrices (12c) and (13c) represent involutions in $PGL(2, q)$ which invert C and C', respectively. When $q \equiv 1 \pmod 4$, τ is represented by (12a) with $a^2 = -1$, and τ' by (13b) with $a = 0$ and $b = 1$. When $q \equiv -1 \pmod 4$, τ is represented by (13a) with $a = 0$ and $b = 1$, and τ' by (12b) with $a = 1$ and $c = -1$.

(C) *The groups* $PSL(2, q)$, q *odd, contain cyclic Hall subgroups of odd orders* $u = (q - \delta)/|S|$ *and* $(q + \delta)/2$. *Two such subgroups of the same order are conjugate, and distinct conjugate subgroups have trivial intersections.*

(D) *There are two conjugate classes of self-centralizing four-subgroups in* $PSL(2, q)$ *if its Sylow* 2*-subgroup has order greater than* 4, *and one class if its Sylow* 2*-subgroup has order* 4. *In* A_7, *there are two classes of four-subgroups, in one of which the subgroups are self-centralizing, and in the other the centralizers have order* 12.

We need the following embedding result.

LEMMA 9. *Let* G *be an* L*-group which contains a normal subgroup* G_0 *isomorphic to* $PSL(2, q)$, $PGL(2, q)$, *or* A_7. *Then* $G_0 O(G)$ *is a direct product*

$G_0 \times O(G)$, and $[G:G_0\, O(G)] \leqq 2$. If $[G:G_0\, O(G)] = 2$, then G_0 is isomorphic to $PSL(2, q)$, and G is the semidirect product $G_1\, O(G)$ where G_1 is isomorphic to $PGL(2, q)$.

Proof. We treat first the case that G_0 is isomorphic to $PSL(2, q)$ or $PGL(2, q)$. For the sake of convenience, we identify G_0 with its isomorphic image. Suppose first that $|G:G_0|$ is even. Then a dihedral Sylow 2-subgroup T of G_0 is a normal subgroup of a dihedral Sylow 2-subgroup S of G. Hence $|S:T| = 2$, and the 2-elements of G/G_0 are involutions. Let xG_0 be such an element; we shall show that $G_1 = \{G_0, x\}$ is isomorphic to $PGL(2, q)$.

We may assume that x is a 2-element, and hence that $x \in S$. If x is an involution, there is another involution y in T such that xy lies in the maximal cyclic subgroup A of S. Replacing xy by x, if necessary, we may suppose $x \in A$.

Now the automorphisms of $PSL(2, q)$ and $PGL(2, q)$ are well known [10, p. 97]; they are induced by a contragredient transformation of $GL(2, q)$ or by conjugation of $GL(2, q)$ by a semilinear transformation t which is defined relative to an automorphism σ_t of the underlying field F_q of q elements.

Let $\theta(x)$ be the outer automorphism of G_0 induced by conjugation by x. If $\theta(x)$ is induced by a contragredient transformation, then relative to a particular choice of the basis of the underlying vector space, $\theta(x)$ is the inverse transpose operation on the matrices (12) and (13). But then $\theta(x)$ will fix the four-group $T_0 \leqq T$ which is generated by τ and the matrix (12c) or (13c). Thus $C_G(T_0)$ contains a noncyclic abelian group of order 8. But this is impossible since the Sylow 2-subgroups of G are dihedral.

Hence $\theta(x)$ is induced by a semilinear transformation t relative to a field automorphism σ_t. Since $x \in A$, t may be assumed to have the matrix form (12a) or (13a) according as $q \equiv 1$ or $-1 \pmod 4$. But then t induces the mapping of the matrices (12) or (13) which sends each matrix with coefficients a and b into the corresponding matrix with coefficients a^{σ_t} and b^{σ_t}.

Clearly $|\sigma_t| \leqq 2$; assume $|\sigma_t| = 2$. In this case F_q is an extension of degree 2 over the fixed subfield F_r of σ_t. If $|F_r| = r$, then $q = r^2$; hence $q \equiv 1 \pmod 4$. Thus the elements of $C_{G_0}(\tau)$ are represented by the matrices (12a) and (13c). Since $x \in A$, x centralizes the elements of T represented by the matrices (12a). This means that t must also have the form (12a). But then it follows that

$$t^{-1}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} t = \begin{pmatrix} a^{\sigma_t} & 0 \\ 0 & a^{-\sigma_t} \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \pm \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

for any 2-element a of F_q. Hence either $a = i$ where $i^4 = 1$, or $a = \pm 1$ and is in F_r. Suppose that $r \equiv 1 \pmod 4$. Then $i \in F_r$. However as $q - 1 = r^2 - 1 = (r + 1)(r - 1)$, there is a 2-element $a \neq i$ in $F_q - F_r$; for this element, $a^{\sigma_t} \neq \pm a$, which is a contradiction. Suppose then that

$r \equiv -1 \pmod 4$. Then 4 divides $r + 1$, and again there is a 2-element $a \neq i$ in $F_q - F_r$. Thus we must have $\sigma_t = 1$, and so t is linear.

Now if $G_0 = PGL(2, q)$, this argument shows that $\theta(x)$ is an inner automorphism. Hence xG_0 contains an element x' of $C(G_0)$, and consequently $x'^2 \in Z(G_0)$. But $Z(G_0) = 1$, and so x' is an involution not in G_0, but centralizing a four-group of G_0. This gives a contradiction.

It follows that $G_0 = PSL(2, q)$. We know that $x \in A$, and hence x generates the maximal cyclic subgroup of $S \cap G_1$. On the other hand, the element $\bar{t}$ of $PGL(2, q)$ which is the image of t satisfies $\bar{t}^2 \in T$, and $\theta(\bar{t}^2) = \theta(x^2)$. But then $\theta(\bar{t}^2x^{-2}) = 1$, and $\bar{t}^2x^{-2}$ is a 2-element in $C(T)$. It follows that $\bar{t}^2 = x^2$. Since the automorphism $\theta(\bar{t})$ induced by $\bar{t}$ is the same as $\theta(x)$ and $\bar{t}^2 = x^2$, the mapping $x \to \bar{t}$ extends the identity mapping of G_0 onto $PSL(2, q)$ to an isomorphism of G_1 onto $PGL(2, q)$.

It remains to consider the case that the coset xG_0 has odd order. Since $G = N_G(T)G_0$ by the Frattini argument, every coset of G_0 contains an element of $N_G(T)$, and so we may assume $x \in N_G(T)$. If $|T| > 4$, $N_G(T)/C_G(T)$ is a 2-group, and so $x \in C_G(T)$. If $|T| = 4$, there exists a 3-element ρ in $N_{G_0}(T) - C'_{G_0}(T)$ and either x, $x\rho$, or $x\rho^2$ is in $C_G(T)$. Thus we may assume $x \in C_G(T) \leqq C_G(\tau)$. But $C_G(\tau) = SU_1$, where U_1 is abelian and contains x. Then x centralizes $C_{G_0}(\tau) = T(U_1 \cap G_0)$. Thus $\theta(x)$ fixes the matrices (12) or (13). If $\theta(x)$ is induced by the semilinear transformation t relative to the field automorphism σ_t, it follows that $a^{\sigma_t} = a$ for all $a \in F_q$. Thus $\sigma_t = 1$, and again t is linear.

We conclude as above that each coset of G_0 of odd order contains an element of $C_G(G_0)$. Thus $G_0 \times C(G_0)$ is a normal subgroup of G of index $\leqq 2$; and, consequently, $C_G(G_0) = O(G)$. If this index is 2, the previous discussion shows that G_0 is isomorphic to $PSL(2, q)$, G_1 is isomorphic to $PGL(2, q)$, and G is the semidirect product $G_1\, O(G)$, as desired.

When $G_0 = A_7$, the lemma still follows by essentially the same argument since A_7 admits only the automorphisms induced from the symmetric group S_7; and these play the role of a contragredient automorphism of $SL(2, q)$ and $GL(2, q)$. We omit the details.

5. Induction assumptions and reduction of the theorem

Henceforth G will denote a fixed but arbitrary L-group. *We shall assume that Theorem* I *is valid for all L-groups of order less than the order of G.*

LEMMA 10. *In proving Theorem* I, *we may suppose that G is simple.*

Proof. If G possesses a normal subgroup G_0 of index 2 but none of index 4, we may apply induction to G_0 to conclude that $G_0/O(G_0)$ is isomorphic to $PSL(2, q)$ or A_7. Since $O(G_0)$ is characteristic in G_0, it is normal in G; hence $O(G_0) = O(G)$. Now Lemma 9 applies to $\bar{G} = G/O(G)$, and we may conclude that G is isomorphic to $PGL(2, q)$ in this case.

If G has a normal subgroup of index 4, then G has a normal 2-complement by Lemma 8.

If G has a normal subgroup $K \neq 1$ of odd order, it follows from Lemma 5 that $\bar{G} = G/K$ is an L-group. By induction, the theorem holds for $\bar{G}$ and hence for G.

Thus there remains only the case where G contains normal subgroups of even order but none of index 2. Let G_0 be a normal subgroup of G, such that $G > G_0 > 1$. By Lemma 8, all involutions in G are conjugate and hence lie in G_0. Thus G_0 contains a Sylow 2-subgroup of G and consequently has odd index. Since G_0 is a normal subgroup of G, $O(G_0)$ is a normal subgroup of G. Hence $O(G_0) = 1$. Then by induction, G_0 is isomorphic to $PSL(2, q)$, $PGL(2, q)$, q odd, or A_7, or G_0 is a Sylow 2-subgroup of G. In any of the first three cases, Lemma 9 implies that $G = G_0 \times O(G)$. Since $O(G) = 1$, $G = G_0$, which is a contradiction.

In the remaining case, G_0 is a dihedral 2-group; and $C(G_0) = G_0$ since otherwise G would have a normal subgroup of odd order. If $|G_0| > 4$, then $G = N_G(G_0) = G_0$, which is a contradiction. If $|G_0| = 4$, the only possibility is that $|G:G_0| = 3$, in which case G is isomorphic to $PSL(2, 3)$. Theorem I then holds in this case. Thus we have reduced the proof of Theorem I to the case that G is simple.

It will be useful to combine Lemma 8 and the induction hypothesis in the following lemma, which is stated without proof.

LEMMA 11. *Let H be a proper subgroup of G containing a dihedral Sylow 2-subgroup S. One of the following cases holds:*

(i) *The group H contains no normal subgroups of index* 2; *and H/OH is isomorphic to $PSL(2, q)$, q odd, or to A_7.*

(ii) *The group H contains a normal subgroup of index 2 but none of index* 4; *$S > S_0$; $N_H(S_i) > C'_H(S_i)$ for exactly one value of $i = 0, 1$; and $H/O(H)$ is isomorphic to $PGL(2, q)$, q odd.*

(iii) *The group H contains a normal subgroup of index* 4; *$N_H(S_0) = C'_H(S_0)$, and, when $S > S_0$, $N_H(S_1) \doteq C'_H(S_1)$; $H/O(H)$ is isomorphic to S.*

Remark. Since all the involutions in G are conjugate, $O(C(\tau))$ is abelian for any involution τ. Hence the Sylow p-subgroups of $C(\tau)$ are unique for all odd p. It follows then from Lemmas 2 and 4 that, *if K is an S_i-invariant subgroup of G of odd order, where $i = 0, 1$, there is a unique S_i-invariant Sylow p-subgroup of K for every prime p dividing $|K|$. This fact is to be used repeatedly.*

PART II. THE STRUCTURE OF THE GROUP G

6. Structure of the centralizer $C_G(\tau_1)$

For brevity we shall henceforth write $C(H)$, $C(x)$, etc., for $C_G(H)$, $C_G(x)$, etc., H being a subgroup, and x an element of G.

By Lemma 8, $C(\tau_1)$ has a normal 2-complement $U = U_1$ which under the hypotheses of Theorem I is abelian. Thus $C(\tau_1) = SU_1$. If ρ_0 is in $N(S_0) - C'(S_0)$, then $C(\tau_2) = S^{\rho_0}U_1^{\rho_0}$ and $C(\tau_3) = S^{\rho_0^2}U_1^{\rho_0^2}$. We set $U_2 = U_1^{\rho_0}$ and $U_3 = U_1^{\rho_0^2}$. Then $U_3^{\rho_0} = U_1$.

Henceforth S will denote a fixed Sylow 2-subgroup of G, and S^* a dihedral subgroup of S. If $| S^* | > 4$, let S_0^*, S_1^* be representatives of the two conjugate classes of four-groups in S^*. To unify the notation, we put $S^* = S_0^*$ when $| S^* | = 4$, and do not define S_1^* in this case. If $S^* = S$, we set $S_0^* = S_0$ and $S_1^* = S_1$ (when it exists).

Whenever there is danger of ambiguity, we shall use the notation τ_{1i}^*, τ_{2i}^*, τ_{3i}^* for the involutions in S_i^*. However, when $S_i^* = S_0$, we shall continue to denote them by τ_1, τ_2, τ_3.

We define $E^* = C(S^*) \cap U_1$, $E_0^* = C(S_0^*) \cap U_1$, and $E_1^* = C(S_1^*) \cap U_1$ when S_1^* exists. When $S = S^*$, we use E, E_0, E_1 in place of E^*, E_0^*, E_1^*.

Let $A^* = \{\alpha^*\}$ be the maximal cyclic subgroup of S^* containing the central involution $\tau_1 = \tau_{10}^* = \tau_{11}^*$. Then $U_1 = D^* \times F^*$ where α^* acts regularly on D^* and $F^* = C(\alpha^*) \cap U_1$. Hence D^* and F^* are uniquely determined. Because A^* is a normal subgroup of S^*, both D^* and F^* are S^*-invariant. In particular, they are invariant under conjugation by the involutions τ_{20}^* and $\tau_{21}^* = \tau_{20}^*\alpha^*$. In fact, both these involutions induce the same automorphism of F^* with E^* as the fixed subgroup. Thus $F^* = E^* \times U_1'^*$ where $U_1'^*$ is inverted by both τ_{20}^* and τ_{21}^*. This decomposition admits the group S^*. We cannot do the same in decomposing D^*. However, setting

$$E_i'^* = C(\tau_{2i}^*) \cap D^*,$$

$i = 0, 1$, we obtain S_i^*-invariant subgroups of D^*. Since $E_0'^* \cap E_1'^* \leq E^*$, $E_0'^* \cap E_1'^* = 1$. Furthermore, $E_i^* = C(\tau_{2i}^*) \cap D^* \times C(\tau_{2i}^*) \cap F^* = E^* \times E_i'^*$.

LEMMA 12. *The subgroup U_1 admits the decomposition*

$$U_1 = E^* \times U_1'^* \quad or \quad U_1 = E^* \times E_0'^* \times E_1'^* \times U_1'^* \tag{14}$$

according as $S^* = S_0^*$ or $S^* > S_0^*$. Furthermore, the commutator subgroup of S^* centralizes U_1 if $E_0^* = 1$ or $E_1^* = 1$.

Proof. In the case $S^* = S_0^*$, $D^* = 1$, and the first case of (14) follows. In the case $S^* > S_0^*$, we have shown that $E^* \times E_0'^* \times E_1'^* \times U_1'^*$ is a subgroup of U_1. Using Wielandt's formula for the order of a group that is normalized by a dihedral group of automorphisms [23, Beispiel (3.1)], we see that equality holds in the second case of (14).

To prove the last statement, we may assume that $E_0^* = 1$. Then $\tau_{22}^* = \tau^*$ inverts U_1. For $x \in U_1$, we have

$$(x^{\alpha^*})^{-1} = x^{\alpha^*\tau^*} = x^{\tau^*\alpha^{*-1}} = (x^{-1})^{\alpha^{*-1}} = (x^{\alpha^{*-1}})^{-1}.$$

Hence the commutator subgroup $\{\alpha^{*2}\}$ centralizes U_1, and the proof is complete.

Remark. In the special case that $S = S^*$ and B centralizes U_1, $E_0' = E_0'^*$ is inverted by $\tau_2 \alpha$ and α, $E_1' = E_1'^*$ is inverted by τ_2 and α (when it is defined), and $U_1' = U_1'^*$ is inverted by τ_2 and $\tau_2 \alpha$ and consequently is centralized by α. Furthermore, if $\rho_i \in N(S_i) - C'(S_i)$, $U_1 \cap U_1^{\rho_i^k} = E_i'$, $i = 0, 1$, $k = 1, 2$. Hence if $x \in U_1'$ and $H = C^*(x)$, it follows that $N_H(S_i) = C_H'(S_i)$, $i = 0, 1$. Thus by Lemma 11, $C^*(x)$ has a normal 2-complement; also $C^*(x)$ contains S. These observations will be used repeatedly.

We shall further decompose $U_1'^*$ into the direct product $V_1^* \times W_1^*$, where V_1^* is the maximal Hall subgroup of $U_1'^*$ in which every element $\sigma \neq 1$ has its centralizer $C(\sigma)$ contained in $C(\tau_1)$, and W_1^* is the complementary subgroup. Consequently for each prime p dividing $|W_1^*|$, there exists a p-element $\sigma \neq 1$ in W_1^* whose centralizer is not contained in $C(\tau_1)$. Setting $X_1^* = E^* \times W_1^*$ or correspondingly $X_1^* = E^* \times E_0'^* \times E_1'^* \times W_1^*$, we obtain the decomposition

$$U_1 = V_1^* \times X_1^* . \tag{15}$$

When $S = S^*$, we use V_1 for V_1^*, W_1 for W_1^*, etc. This notation will be preserved throughout the remainder of the paper.

7. The S^*-invariant p-subgroups of G

For some odd prime p let P be an S_i^*-invariant p-subgroup of G, $i = 0$ or 1, and let $P = P_1 P_2 P_3$ be an S_i^*-decomposition of P, where P_μ is the fixed subgroup of the involution $\tau_{\mu i}^*$. Because G has only one class of involutions, each component P_μ is a subgroup of a conjugate of U_1 and hence is abelian. By Lemma 6,

$$P = P_0 \times P_1' P_2' P_3', \tag{16}$$

where $P_0 = C(S_i^*) \cap P$ and P_μ' is the subgroup of P_μ inverted by $\tau_{\nu i}^*$, $\nu \neq \mu$.

We shall use the notation $E^*(p)$, $E(p)$, $E_i^*(p)$, etc., for the unique Sylow p-subgroups of E^*, E, E_i^*, etc.

LEMMA 13. *Let H be a proper subgroup of G containing $S^* U_1$, where S^* is a dihedral subgroup of S, and S^* is a Sylow 2-subgroup of H. Assume that $N_H(S_i^*) > C_H'(S_i^*)$ for exactly one value of $i = 0, 1$. Then*

(i) *for every prime p dividing $|O(H)|$, there exists a unique S^*-invariant Sylow p-subgroup P of $O(H)$;*

(ii) *P has the S_i^*-decomposition $P = P_1 P_2 P_3 = E_i^*(p) \times P_1' P_2' P_3'$, and*

$$|P_1| = |P_2| = |P_3| \quad \textit{and} \quad |P_1'| = |P_2'| = |P_3'|; \tag{17}$$

(iii) *either $P_\mu = E_i^*(p)$, or P_μ is a Sylow p-subgroup of $C(\tau_{\mu i}^*)$, $\mu = 1, 2, 3$;*

(iv) *$O(H) \cap U_1 = X_1^*$, and every prime dividing $|O(H)|$ divides $|X_1^*|$.*

Proof. Some conjugate of S^* in S contains S_0 or S_1. Hence without loss we may assume that $i = 0$ and that $S_0^* = S_0$. Thus $N_H(S_0) > C'_H(S_0)$ and $E_0^* = E_0$.

(i) The existence of an S^*-invariant Sylow p-subgroup P of $O(H)$ follows from Lemma 1. Its uniqueness follows from the remark at the end of §5.

(ii) Denote by $\bar{K}$ the image of a subgroup K of H in $\bar{H} = H/O(H)$. Since by assumption $U_1 \leqq H$, it follows from Lemma 5 that $\bar{S}_0 \bar{E}_0 = C_{\bar{H}}(\bar{S}_0)$. On the other hand, by Lemma 11, $\bar{H}$ is isomorphic to $PGL(2, q)$, q odd, if $|S^*| > 4$ and to $PSL(2, q)$, q odd, if $|S^*| = 4$; consequently $C_{\bar{H}}(\bar{S}_0) = \bar{S}_0$ by §4(D). Then E_0 and hence $E_0(p)$ is contained in $O(H)$. By Lemma 1, $E_0(p)$ lies in the unique S_0-invariant Sylow p-subgroup of $O(H)$, which must be P. Hence P has the S_0-decomposition given in (ii).

Now by the Frattini argument, $H = O(H)K$, where $K = N_H(P)$; hence $N_K(S_0) > C'_K(S_0)$. There exists an element ρ_0 in K which conjugates τ_1 into τ_2, τ_2 into τ_3, and τ_3 into τ_1. It follows that $P_1^{\rho_0} = P_2$, $P_2^{\rho_0} = P_3$; this gives (17) at once.

(iii) Suppose that $P_1 > E_0(p)$; it suffices to show that P_1 is the Sylow p-subgroup R_1 of U_1 since $P_2 = P_1^{\rho_0}$ and $P_3 = P_2^{\rho_0}$. Now $R_2 = R_1^{\rho_0}$ and $R_3 = R_2^{\rho_0}$ are contained in H. Assume that $\bar{R}_1 \neq 1$; then $\bar{R}_2 \neq 1$ and $\bar{R}_3 \neq 1$ since $O(H)^{\rho_0} = O(H)$. Each subgroup $\bar{R}_\mu$ is a Sylow p-subgroup of $C_{\bar{H}}(\bar{\tau}_\mu)$ and is cyclic since $\bar{H}$ is isomorphic to $PGL(2, q)$ or $PSL(2, q)$. Furthermore $\bar{R}_\mu \cap \bar{R}_\nu = 1$ for $\mu \neq \nu$. Consequently $\bar{R}_1$ and $\bar{R}_2$ do not generate a p-group. We shall now contradict this in order to obtain $\bar{R}_\mu = 1$ and $R_\mu = P_\mu$, as desired.

Now $R_1 O(H)$ is S^*-invariant, and it follows that the unique S_0-invariant Sylow p-subgroup of $R_1 O(H)$ is necessarily S^*-invariant; by Lemma 4, it has the S_0-decomposition $R_1 P_2 P_3 = R_1 P'_2 P'_3$. By (17), $R_1 P_2 P_3 > R_1$; hence we may apply Lemma 7 to obtain an element σ in, say, P'_2 such that $M = C_H^*(\sigma)$ contains R_1. But $N_M(S_0) = C'_M(S_0)$ since σ is in P'_2; also $N_M(S_1^*) = C'_M(S_1^*)$ (if S_1^* exists) since $N_H(S_1^*) = C'_H(S_1^*)$. Hence M has a normal 2-complement $O(M)$. It is clear that $M > S_0$. Hence $O(M)$ is S_0-invariant. On the other hand, $R_1 \leqq O(M)$. As U_1 is abelian, $R_2 \leqq M$; hence $R_2 \leqq O(M)$. By Lemmas 1, 2, and 4, there is a unique S_0-invariant Sylow p-subgroup R of $O(M)$ with the S_0-decomposition $R_1 R_2 R'_3$ where $R'_3 \leqq R_3$.

In $\bar{H}$ the image $\bar{R}$ of R has the decomposition $\bar{R} = \bar{R}_1 \bar{R}_2 \bar{R}'_3$, and consequently $\bar{R}_1$ and $\bar{R}_2$ generate a p-group, which is a contradiction and establishes (iii).

(iv) It follows from (ii) that if p divides $|O(H)|$, then p divides $|O(H) \cap U_1|$. It remains to show that $O(H) \cap U_1 = X_1^*$. We have already shown in the first paragraph of (ii) that $E_0 \leqq O(H)$, and the same argument yields that $\bar{E}_1^* = 1$, and hence that $E_1^* \leqq O(H)$, whence $E_0 E_1^* \leqq O(H)$. Now if $V_1^* \cap O(H) \neq 1$, there exists a p-element $\sigma \neq 1$ in $V_1^* \cap O(H)$ for some prime p. By (ii) and (iii), σ belongs to the subgroup P_1' of the S^*-invariant Sylow p-subgroup P of $O(H)$ with S_0-decomposition (16). By (17), $P_1' \neq 1$; whence by Lemma 7 there exists an element in P_2' which centralizes P_1. Hence σ commutes with an element not in U_1. This is a contradiction, and we conclude that $V_1^* \cap O(H) = 1$.

It thus suffices to show that $W_1^* \leqq O(H)$. Hence suppose that there is a p-element in W_1^*, but not in $O(H)$. Then P_1 is not a Sylow p-subgroup of U_1, so that, by (iii), $P_1 = E_0(p)$. In other words, the subgroup P_1' in (16) is the identity. Thus the Sylow p-subgroup of W_1^* is disjoint from $O(H)$. Hence there exists an element $\sigma \in W_1^*$, but not in $O(H)$, such that $C(\sigma)$ is not contained in S^*U_1.

Since $\sigma \in W_1^*$, $C_H(\sigma)$ has a normal 2-complement F with S_0-decomposition

$$F = F_0 F_1' F_2' F_3' = F_1 F_2 F_3 .$$

Since U_1 is abelian, $F_1 = U_1$. By our assumption on σ, $F_2' F_3' \neq 1$. Let $\lambda_2 \neq 1$ be an element of, say, F_2', and set $\lambda_1 = \lambda_2^{\rho_0^{-1}}$. Now $K = C_H^*(\lambda_1)$ contains σ and $\sigma^{\rho_0^{-1}}$. Since $\lambda_1 \in F_1'$, $N_K(S_0) = C_K'(S_0)$, and $C_H^*(\lambda_1)$ has a normal 2-complement Y which contains σ and $\sigma^{\rho_0^{-1}}$. The unique S_0-invariant Sylow p-subgroup Q of Y has the S_0-decomposition

$$Q = Q_1 Q_2 Q_3 = Q_0 Q_1' Q_2' Q_3' ,$$

and hence $\sigma \in Q_1'$, $\sigma^{\rho_0^{-1}} \in Q_3'$. Thus $\{\sigma, \sigma^{\rho_0^{-1}}\}$ is a p-group. Since $\bar{H}$ is isomorphic to $PGL(2, q)$ and $\bar{\sigma}$ and $\bar{\sigma}^{\rho_0^{-1}}$ are in the centralizers of different involutions, $\bar{\sigma}$ and $\bar{\sigma}^{\rho_0^{-1}}$ do not generate a p-group in $\bar{H}$, which is a contradiction. Thus $W_1^* \leqq O(H)$, and the lemma is proved.

Remark. If $P > E_0(p)$, we have shown in (iii) that P_μ is the Sylow p-subgroup of $C(\tau_\mu)$, $\mu = 1, 2, 3$. It follows, therefore, from Lemma 4 that P is a maximal S_0-invariant and, when $S = S^*$, a maximal S-invariant p-subgroup of G. However, as we shall see later, P need not be a maximal S_0-invariant subgroup of G.

8. The structure of G, Case I

In the remainder of the paper we shall distinguish the following cases:

Case I. $E_0 \neq 1$, $E_1 \neq 1$.
Case II. $E_0 = 1$, $E_1 \neq 1$.
Case III. $E_0 = 1$, $E_1 = 1$.

By symmetry we need not consider the case $E_0 \neq 1$, $E_1 = 1$. Case II can occur only if $|S| \geqq 8$; and the same is true in Case I if $E_i > E$, $i = 0$ or 1.

To analyze Case I we need a lemma which is closely related to Lemma 13.

LEMMA 14. *Let H_0, H_1 be proper subgroups of G containing S^*U_1, where S^* is a dihedral subgroup of S and a Sylow 2-subgroup of both H_0 and H_1. Assume that $N_{H_i}(S_i^*) > C_{H_i}'(S_i^*)$, $N_{H_i}(S_j^*) = C_{H_i}'(S_j^*)$, $i, j = 0, 1, j \neq i$. Then $O(H_0) = O(H_1)$.*

Proof. As in Lemma 13, we may assume for convenience that $S_0^* = S_0$. We first show that for every prime p dividing $|U_1|$, $E_0^*(p) = E^*(p)$ if and only if $E_1^*(p) = E^*(p)$. We may suppose that $E_0^*(p) = E^*(p)$ and $E_1^*(p) > E^*(p)$. By Lemma 13(iv), $O(H_0) \cap U_1 = X_1^*$, and hence $E_1^* \leqq O(H_0)$. Hence by that lemma, $O(H_0)$ contains an S^*-invariant Sylow p-subgroup $P \neq 1$ with S_0-decomposition

$$P = P_1 P_2 P_3 = E^*(p) \times P_1' P_2' P_3', \tag{18}$$

where P_1 is a Sylow p-subgroup of U_1 and $|P_1| = |P_2| = |P_3|$. Thus $|P| = |P_1|^3/|E^*(p)|^2$.

On the other hand, let

$$P = R_1 R_2 R_3 \tag{19}$$

be the S_1^*-decomposition of P. Now $R_1 = P_1$ is the Sylow p-subgroup of U_1; it follows, therefore, from Lemma 3 that

$$|P| = |R_1|\,|R_2|\,|R_3|/|E_1^*(p)|^2 \leqq |P_1|^3/|E_1^*(p)|^2.$$

But $|P| = |P_1|^3/|E^*(p)|^2$ and $E_1^*(p) > E^*(p)$, which gives a contradiction. Thus $E_1^*(p) = E^*(p)$, as we asserted.

Now we prove the lemma. We know from Lemma 13 that $O(H_i) \cap U_1 = X_1^*$, and also that every prime dividing $|O(H_i)|$ divides $|X_1^*|$, $i = 0, 1$. Let then P, Q be the unique S^*-invariant Sylow p-subgroups of $O(H_0)$, $O(H_1)$ respectively for some prime p dividing $|X_1^*|$, with respective S_0- and S_1^*-decompositions

$$P = P_1 P_2 P_3 = E_0^*(p) \times P_1' P_2' P_3', \tag{20}$$

$$Q = Q_1 Q_2 Q_3 = E_1^*(p) \times Q_1' Q_2' Q_3'. \tag{21}$$

If $E_0^*(p) = E_1^*(p) = E^*(p)$, it follows from Lemma 13 that either $P = Q = E^*(p)$, or that $P_1 = Q_1$ is the Sylow p-subgroup of U_1. Moreover, in the latter case P_μ is the Sylow p-subgroup of U_μ, $\mu = 1, 2, 3$, and

$$|P| = |Q| = |P_1|^3/|E^*(p)|^2.$$

Since Q is S_0-invariant, Q must then be contained in P. But as they have the same order, $Q = P$ in this case as well.

Suppose then that $E_0^*(p) > E^*(p)$. By what has been shown above, $E_1^*(p) > E^*(p)$. Now $E_1^*(p) \leqq O(H_0)$, and hence $E_1'^*(p) \leqq P_1'$. Thus again by Lemma 13(iii), P_1 is the Sylow p-subgroup of U_1. By the same argument so is Q_1, whence $P_1 = Q_1$. But then $P_2 = P_1^{\rho_0}$ and $P_3 = P_1^{\rho_0^2}$ are the Sylow p-subgroups of U_2 and U_3, respectively. It then follows that $P = P_1 P_2 P_3$ is the maximal S_0^*-invariant p-subgroup of G. Similarly $Q = Q_1 Q_2 Q_3$ is the maximal S_1^*-invariant p-subgroup of G. By Lemma 6, the S^*-invariant group P admits an S_1^*-decomposition

$$P = E_1^*(p) \times P_1'' P_2'' P_3''. \tag{22}$$

But then $E_1^*(p) \times P_i'' \leqq Q_i$, $i = 1, 2, 3$. Hence $P \leqq Q$. By symmetry, $Q \leqq P$. Thus $P = Q$ and $O(H_0) = O(H_1)$, and the proof is complete.

PROPOSITION 15. *Case* I. *There exists a proper subgroup H of G containing $C(\tau_1)$ and having no normal subgroups of index* 2.

Proof. Define the class of proper subgroups $\mathfrak{K}_0 = \{H_0\}$ of G as follows: $H_0 \in \mathfrak{K}_0$ if $H_0 \geqq S^*U_1$, where S^* is a dihedral subgroup of S containing S_0, S^* is a Sylow 2-subgroup of H_0, and $N_{H_0}(S_0) > C'_{H_0}(S_0)$.

Since $E_0 \neq 1$, $N(E_0) \in \mathfrak{K}_0$, and hence $\mathfrak{K}_0$ is nonempty. Choose H_0 in $\mathfrak{K}_0$ so that S^* has maximal order. We shall show first that $S = S^*$.

Suppose then that $S^* < S$. Let $T = N_S(S^*)$. Then T is a dihedral subgroup of S and $|T:S^*| = 2$. Since $|S| \geqq 8$, $|N(E_0) \cap S| \geqq 8$, and hence $|S^*| \geqq 8$. Then for some δ in $T - S^*$, it follows that $S_0^{*\delta} = S_1^*$, and $S_1^{*\delta} = S_0^*$, where $S_0^* = S_0$.

Assume that $N_{H_0}(S_1^*) > C'_{H_0}(S_1^*)$. Since $U_1 < H_0$,

$$N(S_0^*) = N_{H_0}(S_0^*) \leqq H_0 \quad \text{and} \quad N(S_1^*) = N_{H_0}(S_1^*) \leqq H_0 .$$

But $N(S_0^*)^\delta = N(S_1^*)$, $N(S_1^*)^\delta = N(S_0^*)$, and $U_1^\delta = U_1$. It follows that δ normalizes the subgroup $K = \{N_{H_0}(S_0^*),\ N_{H_0}(S_1^*),\ U_1\}$ of H_0. Thus $H^* = N(K)$ is in $\mathfrak{K}_0$ and contains $T > S^*$, contrary to the maximal choice of H_0.

Hence $N_{H_0}(S_1^*) = C'_{H_0}(S_1^*)$. Set $H_1 = H_0^\delta$. Then $N_{H_1}(S_1^*) > C'_{H_1}(S_1^*)$, $N_{H_1}(S_0^*) = C'_{H_1}(S_0^*)$, $H_1 \geqq S^*U_1$, and S^* is a Sylow 2-subgroup of H_1. Thus by Lemma 14 it follows that $O(H_0) = O(H_1)$. Set $H = N(O(H_0))$; then H contains H_0 and H_1. Since $N_H(S_i^*) > C'_H(S_i^*)$ $i = 0, 1$, we conclude as above that $H^* = N(H)$ is in $\mathfrak{K}_0$ and contains T, contradicting the maximal choice of H_0.

Thus $S^* = S$. If $N_{H_0}(S_1) > C'_{H_0}(S_1)$, the proposition follows with $H = H_0$. Hence we may assume that $N_{H_0}(S_1) = C'_{H_0}(S_1)$.

By symmetry we define a class $\mathfrak{K}_1$ of proper subgroups of G. Since $E_1 \neq 1$, we may show by an entirely analogous argument that there exists a subgroup H_1 of G containing SU_1 and such that $N_{H_1}(S_1) > C'_{H_1}(S_1)$. If $N_{H_1}(S_0) > C'_{H_1}(S_0)$, the proposition follows once again with $H_1 = H$. Thus we may assume that $N_{H_1}(S_0) = C'_{H_1}(S_0)$. The conditions of Lemma 14 are again satisfied, whence $O(H_0) = O(H_1)$. The proposition thus follows as above with $H = N(O(H_0))$.

9. Maximal S_0-invariant p-subgroups of G in Cases II and III

In the next two sections, we shall assume that $E = E_0 = 1$. Thus $C(S_0) = S_0$, and $N(S_0)$ is isomorphic to A_4 or S_4. Furthermore, τ_2 inverts U_1 and, by Lemma 12, α^2 centralizes U_1. We construct in this section, for each prime p dividing $|W_1|$, an $N(S_0)$-invariant p-group in G which contains the Sylow p-subgroup of U_1. To begin this construction we prove the following lemma.

LEMMA 16. *Let $\lambda \in U_1$. Then either $H = C^*(\lambda)$ has a normal 2-complement, or $\lambda \in E_1$ and H satisfies the conditions of Lemma 13 with $i = 1$ and $S^* = S$.*

Proof. If $\lambda \in U_1'$, $C^*(\lambda)$ has a normal 2-complement by virtue of §6. If $\lambda \in U_1 - E_1 - U_1'$, a Sylow 2-subgroup S' of H is generated by τ_2 and α^2. As representatives of the two conjugate classes of four-groups in S', we may take S_0 and $S_1' = \{\tau_1, \tau_2\alpha^2\}$. But S_0 and S_1' are conjugate in G; hence $N_H(S_1') = C_H'(S_1')$. Thus H has a normal 2-complement by Lemma 11.

Finally take $\lambda \in E_1$. Since B centralizes U_1, SU_1 is contained in H. If $\rho_0 \in N(S_0) - C'(S_0)$, $\lambda^{\rho_0} \notin U_1$ since $C(S_0) = S_0$. Thus $N_H(S_0) = C_H'(S_0)$. If $N_H(S_1) = C_H'(S_1)$, again H has a normal 2-complement. If this is not the case, H satisfies the conditions of Lemma 13 with $i = 1$ and $S^* = S$. The lemma is proved.

Suppose that G contains a subgroup H satisfying the conditions of Lemma 13 with $i = 1$ and $S^* = S$. Let P be an S-invariant Sylow p-subgroup of $O(H)$; then P satisfies (16) and (17). Assume further that $P_1' \neq 1$, in which case P_1 is a Sylow p-subgroup of U_1 by Lemma 13(iii). By the remark in §7, P is a maximal S-invariant p-subgroup of H; but it need not be a maximal S_0-invariant subgroup when $S > S_0$. In fact, set $\bar{H} = H/O(H)$, and let $\bar{M}$ and $\bar{\sigma}$ be the images of a subgroup M and an element σ of H in $\bar{H}$. By Lemma 11(ii), $\bar{H}$ is isomorphic to $PGL(2, q)$. Since $N_{\bar{H}}(\bar{S}_0) = C_{\bar{H}}'(\bar{S}_0)$, the involution $\bar{\tau}_2$ of $\bar{S}_0$ lies outside the normal subgroup of $\bar{H}$ which is isomorphic to $PSL(2, q)$. Then by §4(B), $|C_{\bar{H}}(\bar{\tau}_2)| = 2(q + \delta)$ and $|C_{\bar{H}}(\bar{\tau}_1)| = 2(q - \delta)$ where $\delta = \pm 1$ and $\delta \equiv q \pmod 4$. Hence if p divides $q + \delta$, it is possible to form a maximal S_0-invariant subgroup $P^{(2)} > P$ in the S_0-invariant subgroup $C_H(\tau_2)O(H)$, which is the inverse image of $C_{\bar{H}}(\bar{\tau}_2)$ in H by virtue of Lemma 5. By the Frattini argument, $K = C_H(\tau_2)O(H) = N_K(P)O(H)$. Hence $P^{(2)}$ has the form $P_2^* P$ where P_2^* is the Sylow p-subgroup of $C_H(\tau_2)$. In a similar manner we may form $P^{(3)} = P_3^* P$ where P_3^* is the Sylow p-subgroup of $C_H(\tau_3)$. We are now in a position to complete the proof of the following lemma.

LEMMA 17. *Let p be a prime dividing $|W_1|$. Let $\lambda \neq 1$ be in U_1, and set $H = C^*(\lambda)$. Then H contains two S_0-invariant p-subgroups $P^{(2)}$ and $P^{(3)}$ with S_0-decompositions*

$$P^{(2)} = P_1 P_2^* P_3^{**} \quad \text{and} \quad P^{(3)} = P_1 P_2^{**} P_3^*, \tag{23}$$

where P_1 is the Sylow p-subgroup of U_1, and P_μ^* is the Sylow p-subgroup of $C_H(\tau_\mu)$, $\mu = 2, 3$. Furthermore, if λ is in the center of an S_0-invariant p-subgroup Q, then Q is contained in either $P^{(2)}$ or $P^{(3)}$.

Proof. If H has a normal 2-complement, then $O(H)$ contains a unique S_0-invariant Sylow p-subgroup P^*, by the remark in §7. Hence (23) holds with $P^{(2)} = P^{(3)} = P^*$. If $\lambda \in Z(Q)$, then $Q \leqq O(H)$ and $Q \leqq P$.

Thus, by virtue of Lemma 16, we may suppose that H satisfies the conditions of Lemma 13 with $i = 1$ and $S^* = S$. By Lemma 13(iv), $O(H) \cap U_1 = X_1$; hence $O(H) \geqq W_1$. Since p divides $|W_1|$, $P > E_1(p)$. Then by Lemma 13(iii), $P \cap U_1 = P_1$ is the Sylow p-subgroup of U_1. Thus the existence of $P^{(2)}$ and $P^{(3)}$ with the decompositions (23) follows from our preceding discussion.

To prove the final statement of the lemma, note first that $Q \leqq H$. Let $Q = Q_1 Q_2 Q_3$ be the S_0-decomposition of Q, and let $\bar{Q} = \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 = \bar{Q}_2 \bar{Q}_3$ be the image of Q in $H/O(H)$. Then $\bar{Q}_\mu \leqq C_{\bar{H}}(\bar{\tau}_\mu)$, $\mu = 2, 3$, and $\bar{\tau}_2$ and $\bar{\tau}_3$ are not conjugate to $\bar{\tau}_1$ in $\bar{H}$. Since $\bar{H}$ is isomorphic to $PGL(2, q)$, it follows from §4(B) that $O(C_{\bar{H}}(\bar{\tau}_\mu))$ is cyclic and its Sylow subgroups are Sylow subgroups of $\bar{H}$. But $\bar{Q}_2 \cap \bar{Q}_3 = 1$. Hence $\bar{Q}$ can be a p-group only if $\bar{Q}_2 = 1$ or $\bar{Q}_3 = 1$. If $\bar{Q}_\mu = 1$, then $\bar{P}_\nu^* \geqq \bar{Q}$, $\nu \neq \mu$, and $P^{(\nu)} \geqq Q$.

LEMMA 18. *Let p be a prime dividing $|W_1|$. Then G contains an S_0-invariant p-subgroup P with the S_0-decomposition*

$$(24) \qquad P = P_1 P_2 P_3' \quad or \quad P = P_1 P_2' P_3,$$

where P_μ, $\mu = 1, 2, 3$, is a Sylow p-subgroup of U_μ, $\mu = 1, 2, 3$, and $P_\nu' \leqq P_\nu$, $\nu = 2, 3$. Either P is $N(S_0)$-invariant, or P is a Sylow p-subgroup of G and S_0 is a Sylow 2-subgroup of $N(P)$.

Proof. Because of the definition of W_1, there exists an element $\sigma_1 \neq 1$ in $P_1 \cap W_1$ for which $C(\sigma_1)$ is not contained in $C(\tau_1)$. Since $W_1 \leqq U_1'$, $C^*(\sigma_1)$ has a normal 2-complement M with S_0-decomposition $M = M_1 M_2 M_3$. By the condition on σ_1, $M_1 = U_1$ and $M > M_1$. Thus there exists an element λ_3 in, say, M_3. If $\lambda_1 = \lambda_3^{\rho_0}$, where $\rho_0 \in N(S_0) - C'(S_0)$, then $C^*(\lambda_1)$ contains the p-element $\sigma_2 = \sigma_1^{\rho_0} \in M_2$. Furthermore, $\sigma_2 \notin U_1$.

By Lemma 17, $H = C^*(\lambda_1)$ contains an S_0-invariant subgroup Q with S_0-decomposition $Q = P_1 Q_2 Q_3$, where Q_2 is the Sylow p-subgroup of $U_2 \cap H$. Hence $\sigma_2 \in Q_2$. Then by Lemma 7, there exists an η in Q_2 or Q_3 such that $C(\eta) \geqq P_1$. If correspondingly $\eta_1 = \eta^{\rho_0^{-1}}$ or $\eta_1 = \eta^{\rho_0}$, it follows that $H^* = C^*(\eta_1)$ contains P_1 and either P_3 or P_2. Thus by Lemma 17, H^* contains an S_0-invariant p-group P^* with S_0-decomposition given by one of the two relations in (24). Finally let P be a maximal S_0-invariant p-group of G containing P^*. Then P also has an S_0-decomposition given by (24).

If $P = P_1 P_2 P_3$, then clearly $P^{\rho_0} = P$, and P is $N(S_0)$-invariant. Suppose then that, say, $P_3' < P_3$. Then P is not ρ_0-invariant, and hence if $K = N(P)$, $N_K(S_0) = C_K'(S_0)$. Let T be a Sylow 2-subgroup of K. Since T is dihedral, we can assume $S_0 \leqq T$. Suppose $T > S_0$. Then there exists an element γ' of order 4 in $T \leqq N_K(S_0)$. Hence $P_3 = P_2^{\gamma'} = P_3'$, which is contrary to our assumption. Therefore, $T = S_0$; and Burnside's theorem implies that K has a normal 2-complement $O(K)$. By the maximal choice of P, P must be a Sylow p-subgroup of $O(K)$ and hence of K. Thus P is a Sylow p-subgroup of G, as required.

LEMMA 19. *Let p be a prime dividing $|W_1|$. Then G contains an $N(S_0)$-invariant p-group P with S_0-decomposition*

$$P = P_1 P_2 P_3, \tag{25}$$

where P_μ is the Sylow p-subgroup of U_μ, $\mu = 1, 2, 3$.

Proof. We argue by contradiction. By the preceding lemma, we may assume G contains an S_0-invariant Sylow p-subgroup P of G with S_0-decomposition $P = P_1 P_2 P_3'$, where $P_3' < P_3$. Using Lemma 7, we can find an element λ_2 in, say, P_2 which centralizes P_1, and in particular commutes with $\lambda_1 = \lambda_2^{\rho_0 - 1}$. Then λ_2 and $\lambda_3 = \lambda_2^{\rho_0}$ commute, as do λ_2 and λ_1. It follows that $\{\lambda_1, \lambda_2, \lambda_3\}$ is a nontrivial $N(S_0)$-invariant p-subgroup of G. Define Q as the maximal $N(S_0)$-invariant p-subgroup of G, and let it have the S_0-decomposition

$$Q = Q_1 Q_2 Q_3, \tag{26}$$

where $Q_\mu \leqq P_\mu$, $\mu = 1, 2, 3$, $Q_2 = Q_1^{\rho_0}$, $Q_3 = Q_2^{\rho_0}$. If any $Q_\mu = P_\mu$ then $Q > P'$ which contradicts the fact that P is a Sylow p-subgroup of G. Thus $Q_\mu < P_\mu$, $\mu = 1, 2, 3$.

The center $Z(Q)$ is also $N(S_0)$-invariant and has an S_0-decomposition of the form

$$Z(Q) = Z_1 Z_2 Z_3, \tag{27}$$

where $Z_2 = Z_1^{\rho_0}$, $Z_3 = Z_2^{\rho_0}$. Hence $Z_\mu \neq 1$, $\mu = 1, 2, 3$. Let $\lambda \neq 1$ be in Z_1, and consider $H = C^*(\lambda)$. By Lemma 17, H contains an S_0-invariant p-group Q^* with S_0-decomposition

$$Q^* = P_1 Q_2^* Q_3^*, \tag{28}$$

where $Q^* \geqq Q$, since $\lambda \in Z(Q)$. Since $P_1 > Q_1$, $Q^* > Q$.

Thus we can consider a maximal S_0-invariant p-subgroup of G such that $R \geqq Q^* > Q$. We claim that R is also a Sylow subgroup of G. Indeed, let $K = N(R)$, and let $T \geqq S_0$ be a Sylow 2-subgroup of K. If $T = S_0$, then by the maximal choice of Q, $N_K(S_0) = C_K(S_0)$. Hence Burnside's theorem implies $O(K)$ is a normal 2-complement. By Lemma 3, R is a Sylow subgroup of $O(K)$ and hence of G.

Hence suppose that $T > S_0$. Then T has two classes of four-groups, represented by S_0 and S_1^*. If T is not a Sylow subgroup of G, there exists an element δ in $N(T)$ such that $S_0^\delta = S_1^*$. Then $N_K(S_0) = C_K'(S_0)$ implies $N_K(S_1^*) = C'(S_1^*)$. Hence by Lemma 11, K has a normal 2-complement $O(K)$. Again R is a Sylow subgroup of G in this case. Thus we may, after a conjugation, if necessary, suppose that $T = S$ and $N_K(S_1) > C_K'(S_1)$, where now $S_1^* = S_1$.

If $R = P_1 R_2 R_3$ is the S_0-decomposition of R, $R_2 \neq P_2$ and $R_3 \neq P_3$; for otherwise, the fact that $R_2 = R_3^\gamma$ and $R_3 = R_2^\gamma$ would imply that $R = P_1 P_2 P_3$.

By Lemma 6, R has the S_1-decomposition

$$R = E_1(p) \times R_1' R_2' R_3', \tag{29}$$

where $E_1(p) \times R_1' = P_1$ and, for some $\rho_1 \in N_K(S_1) - C_K'(S_1)$, $R_1'^{\rho_1} = R_2'$, $R_2'^{\rho_1} = R_3'$. Now $Z(R)$ also has an S_1-decomposition

$$Z(R) = E_1(p) \times Z_1' Z_2' Z_3', \tag{30}$$

where again $Z_1'^{\rho_1} = Z_2'$, $Z_2'^{\rho_1} = Z_3'$, and $Z_\mu' \neq 1$, $\mu = 1, 2, 3$. On the other hand, $Z(R)$ has an S_0-decomposition

$$Z(R) = Z_1 Z_2 Z_3 .$$

Here $Z_1 = E_1(p) \times Z_1'$. Since $Z_2' Z_3' \neq 1$, $Z_2 Z_3 \neq 1$.

Thus there exists $\lambda \neq 1$ in Z_μ, $\mu = 2$ or 3. Now both R and Z_μ are contained in $H = C^*(\lambda)$. Applying Lemma 17 to H (actually to H^{ρ_0} or $H^{\rho_0^2}$ since $\lambda \in U_2$ or U_3), we obtain a maximal S_0-invariant p-subgroup R^* of H such that $R^* \geqq R$ and $R^* > P_2$. But $R_2 < P_2$, and hence $R^* > R$. This contradicts the maximal choice of R and shows that R must, in fact, be a Sylow p-subgroup of G.

To conclude the proof, note that $R^\sigma = P$ for some $\sigma \in G$. By Lemma 18, S_0 is a Sylow 2-subgroup of $N(P)$. Hence S_0 is also a Sylow 2-group of $N(R)$. This implies that S_0^σ is a Sylow subgroup of $N(P)$. Thus there exists $\eta \in N(P)$ so that $R^{\sigma\eta} = P$ and $\sigma\eta \in N(S_0)$. Since Q is $N(S_0)$-invariant, we obtain that $Q = Q^{\sigma\eta} \leqq P$. Furthermore, $Q_3 = P_3'$; in fact, P_3' generates with $P_3'^{\rho_0}$ and $P_3'^{\rho_0 2}$ an $N(S_0)$-invariant subgroup X of P. Thus $Q \geqq X$, and comparing components we obtain $P_3' \geqq Q_3 \geqq P_3'$. Thus $Q = X$.

Next form $N = N(Q)$. As $Q < P$, $N \cap P > Q$. Set $\bar{N} = N/Q$, and let $\bar{P}$ be the image of $N \cap P$ in $\bar{N}$; $\bar{P}$ is then $\bar{S}_0$-invariant. Since $P_3' \leqq Q$, $\bar{P} \cap C_{\bar{N}}(\bar{\tau}_3) = 1$ by Lemma 5. Hence $\bar{\tau}_3$ inverts $\bar{P}$, and so $\bar{P}$ is abelian. Thus $\bar{P} = \bar{P}_1 \times \bar{P}_2$, where $\bar{P}_\mu = C_{\bar{P}}(\bar{\tau}_\mu)$, $\mu = 1, 2$. We may assume that $\bar{P}_1 \neq 1$. Then there exists x_1 in, say, P_1 which normalizes Q but is not in Q. Then $x_2 = x_1^{\rho_0}$ normalizes Q and is in P_2. Thus the images $\bar{x}_1$ and $\bar{x}_2$ in $\bar{N}$ of x_1 and x_2, respectively, commute. Set $x_3 = x_2^{\rho_0}$. Then $x_3 \in N$, and $\bar{x}_2$ and $\bar{x}_3$ commute as well as $\bar{x}_3$ and $\bar{x}_1$. Hence $\{x_1, x_2, x_3, Q\}$ is a larger $N(S_0)$-invariant p-subgroup than Q. This is a contradiction. Hence $P_3' = P_3$, and we have $P = P_1 P_2 P_3$. This proves the lemma.

10. The structure of G, Cases II and III

We assume $E = E_0 = 1$ in this section.

LEMMA 20. *Let p be a prime dividing $|W_1|$. Then $E_1(p) = 1$. Thus $(|E_1|, |W_1|) = 1$.*

Proof. Suppose that $E_1(p) \neq 1$. Then $S > S_0$. Let $H = N(E_1(p))$. Since $E_0 = 1$, B centralizes U_1, and so $S \leqq H$. Furthermore, Lemma 13 applies to H with $i = 1$. Hence an S-invariant Sylow p-subgroup Q of $O(H)$ has the S_1-decomposition

$$Q = E_1(p) \times Q_1' Q_2' Q_3', \tag{31}$$

where $E_1(p) \times Q_1' = P_1$ is the Sylow p-subgroup of U_1 and $Q_1' = P_1 \cap W_1$. Thus $Q = P_1 Q_2' Q_3'$. Furthermore, Q is a maximal S-invariant p-subgroup of G by the remark in §7.

Let P be the maximal $N(S_0)$-invariant p-subgroup of G, constructed in Lemma 19 and having the S_0-decomposition (25). On the other hand, Q has an S_0-decomposition $Q = P_1 P_2' P_3'$. Here $P_\mu' \leqq P_\mu$, $\mu = 2, 3$, and consequently $Q \leqq P$. Now using (17), we have $|Q:P_1| = |Q_2' Q_3'| = |Q_1'|^2 = |P_2' P_3'|$. Thus $|P_2' P_3'| < |P_2 P_3|$, and so $Q < P$ and Q is not $N(S_0)$-invariant.

If $K = N(Q)$, K contains S and $N_K(S_1) > C_K'(S_1)$. Since Q is not $N(S_0)$-invariant, $N_K(S_0) = C_K'(S_0)$, and it follows from Lemma 11(ii) that $\bar{K} = K/O(K)$ is isomorphic to $PGL(2, q)$, q odd. Since $O(K)$ contains a unique S-invariant Sylow p-subgroup, this must be Q by the maximal nature of Q.

On the other hand, $K \cap P > Q$ as $P > Q$. Let $\bar{P} = \bar{P}_1 \bar{P}_2 \bar{P}_3$ be the image of $K \cap P$ in $\bar{K} = K/O(K)$. Since $P_1 < Q$, $\bar{P} = \bar{P}_2 \bar{P}_3$. But $\bar{P}_2$ and $\bar{P}_3$ are contained in distinct cyclic Sylow subgroups of $\bar{K}$ and hence generate a p-group only if $\bar{P}_2 = 1$ or $\bar{P}_3 = 1$. Thus $K \cap P_2$ or $K \cap P_3$ is contained in $O(K)$. Since $(K \cap P_2)^\gamma = K \cap P_3$ and $O(K)$ is S-invariant, both $K \cap P_2$ and $K \cap P_3$, and consequently $K \cap P$, are contained in $O(K)$. Thus $K \cap P = Q$, which is a contradiction.

LEMMA 21. *For each prime p dividing $|W_1|$, let P be the $N(S_0)$-invariant p-subgroup of G constructed in Lemma 19. Then $N(P)$ contains S, V_1, and $N(S_1)$.*

Proof. The center $Z(P)$ has the S_0-decomposition $Z(P) = Z_1 Z_2 Z_3$, where each $Z_\mu \neq 1$. Let $\lambda \in Z_1$. Since $E_1(p) = 1$ by Lemma 20, $\lambda \in U_1'$; hence $M = C^*(\lambda)$ has a normal 2-complement. Since both S and P are contained in M, it follows at once that P is S-invariant.

Let π be the set of primes dividing $|V_1|$. Then $E_1 = F_1 \times F_1'$, where F_1 is the Hall π-subgroup of E_1 and F_1' is the complementary subgroup. Thus $V_1 \times F_1$ is a Hall π-subgroup of U_1 since V_1 is, by definition, a Hall π-subgroup of U_1'. Let $q \in \pi$, and let Q_1 be the Sylow q-subgroup of U_1. Let Q be the S-invariant Sylow q-subgroup of $O(M)$. Since $U_1 \leqq O(M)$, $Q \geqq Q_1$. If $Q > Q_1$, it follows from Lemma 7 that there exists a nontrivial element in $Q \cap U_2$, say, which centralizes Q_1, and, in particular, $Q_1 \cap V_1$. This contradicts the definition of V_1 and shows that $V_1 F_1$ is an abelian Hall subgroup of $O(M)$.

Let $Y = N_{O(M)}(V_1 F_1)$, and let $Y = Y_1 Y_2 Y_3$ be the S_0-decomposition of Y. Clearly Y_1 centralizes $V_1 F_1$. Let $\lambda \in V_1 F_1$ and $\sigma \in Y_\mu$, $\mu = 2$ or 3.

Then $\lambda^{\sigma} \in V_1 F_1 \leqq U_1$, whence $\lambda^{\sigma} = \lambda^{\sigma\tau_1} = \lambda^{\tau_1\sigma^{-1}} = \lambda^{\sigma^{-1}}$. Thus $\lambda^{\sigma^2} = \lambda$. Since σ has odd order, it follows that σ commutes with λ and that $V_1 F_1$ is in the center of its normalizer in $O(M)$. Since $V_1 F_1$ is an abelian Hall subgroup of $O(M)$, Burnside's theorem implies that $V_1 F_1$ has a normal complement M_0 in $O(M)$ By Lemma 1, M_0 contains an $SV_1 F_1$-invariant Sylow p-subgroup, and this must be P, since P is the unique S-invariant Sylow p-subgroup of $O(M)$. Thus $V_1 \leqq V_1 F_1 \leqq N(P)$.

Suppose next that $E_1 \neq 1$. Then $H = N(E_1) < G$. Since $N_H(S_0) = C_H(S_0)$ and $N_H(S_1) > C_H'(S_1)$, we may apply Lemma 13 to H. It follows from parts (iii) and (iv) of that lemma and from Lemma 19 that P is the S-invariant Sylow p-subgroup of $O(H)$. Since $H = O(H)K$, where $K = N_H(P)$, $N_K(S_1) > C_K'(S_1)$. Let $\rho_1 \in N_K(S_1) - C_K'(S_1)$, and let P have the S_1-decomposition $P = P_1 P_2' P_3'$. Now $E_1 \leqq U_1 \leqq C(P_1)$. We have $E_1 = E_1^{\rho_1} \leqq C(P_1^{\rho_1}) = C(P_2')$, and similarly $E_1 \leqq C(P_3')$. Thus $E_1 \leqq C(P)$. Since $N(S_1) = \{\rho_1, C'(S_1)\}$, $N(S_1) \leqq N(P)$, which proves the lemma in this case.

Assume finally that $E_1 = 1$. Then $C(S_1) = S_1$, and $N(S_1) = \{\rho_1, C'(S_1)\}$, where ρ_1 has order 3. Since P is S-invariant, it has an S_1-decomposition $P = P_1 P_2' P_3'$, where P_μ' is contained in the Sylow p-subgroup of $C(\tau_\mu \alpha)$, $\mu = 2, 3$. Since G has only one class of involutions, these Sylow p-subgroups are conjugate to P_1. Since $|P| = |P_1|^3$, we must have $|P_1| = |P_2'| = |P_3'|$, and hence P_μ' is the Sylow p-subgroup of $C(\tau_\mu \alpha)$, $\mu = 2, 3$. But then $P_1^{\rho_1} = P_2'$, $P_2'^{\rho_1} = P_3'$, and $P_3'^{\rho_1} = P_1$. Thus $P^{\rho_1} = P$, and $N(S_1) \leqq N(P_1)$. The lemma is proved.

PROPOSITION 22. *Case* II. *If $W_1 \neq 1$, there exists a proper subgroup of G containing $C(\tau_1)$ and having no normal subgroups of index* 2.

Proof. Let $H = N(E_1)$. Since $E_1 \neq 1$, $H < G$. For each prime p dividing $|W_1|$, let P be the $N(S_0)$-invariant p-subgroup constructed in Lemma 19. It follows from Lemma 13 and Lemma 19 that $P \leqq O(H)$. Let K be the subgroup of $O(H)$ generated by the subgroups P for all the primes p dividing $|W_1|$. Then $K \leqq O(H) < G$. By Lemma 21, S, V_1, E_1, $N(S_0)$, and $N(S_1)$ all are contained in $N(K)$. Hence Lemma 11 implies that $N(K)$ has no normal subgroups of index 2. Thus $N(K)$ satisfies the requirements of this proposition.

PROPOSITION 23. *Case* III. *For each prime p dividing $|W_1|$, there exists an S-invariant p-subgroup P of G of order $|P_1|^3$, where P_1 is the Sylow p-subgroup of $C(\tau_1)$. If $H = N(P)$, H contains S and V_1 and has no normal subgroups of index 2. Furthermore, $H/O(H)$ is isomorphic to $PSL(2, q)$, q odd, and either q is determined independently of p, or $S = S_0$ and $V_1 = 1$, in which case $q = 3$ or* 5.

Proof. Let P be the $N(S_0)$-invariant p-subgroup of G which was constructed in Lemma 19. The first assertion has been proved in Lemma 19, and the second in Lemma 21. By Lemma 11, H is isomorphic to $PSL(2, q_p)$ or to A_7. As $E_0 = E_1 = 1$, H cannot be isomorphic to A_7. Thus it remains only to show that q_p is independent of p.

We first claim that $U_1 \cap O(H) = W_1 \cap H$. Indeed, if R_1' is a Sylow r-subgroup of $U_1 \cap H$, R_1', $R_2' = R_1'^{\rho_0}$, and $R_3' = R_2'^{\rho_0}$ normalize P, where $\rho_0 \in N(S_0) - C'(S_0)$. If r divides $|W_1|$, $R_\mu' \leqq R$, the unique maximal $N(S_0)$-invariant r-subgroup of G constructed in Lemma 19. Hence

$$R' = \{R_1', R_2', R_3'\}$$

is an r-group, and $R' > R_1'$. By the usual argument, the image of R' in $\bar{H}$ must be the identity. Hence $R' \leqq O(H)$. Thus $U_1 \cap O(H) \geqq W_1 \cap H$.

Since ρ_0 normalizes $O(H)$, $O(H)$ admits the S_0-decomposition

$$O(H) = K_1 K_2 K_3,$$

where $K_2 = K_1^{\rho_0}$ and $K_3 = K_2^{\rho_0}$. Hence for any prime r dividing $|O(H) \cap V_1|$, the unique S_0-invariant Sylow r-subgroup R of $O(H)$ has the S_0-decomposition $R = R_1 R_2 R_3$ where R_μ is the Sylow r-subgroup of K_μ; and, consequently, $R_\mu \neq 1$, $\mu = 1, 2, 3$. But then by Lemma 7, $C(R_1)$ contains a nontrivial element of R_2 or R_3; this contradicts the definition of V_1. Thus $U_1 \cap O(H) = W_1 \cap H$.

It follows that in $\bar{H} = H/O(H)$, $C_{\bar{H}}(\bar{\tau}_1) = \bar{S}\bar{V}_1$ is isomorphic to SV_1. From §4(A), it follows that

$$|\bar{H}| = \tfrac{1}{2}q_p(q_p + 1)(q_p - 1),$$

where $q_p = 2^{a+1}v + \varepsilon_p$, $\varepsilon_p = \pm 1$, and $v = |V_1|$.

Let $M = \{S, V_1, N(S_0), N(S_1)\}$. Then $M \leqq H$ and by Lemma 11 has no normal subgroups of index 2. Hence $\bar{M} = M/O(M)$ is isomorphic to $PSL(2, q)$, q is odd (A_7 is not allowed as $E_0 = E_1 = 1$). Certainly $M \cap O(H) \leqq M \cap O(M)$; hence $|\bar{M}|$ divides $|\bar{H}|$. As above, $C_{\bar{M}}(\bar{\tau}_1)$ is isomorphic to SV_1. Hence $|\bar{M}| = \tfrac{1}{2}q(q + 1)(q - 1)$, where $q = 2^{a+1}v + \varepsilon$, $\varepsilon = \pm 1$. If $\varepsilon \neq \varepsilon_p$, we must have $\varepsilon_p = 1$ and $\varepsilon = -1$ since $|\bar{M}| \leqq |\bar{H}|$. But then $q_p = q + 2$, and $(q + 3)(q + 2)(q + 1)/(q + 1)q(q - 1)$ must be an integer. This implies that q divides 6. Since q is odd, the only solution is $q = 3$, in which case $\bar{M}$ is isomorphic to $PSL(2, 3)$, and $\bar{H}$ to $PSL(2, 5)$. In this case $S = S_0$ and $V_1 = 1$. For every other choice of q, $|\bar{M}| = |\bar{H}|$ and $q_p = q$. Since M is determined independently of the choice of p, the proof is complete.

COROLLARY 24. *Case* III *with* $W_1 \neq 1$. *Set* $|V_1| = v$, $|W_1| = w$, *and* $q = 2^{a+1}v + \varepsilon$, *where* $\varepsilon = \pm 1$. *When* $V_1 > 1$, *or* $S > S_0$, g *is divisible by* $g_1 = \frac{1}{2}\, q(q^2 - 1)w^3$ *for suitable choice of* ε. *When* $V_1 = 1$ *and* $S = S_0$, g *is divisible by* $12w^3$.

Proof. When $V_1 > 1$ or $S > S_0$, $\frac{1}{2}q_p(q_p^2 - 1)w_p^3$ divides g, where w_p is the highest power of a prime p dividing w. Since q_p is determined independently of p, we may set $q = q_p$ and easily obtain that g_1 divides g.

When $V_1 = 1$ and $S = S_0$, the group $\bar{H}$ constructed in the proof of Proposition 23 is isomorphic to $PSL(2, 3)$ or $PSL(2, 5)$. Hence $|\bar{H}| = 12$ or 60. The result now follows.

Remark. If for each p dividing $|W_1|$ the corresponding $N(S_0)$-invariant p-subgroups of G constructed in Lemma 19 were pairwise permutable, they would generate a group of odd order by a theorem of P. Hall [13, p. 144], and it would follow also in Case III with $W_1 \neq 1$ that G possesses a proper subgroup $H = N(K)$ having no normal subgroups of index 2 and containing $C(\tau_1)$. Indeed, this would be the case if it were known that a group of odd order which admits the four-group as a fixed-point-free group of automorphisms is solvable.[9]

To see this, let p and q be two primes dividing $|W_1|$, and let P, Q be the corresponding $N(S_0)$-invariant subgroups. Then if $\sigma \in Z(P) \cap U_1$, $C^*(\sigma)$ has a normal 2-complement M containing P and U_1; and M would be solvable by the above proposition. But then M would contain a unique S_0-invariant Hall (p, q)-subgroup PQ_1', where $Q \cap U_1 \leqq Q_1' \leqq Q$, and it would follow that P is permutable with $Q_1'^{\rho_0 i}$, where $\rho_0 \in N(S_0) - C(S_0)$. Consequently P would be permutable with $Q = \{Q_1', Q_1'^{\rho_0}, Q_1'^{\rho_0 2}\}$.

Part III. Character Theory

11. General character theory

In this part, we treat the theory of exceptional characters. This theory has been developed principally by R. Brauer and M. Suzuki (cf. [2], [5], and [19]). Its object is to develop at least part of the character table of G. In our case we apply it to compute formulas for the order of G and to obtain certain congruences for the degrees of the irreducible characters of G. In the following sections, this will be done with just the assumption that G contains a dihedral Sylow 2-subgroup and no normal subgroups of index 2. We will also obtain slightly stronger results in particular cases when G is an L-group.

In this section, we will present a refinement of the technique of Suzuki [19] which was obtained by comparing his approach with that of Brauer in [5]. We summarize Suzuki's work omitting the proofs given there.

Let G be a finite group with a subgroup H. We say that a set of conjugate classes of H are *special classes* if the following two conditions are satisfied: (a) If x belongs to a special class, then $C_G(x) \leqq H$. (b) If x and y belong

[9] *Added in proof.* Now that Walter Feit and John Thompson have proved that all groups of odd order are solvable, it follows from this remark that an alternative proof could be given for Case III with $W_1 \neq 1$ by using Proposition 30 instead of Lemma 32 and Proposition 33.

to special classes and are conjugate in G, then they belong to the same class of H.

We will assume that H possesses a set of special classes; designate by D the elements of these classes. We let D^G be the set of elements of G which are conjugate to elements of D. Let M_G and M_H be the modules of generalized characters of G and H, respectively. Let $M_H(D)$ be the submodule of M_H which consists of those generalized characters which vanish outside of D. Let ξ^* be the (generalized) character of G that is induced by a character ξ of H, and let $M_H(D)^*$ be the submodule of M_G consisting of the character ξ^* where $\xi \in M_H(D)$.

The principal results about the characters of $M_H(D)$ are the following. It follows from the definition of the special classes that if $\xi \in M_H(D)$ and $\sigma \in D$,

$$\xi^*(\sigma) = \xi(\sigma). \tag{32}$$

Set

$$\langle \xi, \eta \rangle_H = (1/h) \textstyle\sum_{\sigma \in H} \xi(\sigma)\overline{\eta(\sigma)}, \qquad \langle \xi, \eta \rangle_G = (1/g) \textstyle\sum_{\sigma \in G} \xi(\sigma)\overline{\eta(\sigma)}, \tag{33}$$

where ξ and η are characters of H or G as the case may be and $g = |G|$ and $h = |H|$. The weight $w(\xi)$ of a character is the integer $\langle \xi, \xi \rangle_H$ or $\langle \xi, \xi \rangle_G$ as the case may be. If ξ and η are in $M_H(D)$,

$$\langle \xi, \eta \rangle_H = \langle \xi^*, \eta^* \rangle_G . \tag{34}$$

The rank of the module $M_H(D)$ is the number of special classes that can be formed from the elements of D [19, Theorem 2]. If $\eta \in M_H$, then $\eta(\sigma) = 0$ for all $\sigma \in D$ if and only if $\langle \xi, \eta \rangle_H = 0$ for all $\xi \in M_H(D)$. (See [19, Theorem 3].)

We now wish to make a more careful investigation of the structure of the modules $M_H(D)$ and $M_G(D^G)$. We say that a character ξ of M_H belongs to a p-block B of H if it is the sum of irreducible characters of H belonging to B. The set of such characters form a submodule $M_H(D, B)$ of $M_H(D)$. If $\bar{B}$ is a p-block of G, we similarly form the submodule $M_G(D^G, \bar{B})$. It is our purpose to compare the modules $M_H(D, B)$ and $M_G(D^G, \bar{B})$. To do this we make use of the mapping $B \rightarrow B^G$ of the p-blocks of H into the p-blocks of G established by Brauer [3], [4]. We say that D is *complete* if it contains along with any p-singular element $\pi\sigma = \sigma\pi$ all p-singular elements in the centralizer $C(\pi)$ of the p-component π of $\pi\sigma$.

PROPOSITION 25. *Let D be a subset of the subgroup H which determines a set of special classes and which consists of p-singular elements. Assume that D is complete. Suppose that*

$$M_H(D) = \oplus_{i=1}^{k} M_H(D, B_i), \tag{35}$$

where the summation is over all p-blocks of H. Then $M_G(D^G, \bar{B})$ is generated *by those submodules $M_H(D, B_i)^*$ for which $B_i^G = \bar{B}$, and*

$$M_G(D^G) = \oplus_{i=1}^{l} M_G(D^G, \bar{B}_i), \tag{36}$$

where the summation is over all p-blocks $\bar{B}_i$ of G of the form $\bar{B}_i = B_G$ where B is a p-block of H.

Proof. We may choose a basis ξ_t for $M_H(D)$ such that for $m_{i-1} \leqq t < m_i$, $i = 1, 2, \cdots, k$, ξ_t belongs to the module $M_H(D, B_i)$. Then

$$\xi_t = \sum_{r=1}^{m} a_{tr}\,\phi_r, \tag{37}$$

where the ϕ_r are irreducible characters of H and $a_{tr} \neq 0$ only if ξ_t and ϕ_r belong to the same p-block. Set

$$\xi_t^* = \sum_{r=1}^{n} c_{tr}\,\Phi_r, \tag{38}$$

where the Φ_r are irreducible characters of G. We shall show that $c_{tr} \neq 0$ only if Φ_r belongs to the block B^G where B is the p-block of H to which ξ_t belongs.

Now we have for the restriction of Φ_r to H

$$\Phi_r\,|_H = \sum_{s=1}^{m} y_{rs}\,\phi_s\,. \tag{39}$$

Using the Frobenius reciprocity law, we obtain that

$$C = AX, \tag{40}$$

where $C = (c_{tr})$, $A = (a_{ts})$, and $X = (x_{sr})$ is the transpose of the matrix $Y = (y_{rs})$. In particular, we have for $\sigma \in D$,

$$\Phi_r(\sigma) = \sum_{s=1}^{m} x_{sr}\,\phi_s(\sigma). \tag{41}$$

However, as Suzuki shows, the matrix X is not uniquely determined by the conditions (41) alone. More than this, any matrix $X' = (x'_{sr})$ satisfying (41) also satisfies (40). We shall make use of this and choose an appropriate matrix to analyze the matrix C.

Now if π is a p-element of D and σ is a p-regular element in $C_G(\pi) \leqq H$, then

$$\Phi_r(\pi\sigma) = \sum_{j=1}^{v} d_{rj}^{\pi}\,\zeta_j^{\pi}(\sigma), \tag{42}$$

where the d_{rj}^{π} are the generalized decomposition numbers of G, and ζ_j^{π} are the modular irreducible characters of $C_G(\pi)$. Likewise

$$\phi_s(\pi\sigma) = \sum_{j=1}^{v} e_{sj}^{\pi}\,\zeta_j^{\pi}(\sigma), \tag{43}$$

where now e_{sj}^{π} are the generalized decomposition numbers of H. Using the linear independence of the modular characters and the completeness of D, we obtain from (41), (42), and (43) that

$$d_{rj}^{\pi} = \sum_{s=1}^{m} x_{sr}\,e_{sj}^{\pi}\,. \tag{44}$$

Now let $\tilde{B}$ be a p-block of $C(\pi)$. We can then form $\tilde{B}^H = B$ and $\tilde{B}^G$, the p-blocks of H and G, respectively, determined by $\tilde{B}$, and $\tilde{B}^G = B^G$, according to [4, (2A), (2B)]. Now the Principal Theorem of [4] (Theorem (6A)) asserts that $e_{sj} \neq 0$ only if ϕ_s belongs to the block $B = \tilde{B}^H$ where $\tilde{B}$ is the block to which ζ_j^π belongs. Likewise $d_{rj} \neq 0$ only if Φ_r belongs to the block $B^G = \tilde{B}^G$. Hence if we replace $X = (x_{sr})$ by a matrix $X' = (x'_{sr})$, where $x'_{sr} = 0$ if $\phi_s \in B$ and $\Phi_r \notin B^G$ and $x'_{sr} = x_{sr}$ otherwise, equation (44) still holds, so that we obtain

$$d_{rj} = \sum_{s=1}^{m} x'_{sr}\, e_{sj}\,. \tag{45}$$

Now using (43) and (45) in (42), we obtain for $\sigma \in D$

$$\Phi_r(\sigma) = \sum_{s=1}^{m} x'_{sr}\, \phi_s(\sigma), \tag{46}$$

and hence that $C = AX'$. Then

$$c_{tr} = \sum_{s=1}^{m} a_{ts}\, x'_{sr} \tag{47}$$

is nonzero only if ϕ_s belongs to the block B of H to which ξ_t belongs and Φ_r belongs to the block B^G determined by the block B to which ϕ_s belongs. This shows that if $\xi_t \in B$, then $\xi_t^* \in B^G$. Hence

$$M_G(D^G, \bar{B}) \geqq M_H(D, B_i)^*, \tag{48}$$

where $B_i^G = \bar{B}$. Also

$$M_G(D^G) \geqq \oplus_{i=1}^{l} M_G(D^G, \bar{B}_i). \tag{49}$$

On the other hand, (32) implies that every character of $M_G(D^G)$ agrees on G with the character of G induced by its restriction on H. Thus

$$M_G(D^G) = M_H(D)^* = \sum_{i=1}^{k} M_H(D, B_i)^*. \tag{50}$$

Thus (48), (49), and (50) imply that $M_G(D^G, \bar{B})$ is generated by those submodules $M_H(D, B_i)^*$ for which $B_i^G = \bar{B}$, and (36) follows.

12. Character theory of $C(\tau_1)$

We now apply these results to the case where G contains a dihedral Sylow 2-subgroup S and no normal subgroups of index 2. We also consider the case G is an L-group in Case II with $W_1 = 1$ or in Case III.

LEMMA 26. *The set $D = AU_1 - U_1$ determines a set of special classes of $C(\tau_1)$. When G is an L-group in Case* II *with $W_1 = 1$ or in Case* III, *the set $D' = AU_1 - X_1$ also determines a set of special classes.*

Proof. Let x have even order. Then $x^n = \tau_1$ for some integer n, and $C(x) \leqq C(\tau_1)$.

Let x have odd order. This can occur only if G is an L-group and $x \in D' - D$. Hence we are in Case II or III. First consider Case II, where $W_1 = 1$. Then $x \in V_1 E_1 - E_1$. By Lemma 16, $C^*(x)$ has an S_0-invariant normal 2-complement K. Let $K = K_1 K_2 K_3$ be its S_0-decomposition. Then

$K \geqq U_1$. Hence $K_1 = U_1$, and $|K_2|$ and $|K_3|$ divide $|K_1|$. But then for any prime p dividing $|K_2|$ or $|K_3|$, there exists an S_0-invariant p-subgroup P of K with S_0-decomposition $P = P_1 P_2 P_3$. Since $P_1 \neq 1$ and $P_2 P_3 \neq 1$, Lemma 7 implies p divides $|W_1|$. Hence $K_2 = K_3 = 1$, and $C(x) \leqq SU_1 = C(\tau_1)$.

In Case III, $x \in V_1 W_1 - W_1$. Since by definition $(|V_1|, |W_1|) = 1$, some power $x^n \neq 1$ is in V_1. But then $C(x) \leqq C(x^n) \leqq C(\tau_1)$.

We next show that if $y^{-1}xy \in C(\tau_1)$ for $x \in D'$ and $y \in G$, then $y \in C(\tau_1)$. In any event, $y^{-1}x^n y$ is also in $C(\tau_1)$ for any power x^n of x. Hence if x has even order, it follows that $y \in C(\tau_1)$ since τ_1 is the only involution in AU_1. If x has odd order, then $x \in U_1 - X_1$; hence some power $x^n \neq 1$ is in the unique Sylow p-subgroup P_1 of U_1 corresponding to a prime p which divides $|V_1|$.

We claim that $H = N(P_1) \leqq C(\tau_1)$. Since P_1 is the unique Sylow p-subgroup of U_1, S is a Sylow 2-subgroup of H. Also $N_H(S_0) = C'_H(S_0)$. Should $N_H(S_1) > C'_H(S'_1)$, Lemma 13 implies that $P = E_1(p) \times P'_1 P'_2 P'_3$. Since $P_1 \cap V_1 \neq 1$, $P'_1 \neq 1$, and also $P'_2 P'_3 \neq 1$. Lemma 7 then shows that there exists $\lambda \in P'_2 P'_3$ which centralizes $P_1 \geqq P_1 \cap V_1$. This is a contradiction. Hence $N_H(S_1) = C'_H(S_1)$, and H contains a normal 2-complement K. Then K is S_0-invariant and possesses the S_0-decomposition $K = K_1 K_2 K_3$. Let σ be in K_μ, $\mu = 2$ or 3, and $z \in P_1$. Then $z^\sigma \in P_1$, so that $z^\sigma = z^{\sigma\tau_1} = z^{\tau_1\sigma^{-1}} = z^{\sigma^{-1}}$. Hence σ^2 and thus σ centralize P_1. But as $P_1 \cap V_1 \neq 1$, the definition of V_1 forces $\sigma = 1$. Hence $K = K_1 \leqq U_1$. This means that $H \leqq C(\tau_1)$.

But if $H = N(P_1) \leqq C(\tau_1)$, then P_1 is a Sylow subgroup of $N(P_1)$ and hence of G. Because P_1 is abelian, a theorem of Burnside [14, p. 203] implies that there exists z in $N(P_1)$ such that $y^{-1}xy = z^{-1}xz$. Then both z and zy^{-1} are in $C(\tau_1)$. Thus $y \in C(\tau_1)$.

We now shall obtain a maximal linearly independent subset in $M_{C(\tau_1)}(D)$ and, when G is an L-group in Case II with $W_1 = 1$ and in Case III, in $M_{C(\tau_1)}(D')$.

In order to do this, we first describe the irreducible characters of AU_1 and then form a basis of $M_{AU_1}(D)$. The relations between the irreducible characters of AU_1 and those of the normal subgroup U_1 are described by Clifford [9, p. 547] in the case that AU_1/U_1 is cyclic; we summarize these results here. We shall denote by $\bar{\nu}$ the character of AU_1 induced by a character ν of a subgroup. Associated with each irreducible character μ of U_1 is its stability group $A(\mu)U_1$, where $A(\mu)$ is a subgroup of A and $A(\mu)U_1$ consists of those elements σ of AU_1 such that $\mu^\sigma = \mu$. There exists $|A(\mu)|$ extensions of μ to $A(\mu)U_1$ which are of the form $\lambda\mu'$, where λ is a linear character of $A(\mu)U_1/U_1$ and μ' is the extension of μ to $A(\mu)U_1$ obtained by setting $\mu'(\sigma) = \deg \mu$ for $\sigma \in A(\mu)$. The character $\lambda\mu'$ is irreducible, and $\overline{\lambda_1 \mu'_1} = \overline{\lambda_2 \mu'_2}$ if and only if $\mu_2 = \mu_1^\sigma$, $A(\mu_1) = A(\mu_2)$, and $\lambda_1 = \lambda_2$. All the distinct irreducible characters of AU_1 are obtained in this way. If $|A(\mu)| = 2^{d_\mu}$ and $u = \deg \mu$, $\deg \overline{\lambda\mu'} = |A:A(\mu)|\, u = 2^{a-d_\mu}u$.

The number of irreducible characters of AU_1 is, of course, the number of classes in AU_1. But also the number of irreducible characters of AU_1 of the form $\overline{\mu'}$, where μ is a character of U_1, is the number of classes of AU_1 contained in U_1. This can be seen from the fact that $\overline{\mu_1'} = \overline{\mu_2'}$ if and only if $\mu_1^\sigma = \mu_2$ for some σ in AU_1. Thus the number of generalized characters of AU_1 of the form $\xi(\lambda, \mu) = \overline{\mu'} - \overline{\lambda\mu'}$, where λ and μ are irreducible characters of $A(\mu)U_1/U_1$ and of U_1, respectively, is precisely the number of classes in $AU_1 - U_1$. By [16, Theorem 2], this is the rank of $M_{AU_1}(D)$. Then since $\xi(\lambda, \mu)$ are independent elements of $M_{AU_1}(D)$, they form a maximal linearly independent subset of $M_{AU_1}(D)$.

Denote by $\nu^\#$ the character of $C(\tau_1) = SU_1$ induced by a character ν of a subgroup. Then the characters $\xi(\lambda, \mu)^\#$ induced on SU_1 by the characters $\xi(\lambda, \mu)$ form a basis for $M_{SU_1}(D)$. We wish to describe their decomposition into irreducible characters. Let $\overline{\lambda\mu'}$ be an irreducible character of AU_1. Then $\overline{(\lambda\mu')}^\#$ is reducible if and only if $\overline{(\lambda\mu')}^{\tau_2} = \overline{\lambda\mu'}$. This will be the case if and only if $\lambda^{\tau_2} = \lambda$ and $\overline{\mu'}^{\tau_2} = \overline{\mu'}$.

Let $B(\mu)$ be the normal subgroup of index 2 in $A(\mu)$. Because $\mu^{\tau_1} = \mu$, we must have $A(\mu) \neq 1$. Hence $B(\mu)$ always exists. Let $1_{A(\mu)}$ be the identity character of $A(\mu)U_1$, and let $\varepsilon_{A(\mu)}$ be the character whose kernel is $B(\mu)U_1$. Then $\lambda^{\tau_2} = \lambda$ if and only if $\lambda = 1_{A(\mu)}$ or $\lambda = \varepsilon_{A(\mu)}$.

If $\overline{\mu'}^{\tau_2} = \overline{\mu'}$, then $\mu'^{\sigma} = \mu'$ for some $\sigma \in SU_1 - A(\mu)U_1$. But for $\sigma \in AU_1 - A(\mu)U_1$, $\mu'^{\sigma} \neq \mu'$. Hence $\mu'^{\sigma} = \mu'$ implies that σ may be taken to be in $S - A$. Thus $\sigma = \tau_2\alpha^i$ is an involution. Let $S(\mu)$ be the dihedral subgroup $\{A(\mu), \tau_2\alpha^i\}$. Then μ' may be extended to a character μ'' of $S(\mu)U_1$ of the same degree. Let $\phi_{1\mu}$, $\phi_{2\mu}$, and $\phi_{3\mu}$ be the linear characters of $S(\mu)U_1$ with kernels $A(\mu)$, $\{B(\mu), \tau_2\}$, and $\{B(\mu), \tau_1\tau_2\}$, respectively. In particular, when $S(\mu) = S$, set $\phi_i = \phi_{i\mu}$, $i = 1, 2, 3$. These characters will be extended to SU_1 by setting $\phi_{i\mu}(\sigma) = 0$ for $\sigma \in SU_1 - S(\mu)U_1$ if necessary. An irreducible character $\mu' = 1_{A(\mu)}\mu'$ of $A(\mu)U_1$ now induces the character $(1 + \phi_{1\mu})\mu''$ of $S(\mu)U_1$. Likewise $\varepsilon_{A(\mu)}\mu'$ induces the character $(\phi_{2\mu} + \phi_{3\mu})\mu''$. Thus $\mu'^\# = (1 + \phi_{1\mu})\mu''^\#$; and so

$$\deg \mu''^\# = |S{:}S(\mu)| \deg \mu'' = 2^{a-d_\mu}u,$$

where $2^{d_\mu} = |A(\mu)|$ and $u = \deg \mu$.

When $\mu^{\tau_2} = \mu$ but $\lambda^{\tau_2} \neq \lambda$, $\overline{(\lambda\mu')}^\# = (\lambda\mu')^\#$ is irreducible. Because the characters $\overline{\lambda\mu'}$ induced on AU_1 are distinct for distinct characters λ of $A(\mu)$ and because $\overline{(\lambda^{\tau_2}\mu')}^\# = (\lambda^{\tau_2}\mu')^\# = (\lambda\mu')^\# = \overline{(\lambda\mu')}^\#$, two characters $(\lambda_1\mu')^\#$ and $(\lambda_2\mu')^\#$ are the same if and only if $\lambda_2 = \lambda_1^{\tau_2}$. Thus there are $\frac{1}{2}|A(\mu)| - 1$ distinct irreducible characters of the form $(\lambda\mu')^\#$ where $\lambda^{\tau_2} \neq \lambda$. Here $\deg (\lambda\mu')^\# = 2 \deg (\mu'')^\# = 2^{a-d_\mu+1}u$.

When $\mu^{\tau_2} \neq \mu$, $(\lambda\mu')^\#$ again is irreducible. Suppose that $(\lambda_1\mu')^\# = (\lambda_2\mu')^\#$. Then for $\sigma \in A(\mu)U_1$

$$\lambda_1(\sigma)\mu'(\sigma) + \lambda_1^{\tau_2}(\sigma)\mu'^{\tau_2}(\sigma) = \lambda_2(\sigma)\mu'(\sigma) + \lambda_2^{\tau_2}(\sigma)\mu'^{\tau_2}(\sigma).$$

Hence $\lambda_1 = \lambda_2$. Thus there are $|A(\mu)|$ irreducible characters of the form

$(\lambda\mu')^{\#}$ when $\mu^{\tau_2} \neq \mu$. Also in this case

$$\deg (\lambda\mu')^{\#} = | S:A(\mu)| \deg \mu' = 2^{a-d_\mu+1}u,$$

where $u = \deg \mu$.

Thus the characters $\zeta(\lambda, \mu)^{\#}$ have three types of decompositions, namely

$$\theta(\mu) = 1 + \phi_{1\mu} - \phi_{2\mu} - \phi_{3\mu})(\mu'')^{\#}, \tag{51}$$

$$\eta(\lambda, \mu) = (1 + \phi_{1\mu})(\mu'')^{\#} - (\lambda\mu')^{\#}, \tag{52}$$

$$\pi(\lambda, \mu) = (\mu')^{\#} - (\lambda\mu')^{\#}. \tag{53}$$

Here λ is a character of $A(\mu)$ such that $\lambda \neq 1$ and in (52), $\lambda \neq \varepsilon_{A(\mu)}$. For $\mu^{\tau_2} = \mu$, (51) and (52) apply. For $\mu^{\tau_2} \neq \mu$, (53) applies. There are $\frac{1}{2} | A(\mu)| - 1$ characters of the form (52) for each μ. Hence if $| A(\mu)| = 2$, $\eta(\lambda, \mu)$ does not exist. When $\mu = 1$, $A(\mu) = A$. In this case we distinguish $\theta_0 = \theta(1)$ and $\eta(\lambda) = \eta(\lambda, 1)$:

$$\theta_0 = 1 + \phi_1 - \phi_2 - \phi_3, \tag{54}$$

$$\eta(\lambda) = 1 + \phi_1 - \lambda^{\#}, \tag{55}$$

where $\deg \lambda^{\#} = 2$. The characters $\eta(\lambda)$ exist if and only if $| S | > 4$.

The central character of H that is determined by an irreducible character ϕ of H is the function

$$\omega(\sigma) = \frac{| H | \phi(\sigma)}{| C_H(\sigma) | \phi(1)}. \tag{56}$$

It is known that two characters of H belong to the same 2-block if the central characters which they determine are equivalent modulo a prime ideal divisor of 2 in the field of $|H|^{\text{th}}$ roots of unity. Applying this criterion to the components of $\theta(\mu)$, $\eta(\lambda, \mu)$, and $\pi(\lambda, \mu)$, we see that the characters of the form $\theta(\mu)$ and $\eta(\lambda, \mu)$ for a fixed character μ of U_1 belong to the same 2-block. Likewise the characters $\pi(\lambda, \mu)$ for a fixed character μ belong to the same 2-block. The distinct sets of characters just described belong to distinct 2-blocks of H.

When G is an L-group in Case II with $W_1 = 1$ or in Case III, we modify our development to replace the role of U_1 by $X_1 = W_1 E_1$ and $A(\mu)$ by $A(\mu) V_1$. Since V_1 is cyclic, AV_1 is cyclic and the above analysis applies. We thus consider μ to be a character of the normal subgroup $W_1 E_1 = X_1$. Because of Lemma 12, $V_1 \leqq Z(AU_1)$. Hence $\mu^{\sigma} = \mu$ for all $\sigma \in V_1$. Thus the stability group of μ now always contains V_1 and is of the form $(A(\mu)V_1)X_1$. Because $\tau_1 \in Z(AU_1)$, $A(\mu) \neq 1$. Thus the characters λ now are taken to be characters of the cyclic group $A(\mu)V_1$. Hence, for each μ, there will now be $\frac{1}{2} | A(\mu)V_1 | - 1$ characters of the form (53) for each character μ of X_1 such that $\mu^{\tau_2} = \mu$. There also will be $| A(\mu)V_1 | - 1$ characters of the form (53) for each μ such that $\mu^{\tau_2} \neq \mu$.

13. Induced characters of G

Now form the characters $\theta(\mu)^*$, $\eta(\lambda, \mu)^*$, and $\pi(\lambda, \mu)^*$. By virtue of (34) and (52) they have the respective weights 4, 3, and 2. From the Frobenius reciprocity theorem, the character 1 appears only in θ_0^* and $\eta(\lambda)^*$. Using the fact that $w(\theta_0^* - \eta_0(\lambda)^*) = w(\theta_0 - \eta(\lambda)) = 3$, and that $\eta_0(\lambda)^*(1) = \eta_0(\lambda)(1) = 0$, we obtain the expansions into distinct irreducible characters of G:

$$\theta_0^* = 1 + \delta_1 \chi_1 + \delta_2 \chi_2 + \delta_2 \chi_3 , \tag{57}$$

$$\eta(\lambda)^* = 1 + \delta_1 \chi_1 - \delta_1 \Lambda_\lambda . \tag{58}$$

where $\delta_i = \pm 1$. The remaining characters $\theta(\mu)^*$, $\eta(\lambda, \mu)^*$, and $\pi(\lambda, \mu)$ decompose into sums of 4, 3, and 2 irreducible characters.

Let $f_i = \deg \chi_i$, $i = 1, 2, 3$, and let $f_4 = \deg \Lambda_\lambda$. Equation (58) shows that f_4 is determined independently of λ, and

$$1 + \delta_1 f_1 + \delta_2 f_2 + \delta_3 f_3 = 0, \tag{59}$$

$$1 + \delta_1 f_1 = \delta_1 f_4 . \tag{60}$$

The character χ_1 is distinguished by the fact that it appears in both (57) and (58) when $| S | \geqq 8$. In case $| S | = 4$, we argue that χ_1 can be chosen so that $f_1 \neq f_j$, $j = 2, 3$. Indeed, should $f_1 = f_2 = f_3$, then by (59), $f_i = 1$, $i = 1, 2, 3$. On the other hand as G has no normal subgroups of index 2, $\chi_i(\tau_1) = f_i = 1$ for $i = 1, 2, 3$. But then

$$4 = \theta_0(\tau_1) = \theta_0(\tau_1)^* = 1 + \delta_1 + \delta_2 + \delta_3 = \theta_0^*(1) = \theta_0(1) = 0,$$

which is a contradiction. Hence when $| S | = 4$, we choose χ_1 so that $f_1 \neq f_j$, $j = 1, 2$.

We next show that the characters 1, χ_i and Λ_λ appear only in the decomposition of the two characters θ_0^* and $\eta(\lambda)^*$. The characters θ_0 and $\eta(\lambda)$ are the only characters of the form (51), (52), and (53) which belong to the first 2-block B_1 of H. By virtue of Proposition 25, the characters θ_0^* and $\eta(\lambda)^*$ belong to the block B_1^G of G. Since 1 is a component of θ_0^*, this is the first 2-block $\bar{B}_1$. We must show that none of the characters $\theta(\mu)^*$, $\eta(\lambda, \mu)^*$, or $\pi(\lambda, \mu)^*$ for $\mu \neq 1$ belong to $\bar{B}_1$. To do this it suffices to show that, for any 2-block $B \neq B_1$ of H, $B^G \neq \bar{B}_1$ by virtue of Proposition 25.

An important result of Brauer ([13, (12A)] and [4, (2D)]) shows in the case we are considering that to each block with defect group S there corresponds a character of $C(S)/S$. Furthermore, two blocks of $C(\tau_1)$ with defect group S will correspond to the same block in G if and only if the corresponding characters of $C(S)/S$ are associated in $N(S)/S$. But, of course, the principal character, which determines B_1, is associated only with itself. Since $B^G = \bar{B}_1$ implies that B has defect group S, we have that $B = B_1$.

Now we are in a position to prove the following result.

Lemma 27. *The characters χ_1, χ_2, and χ_3 have the values*

$$(61)\qquad \chi_1(\sigma) = \delta_1 = \delta_1\phi_1\,; \qquad \chi_2(\sigma) = -\delta_2\phi_2(\sigma)\,; \qquad \chi_3(\sigma) = -\delta_3\phi_3(\sigma)$$

for $\sigma \in D$ or for $\sigma \in D'$ if G is an L-group in Case II *with $W_1 = 1$ or in Case* III. *In particular, when $|S| = 4$,*

$$(62)\qquad \chi_1(\tau_1) = \delta_1\,; \qquad \chi_2(\tau_1) = \delta_2\,; \qquad \chi_3(\tau_1) = \delta_3\,.$$

When $|S| > 4$,

$$(63)\qquad \chi_1(\tau_1) = \delta_1\,; \qquad \chi_2(\tau_1) = -\delta_2\,; \qquad \chi_3(\tau_1) = -\delta_3\,.$$

Proof. We have shown that χ_1, χ_2, χ_3 appear only in the characters θ_0^* and $\eta(\lambda)^*$. Also ϕ_1, ϕ_2, and ϕ_3 appear only in the characters θ_0 and $\eta(\lambda)$. Thus from the Frobenius reciprocity law, we have that

$$(64)\qquad \langle \chi_1\,|_{C(\tau_1)} - \delta_1\phi_1\,,\ \theta(\mu)\rangle_{C(\tau_1)} = \langle \chi_1\,,\ \theta(\mu)^*\rangle_G - \delta_1\langle\phi_1\,,\ \theta(\mu)\rangle_{C(\tau_1)} = 0.$$

Similarly,

$$\langle \chi_1\,|_{C(\tau_1)} - \delta_1\phi_1\,,\ \eta(\lambda,\mu)\rangle_{C(\tau_1)} = \langle \chi_1\,|_{C(\tau_1)} - \delta_1\phi_1\,,\ \pi(\lambda,\mu)\rangle = 0.$$

Thus $\chi_1\,|_{C(\tau_1)} - \delta_1\phi_1$ is orthogonal to a maximal linearly independent subset of $M_{C(\tau_1)}(D)$. Then by [19, Theorem 3], $\chi_1(\sigma) = \delta_1\phi_1(\sigma)$ for $\sigma \in D$. In the case that G is an L-group in Case II with $W_1 = 1$ or in Case III, we can form a maximal linearly independent subset for $M_{C(\tau_1)}(D')$ by adding to the set of characters $\theta(\mu)$, $\eta(\lambda,\mu)$, and $\pi(\lambda,\mu)$, the characters $\eta(\lambda) = 1 + \phi_1 - \lambda^{\#}$ where now λ is an irreducible character of AU_1/X_1. Using the orthogonality relations (34), we obtain

$$\eta(\lambda)^* = 1 + \delta_1\chi_1 - \delta_1\Lambda_\lambda\,.$$

One may verify that $\chi_1\,|_{C(\tau_1)} - \delta_1\phi_1$ is now orthogonal to all elements of $M_{C(\tau_1)}(D')$. Thus $\chi_1(\sigma) = \delta_1\phi_1(\sigma)$ for all $\sigma \in D'$. A similar argument verifies the other equalities in (61). Of course, (62) and (63) are special cases of (61).

14. Formulas for the order of G

The groups E, E_0, and E_1 are defined in general in the same way as in §6 where U_1 is abelian. Throughout the remainder of the paper we use the following notation:

$$(65)\qquad e = |E|, \qquad e_0 = |E_0{:}E|, \qquad e_1 = |E_1{:}E|, \qquad u = |C_{U_1}(\alpha){:}E|.$$

When $|S| = 4$, $e_0 = e_1 = 1$. When G is an L-group, we also have

$$(66)\qquad u = |U_1'|, \qquad v = |V_1|, \qquad w = |W_1|, \qquad u = vw.$$

Using [23, Beispiel (3.1)], we have

$$|C(\tau_1)| = 2^{a+1}ue_0e_1e, \qquad |C(S_0)| = 4e_0e, \quad \text{and} \quad |C(S_1)| = 4e_1e.$$

We shall employ Suzuki's formula (**) developed in [19], which we repeat here:

$$(67)\quad g\sum_i (T_G(\chi_i)^2/D_g\,\chi_i)\langle\chi_i\,,\ \zeta(\mu)^*\rangle_G = |\,C(\tau_1)\,|\sum_j (T_{C(\tau_1)}(\phi_j)^2/D_g\,\phi_j)\langle\phi_j\,,\zeta(\mu)\rangle_{C(\tau_1)}\,,$$

where the summations are taken over all the irreducible characters χ_i of G and all the irreducible characters ϕ_j of $C(\tau_1)$, and

$$(68)\quad T_G(\chi_i) = \chi_i(\tau_1)/|\,C(\tau_1)\,|,$$

$$(69)\quad T_{C(\tau_1)}(\phi_j) = \phi_j(\tau_1)/|\,C(\tau_1)\,| + \phi_j(\tau_2)/|\,C(S_0)\,| + \phi_j(\tau_3)/|\,C(S_1)\,|.$$

PROPOSITION 28. *When* $|\,S\,| > 4$ *or* G *is an* L*-group in Case* III *with* $|\,SV_1\,| > 4$, *we have*

$$(70)\quad g = 2^{3a}u^3e_0\,e_1(e_0+e_1)^2e\,\frac{f_1(f_1+\delta_1)}{(f_1-\delta_1)^2}\,.$$

In general, we have

$$(71)\quad g = 2^{3a+2}u^3e_0^2\,e_1^2\,e\,\frac{f_1f_2f_3}{(f_1+\delta_1)(f_2+\delta_2)(f_3+\delta_3)}\,.$$

Proof. To derive these formulas, we shall use the characters θ_0 and $\eta(\lambda)$ for $\zeta(\mu)$ in (67), where λ is a linear character of A for which $\lambda(\tau_1) = -1$. Such a character $\eta(\lambda)$ exists if either $|\,S\,| > 4$ or G is an L-group in Case III with $|\,SV_1\,| > 4$.

Now $\phi_1(\tau_1) = \phi_2(\tau_2) = \phi_3(\tau_3) = 1, \phi_1(\tau_2) = \phi_1(\tau_3) = \phi_2(\tau_3) = \phi_3(\tau_2) = -1$, and $\phi_2(\tau_1) = \phi_3(\tau_1) = \pm1$ according as $|\,S\,| > 4$ or $|\,S\,| = 4$. By Lemma 27, $\chi_1(\tau_1) = \delta_1$. Furthermore $\eta(\lambda)^*(\tau_1) = -2$, and hence by (32) and (58), we obtain $\Lambda_\lambda(\tau_1) = 2\delta_1$.

Substituting $\eta(\mu) = \zeta(\lambda)$ in (67) and using these values, we obtain after simplification

$$(72)\quad g(1+\delta_1/f_1-\delta_1/f_4) = 2^{3a}u^3e_0\,e_1(e_0+e_1)^2e.$$

Formula (70) now follows at once if we use (60).

On the other hand, in all cases it follows from Lemma 27 that $\chi_i(\tau_1) = \pm\delta_i\,, i = 1, 2, 3$. Substituting $\zeta(\mu) = \theta_0$ in (67) and using these values, we obtain

$$(73)\quad g(1+\delta_1/f_1+\delta_2/f_2+\delta_3/f_3) = 2^{3a+2}u^3e_0^2e_1^2\,e.$$

Formula (71) now follows at once if we use (59).

Formula (70) has been derived by Suzuki in [19] under the assumption that $a = 2$, $e_0 = e = 1$ and $u = 1$. Formulas (70) and (71) have been derived previously by Brauer in [5].

When G is an L-group in Case II with $W_1 = 1$ and in Case III, we shall need congruences for the degrees f_i of $\chi_i\,, i = 1, 2, 3$. These are easily obtained by using the orthogonality relations in evaluating $\sum_{\sigma \in SV_1} \chi_i(\sigma)$ and using Lemma 27. These calculations yield

$$f_1 = \delta_1 + 2^{a+1}vr, \tag{74}$$

$$f_2 = -\delta_2 + 2^a v(2s + 1), \tag{75}$$

$$f_3 = -\delta_3 + 2^a v(2t + 1), \tag{76}$$

where r is a positive integer and s, t are nonnegative integers.

When G is isomorphic to $PSL(2, q)$, we remark that $r = 1$, $s = t = 0$, and $\delta_1 = -\delta_2 = -\delta_3 = \varepsilon$, where $\varepsilon = \pm 1$ and $\varepsilon \equiv q \pmod 4$. Also in this case $u = v$, $w = e_0 = e_1 = e = 1$.

Part IV. Completion of the Proof of Theorem I and an Application

15. Application of the formulas for the order of G

We shall apply the results of Part III to the three cases which we have considered in Part II. We shall continue to use the notation χ_1, χ_2, χ_3, and Λ_λ for the irreducible characters of G constructed in §12. We shall still denote their degrees by f_1, f_2, f_3, and f_4, and use δ_1, δ_2, and δ_3 for the signs occurring in (59) and (60). If H is a proper subgroup of G containing SV_1 and having no normal subgroups of index 2, the preceding discussion applies to H, and we shall use the notation χ_1', χ_2', χ_3', Λ_λ' for the irreducible characters of H, f_1', f_2', f_3', and f_4' for their degrees, and δ_1', δ_2', δ_3' for the corresponding signs.

Lemma 29. *In proving Theorem* I *we may assume that*

$$f_i \geqq 5 \tag{77}$$

for $i = 1, 2, 3$.

Proof. Since G is simple, each $f_i > 1$. By (74), (75), and (76) each f_i is odd. If G possesses an irreducible representation of degree 3, a result given by Blichfeldt [1, p. 112] shows that G is isomorphic to $PSL(2, q)$ where $q = 5, 7$, or 9. The theorem is verified in this case, and the lemma is proved.

Proposition 30. *There exists no proper subgroup of G which contains the centralizer of an involution and no normal subgroups of index* 2. *Consequently Case* I *and Case* II *where $W_1 \neq 1$ do not occur.*

Proof. Let H be a proper subgroup of G containing $C(\tau_1)$ and having no normal subgroups of index 2. Suppose that $|S| > 4$. Then (70) applies to H as well as G. Set

$$\psi(f_1, \delta_1) = f_1(f_1 + \delta_1)/(f_1 - \delta_1)^2, \tag{78}$$

$$\psi(f_1', \delta_1') = f_1'(f_1' + \delta_1')/(f_1' - \delta_1')^2. \tag{79}$$

Then by (71), inasmuch as $C_H(\tau_1) = C_G(\tau_1)$, $|G:H| = \psi(f_1, \delta_1)/\psi(f_1', \delta_1')$.

From Lemma 29, we see that $\psi(f_1, \delta_1) \leqq \frac{15}{8}$. On the other hand, the minimum value of $\psi(f_1', \delta_1')$ occurs when $H/O(H)$ is isomorphic to $PSL(2, 3)$, in which case $f_1' = 3$ and $\delta_1 = -1$. Hence $\psi(f_1', \delta_1') \geqq \frac{3}{8}$. Thus

$$|G:H| \leqq 5;$$

and it follows that G has a representation in the symmetric group S_5, which must be faithful as G is simple. But S_5 contains no L-subgroup H with $O(H) \neq 1$. When $|S| = 4$, a similar argument using (71) yields the same contradiction.

The last statement of the proposition now follows from Propositions 15 and 22.

PROPOSITION 31. *Case* II, $W_1 = 1$. *The group G is isomorphic to the alternating group A_7.*

Proof. Equating (70) and (71) and setting $e = e_0 = w = 1$, we obtain after simplification

$$(80) \qquad 4e_1 = (1 + e_1)^2(1 + \delta_2/f_2)(1 + \delta_3/f_3)(1 + 2\delta_1/(f_1 - \delta_1)).$$

Since $E_0 = 1$ and $E_1 \neq 1$, $S > S_0$; hence $2^{a+1} \geqq 8$. We first treat the case $2^{a+1}v > 8$. Using (74), (75), and (76), we find that

$$(81) \quad 1 + \delta_2/f_2 \geqq \tfrac{8}{9}, \qquad 1 + \delta_3/f_3 \geqq \tfrac{8}{9}, \quad \text{and} \quad 1 + 2\delta_1/(f_1 - \delta_1) \geqq \tfrac{7}{8}.$$

However, by (59), at least one $\delta_i = 1$, $i = 1, 2, 3$; so at least one of the expressions in (81) is actually greater than 1. Thus (80) yields the inequality

$$(82) \qquad 4e_1 \geqq \tfrac{8}{9}(\tfrac{7}{8})^2(1 + e_1)^2.$$

Since the resulting quadratic equation has a largest solution less than 5, the only possibility for e_1 is $e_1 = 3$.

From (70) and (74) we obtain

$$(83) \qquad g = 2^{a-1}ve_1(e_1 + 1)^2(2^{a+1}vr + \delta_1)(2^a vr + \delta_1)/r^2.$$

Now $|G:C(\tau_1)|$ is an *odd* integer. Since $|C(\tau_1)| = 2^{a+1}ve_1$, it follows from (83) that $(e_1 + 1)/2r$ must be an odd integer. Since $e_1 = 3$, this forces $r = 2$.

Substituting $e_1 = 3$ in (80), we find that

$$(84) \qquad \begin{aligned} &\tfrac{3}{4} \geqq (1 + \delta_2/f_2)(1 + \delta_3/f_3) \quad \text{or} \\ &\tfrac{3}{4} \geqq (1 + \delta_2/f_2)(1 + 2\delta_1/(f_1 - \delta_1))^2 \end{aligned}$$

according as $\delta_1 = 1$ or $\delta_3 = 1$. The first inequality gives $\frac{3}{4} \geqq (\frac{8}{9})^2$, and the second $\frac{3}{4} \geqq (\frac{8}{9})(\frac{15}{16})^2$, since $r = 2$, both of which are contradictions.

Hence $2^{a+1}v \leqq 8$. Since $2^{a+1} \geqq 8$, we must have $2^{a+1} = 8$, $v = 1$. But this is precisely the case considered by Suzuki in [16, pp. 265–266]; and there he showed that G is isomorphic to A_7.

Thus to complete the proof of Theorem I, it remains to treat Case III. We begin with the following lemma.

LEMMA 32. *Assume $SV_1 > S_0$, and let H be a subgroup of G containing SV_1 and having no normal subgroups of index* 2. *If $f_1 = f_1'$ and $\delta_1 = \delta_1'$, then* $O(H) = 1$.

Proof. Let W_1' be the S-invariant complement of $W_1 \cap H$ in W_1. Then $U_1 = (U_1 \cap H) \times W_1'$, and W_1' is a normal subgroup of $C(\tau_1)$. Hence $C_H(\tau_1)$ is isomorphic to $C(\tau_1)/W_1'$. This means that the module $M' = M_{C_H(\tau_1)}(D' \cap H)$ of characters of $C_H(\tau_1)$ which vanish outside of the set $D' \cap H$ is the submodule of $M_{C(\tau_1)}(D')$ which is generated by the characters which vanish on W_1'. In particular the character $\eta(\lambda)$ vanishes on $U_1 \geqq W_1'$ and belongs to $M_{C(\tau_1)}(D')$ since $SV_1 > S_0$. Thus $\eta(\lambda)$ is in M'. Let χ_1' and Λ_λ' be the irreducible characters of H appearing in the expansion of the induced character $\tilde{\eta}(\lambda)$ of $\eta(\lambda)$ to H, so that

$$\tilde{\eta}(\lambda) = 1 + \varepsilon'\chi_1' - \varepsilon'\Lambda_\lambda' . \tag{85}$$

We next observe that if an element σ of H is conjugate in G to an element σ' of a special class, then σ is already conjugate to that element in H. Indeed, σ will be in the centralizer of an involution τ' in G. Since H has no normal subgroups of index 2, τ' is conjugate to τ_1 in H, and σ is conjugate to an element of $C(\tau_1) \cap H = C_H(\tau_1)$. Thus we may suppose that σ is in $C_H(\tau_1)$. By Lemma 26, σ and σ' already are conjugate in $C(\tau_1)$. But then they will be conjugate by an element of $S \leqq H$, which is what we asserted.

Thus because of (33), $\eta(\lambda)$ and $\eta(\lambda)^*$ agree on all elements of H which are conjugate to an element of a special class, and they vanish on the remaining elements of H. Then $\eta(\lambda)^*|_H = \tilde{\eta}(\lambda)$. Because $f_1 = f_1'$ and $\delta_1 = \delta_1'$, it follows from (60) that $f_4 = f_4'$. But then neither $\chi_1|_H$ nor $\Lambda_\lambda|_H$ can be reducible. Hence $\chi_1|_H = \chi_1'$.

Now χ_1' is the character of H defined from a character of $H/O(H)$. Hence χ_1' has $O(H)$ in its kernel. Then the same is true of χ_1. Therefore, since G is simple, $O(H) = 1$.

PROPOSITION 33. *Case* III. *The subgroup $W_1 = 1$.*

Proof. Suppose that $W_1 \neq 1$, and let p be any prime dividing $|W_1|$. Let H be the subgroup constructed in Proposition 23 for the prime p. Then H contains SV_1 and has no normal subgroups of index 2. Now $\bar{H} = H/O(H)$ is isomorphic to $PSL(2, q)$ where $q = 2^{a+1}v + \varepsilon$ by Corollary 24. Hence $\varepsilon = \delta_1' = -\delta_2' = -\delta_3'$.

$$f_1' = 2^{a+1}v + \varepsilon, \qquad f_2' = 2^a v + \varepsilon, \qquad f_3' = 2^a v + \varepsilon. \tag{86}$$

We now divide the proof into two cases.

Case A. Assume $|SV_1| > 4$. By Corollary 24, g/g_1 is an integer. Using the value of g given by (70) (with $e = e_0 = e_1 = 1$), the value of

f_1 given by (74), and the value of g_1 given by Corollary 24, we obtain that

$$\frac{(2^{a+1}vr + \delta_1)(2^a vr + \delta_1)}{r^2(2^{a+1}v + \varepsilon)(2^a v + \varepsilon)} \tag{87}$$

is an integer. But this is possible only if $r = 1$ and $\delta_1 = \varepsilon$. Hence by (74) and (86), $f_1 = f_1'$ and $\delta_1 = \delta_1' = \varepsilon$. But Lemma 32 implies $O(H) = 1$, contrary to the fact that $O(H)$ contains the p-group P constructed in Proposition 23.

Case B. *Assume* $|SV_1| = 4$. Then $|S| = 4$ and $|V_1| = 1$. By Corollary 24, $g/12w^3$ is an integer. It follows then from (71) that

$$\frac{32}{12}\frac{f_1 f_2 f_3}{(f_1 + \delta_1)(f_2 + \delta_2)(f_3 + \delta_3)} = \frac{32}{12}\frac{(\delta_1 f_1)(\delta_2 f_2)(\delta_3 f_3)}{(\delta_1 f_1 + 1)(\delta_2 f_2 + 1)(\delta_3 f_3 + 1)} \tag{88}$$

is an odd integer. In (74), (75), and (76), set $x = 2r\delta_1$, $y = (2s + 1)\delta_2$, and $z = (2t + 1)\delta_3$. From (88) we obtain that

$$\begin{aligned} \phi(x, y, z) &= \frac{2x + 1}{x + 1} \cdot \frac{2y - 1}{y} \cdot \frac{2z - 1}{z} \\ &= \left(2 - \frac{1}{x - 1}\right)\left(2 + \frac{1}{y}\right)\left(2 + \frac{1}{z}\right) \end{aligned} \tag{89}$$

is an odd multiple of 3.

From Lemma 29 we see that $x \neq 0, -2$, $y \neq \pm 1$, and $z \neq \pm 1$. Also formula (59) gives

$$x + y + z = 0. \tag{90}$$

Now from (89), $3 < (\frac{5}{3})^3 \leqq \phi(x, y, z) \leqq (\frac{7}{3})^3 < 13$. Hence

$$\phi(x, y, z) = 9. \tag{91}$$

Suppose that $x > 0$. Because of (90) we may suppose $z < 0$. If $y < 0$, then certainly $\phi(x, y, z) < 8$ by (89), which contradicts (91). Hence $y > 0$. Then from (89) and (91), $2 + 1/y > \frac{9}{4}$, whence $0 < y < 4$. Thus $y = 3$. This gives $(2 - 1/(x + 1))(2 + 1/z) = \frac{27}{7}$. Substituting $z = -x - 3$, we obtain a quadratic equation with nonintegral solutions. Consider then the case $x < 0$. As above we immediately reduce to the case $y > 0$, $z < 0$. This time (89) and (91) yield

$$\left(2 - \frac{1}{x + 1}\right)\left(2 - \frac{1}{y}\right) > \frac{9}{2}. \tag{92}$$

Now by (90), $(1 + x) + y = 1 - z > 0$. Hence $2 - 1/(x + 1) > 2 - 1/y$, and so $(2 - 1/(x + 1))^2 > \frac{9}{2}$. This yields $x > -9.2$, whence $x \geqq -8$, and the possibilities for x are $-4, -6, -8$. Substituting $x = -4, -6, -8$ in (89) and using (90) and (91), we again obtain quadratic equations for y with nonintegral solutions. Hence this case is not possible.

PROPOSITION 34. *Case* III. *The group* G *is isomorphic to* $PSL(2, q)$, q *odd.*

Proof. Since $W_1 = 1$, $C(\tau_1) = SV_1$. Hence for any element $\sigma \neq 1$ of V_1 we have $C(\sigma) \leqq C(\tau_1)$. Since τ_2 inverts V_1, the conditions of the theorem of Brauer, Suzuki, and Wall[5] are satisfied, and it follows that G is isomorphic to $PSL(2, q)$, q odd.

This completes the proof of Theorem I.

16. An application of Theorem I

In [16] Suzuki has investigated groups which contain a cyclic subgroup of order 4 which is its own centralizer. Combining his results with Theorem I and a theorem of Brauer and Suzuki [6] on the structure of groups whose Sylow 2-subgroups are generalized quaternion groups, we are able to obtain the following generalization of Suzuki's results:

THEOREM II. *Let* G *be a finite group containing a subgroup of order* 4 *which is its own centralizer in* G. *Then either*

(i) *G has no normal subgroups of index* 2, *and a Sylow* 2-*subgroup of* G *is generated by elements* α, β *satisfying the relations*

$$\beta^2 = 1, \qquad \alpha^{2^a} = 1, \qquad \beta\alpha\beta^{-1} = \alpha^{-1+2^{a-1}};$$

(ii) *G contains a normal subgroup* G_0 *of index less than or equal to* 2, *and* $G_0/O(G_0)$ *is isomorphic to one of the groups* $SL(2, q)$, $PGL(2, q)$, $PSL(2, q)$, q *odd, or* A_7 ; *or*

(iii) *G possesses a normal* 2-*complement.*

Proof. Let S_0 be a self-centralizing subgroup of order 4 in G. In [14] and [16] Suzuki has determined the structure of a 2-group S which contains such a subgroup S_0. Taking each of these possibilities for S in turn as a Sylow 2-subgroup of G, Suzuki shows in [14] and [16] by means of Grün's theorem that one of the following conditions must hold:

(a) $S = \{\alpha, \beta\}$, where $\beta^2 = 1$, $\alpha^{2^a} = 1$, and $\beta\alpha\beta^{-1} = \alpha^{-1+2^{a-1}}$, and G has no normal subgroups of index 2;

(b) G contains a normal subgroup G_0 of index less than or equal to 2 whose Sylow 2-subgroups are dihedral groups;

(c) The Sylow 2-subgroups of G are of the form (a), and G contains a normal subgroup G_0 of index 2 whose Sylow 2-subgroups are generalized quaternion groups;

(d) G has no normal subgroups of index 2, and the Sylow 2-subgroups of G are quaternion groups of order 8;

(e) G has a normal 2-complement.

In Case (b) if τ_1 denotes the central involution of $S \cap G_0$ (which is a Sylow 2-subgroup of G_0), then $| C_G(\tau_1) : C_{G_0}(\tau_1) | \leqq 2$, and $C_{G_0}(\tau_1)$ has a normal 2-complement U_1 which is normalized by S_0. Since $C_G(S_0) = S_0$ and τ_1 centralizes U_1, S_0 must contain an element π which inverts U_1, so U_1 is abelian. This holds whether S_0 is cyclic or a four-group. It follows then from Theorem I that $G_0/O(G_0)$ is isomorphic to $PGL(2, q)$, $PSL(2, q)$, q odd, or A_7, or else has a normal 2-complement.

On the other hand, in Case (c) it follows from [6] that $G_0/O(G_0)$ contains a unique element of order 2, and hence that G possesses a normal subgroup $K > O(G_0)$ such that $|K:O(G_0)| = 2$. Furthermore $\bar{G} = G/K$ has a dihedral Sylow 2-subgroup and contains a normal subgroup $\bar{G}_0$ of index 2. It follows as in (b) that $\bar{G}$ satisfies the hypotheses of Theorem I. Since $O(\bar{G}) = 1$, $\bar{G}_0$ is either isomorphic to $PSL(2, q)$ or else has a normal 2-complement. In the first case $G_0/O(G_0)$ is isomorphic to $SL(2, q)$ by a theorem of Schur, and in the second G has a normal 2-complement.

Finally in Case (d), $G/O(G)$ is isomorphic to $SL(2, 3)$ or $SL(2, 5)$ by [16].

Remarks. Theorem I is not sufficient to classify groups which satisfy condition (i), for it is known that these include the groups $PSL(3, q)$, $q \equiv 3 \pmod 4$, among others.

Theorem II does give, however, as a special case, a classification of groups which admit an automorphism ϕ of order 2 with exactly two fixed points, which was previously obtained by Zassenhaus [24]. In fact, if G_0 is such a group and G denotes the holomorph of G_0 and ϕ, then G possesses a subgroup of order 4 which is its own centralizer in G; and hence G satisfies either condition (ii) or (iii) of the theorem.

Finally we remark that when $S_0 = \{\pi\}$ is cyclic, Theorem II can be considerably sharpened. In fact, in this case Suzuki shows that when G has no normal subgroups of index 2, either G is isomorphic to A_7, or $G/O(G)$ is isomorphic to $PSL(2, 7)$, $PSL(2, 9)$, $SL(2, 3)$, or $SL(2, 5)$, and that $O(G)$ is abelian. Furthermore π induces an automorphism of $O(G)$ leaving only the identity element fixed, so that by a result of Gorenstein and Herstein [12], $O(G)$ is always solvable, and its commutator subgroup is nilpotent. If G has a normal subgroup of index 2 but not a normal 2-complement, it follows easily from Theorem I that $G/O(G)$ is isomorphic to S_4 or S_5.

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CLARK UNIVERSITY
WORCESTER, MASSACHUSETTS
THE UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS