MATHEMATICAL CHARACTERIZATION OF THE PHYSICAL VACUUM FOR A LINEAR BOSE-EINSTEIN FIELD[1]

(Foundations of the dynamics of infinite systems III)

BY

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The mathematical treatment of the physical vacuum is one of the most challenging and basic problems of quantum field theory. In the case of "interacting" fields this problem is in substantial part one of the formulation of the underlying theory, but in the case of "free" fields—which are theoretically enlightening as well as physically relevant by virtue of the mathematical identity of their structure with that of an interacting field in the "interaction" representation at a particular time, this structure being empirically manifested, according to the usual postulates, at the times $\pm\infty$— the necessary basic formulations are now at hand. To take the presently most conservative and general position, such a field is mathematically a somewhat structured C^*-algebra of "observables" (representable, but not at all in any unique way, as a uniformly closed self-adjoint algebra of bounded linear operators on a Hilbert space), and the physical states are certain linear forms on this algebra, having the conventional interpretation of the expectation value form corresponding to the given state. This assumption is entirely independent of any assumptions as to the nature of space-time, or group-invariance of the theory.

The present paper is concerned with the problem of characterizing, in terms which are both mathematically rigorous and physically meaningful, the physical vacuum for such general types of field. The conventional formulation of the vacuum in theoretical physics as "the state of lowest energy" can be immediately transcribed mathematically, but in such a nonunique manner as to be ineffective except for certain formal purposes. The well-known divergences of quantum field theory signify essentially that the energy in conventional theories of interacting fields is mathematically highly ambiguous. It is less familiar, but equally troublesome, that the Hilbert space on which the energy is supposed to act as a self-adjoint operator, has no explicit formulation in the conventional theories. Both of these difficulties are connected with the existence of many inequivalent representations for the canonical variables of a quantum field [1], although in the relatively transparent case of a free field there are no nontrivial divergences. Nevertheless, the conventional description of the free field involves either mathematical ambiguity or technical requirements lacking in physical interpretation. The more conservative approach described above avoids the problem posed by the ambiguity of the

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representation for the canonical variables used in conventional theory, but its utilization of "state" as defined by its expectation value form, rather than as a vector in a Hilbert space, means that, a priori, the energy of a state is a somewhat vague mathematical object. The problem is thus that of dealing, in a fashion which is effective from a quite broad point of view, with the vacuum for free fields, the ultimate aim being to arrive at a simple characterization which may be adapted with a reasonable degree of confidence to the case of interacting fields.

From this point of view the main result is essentially that, although contrary to common intuitive belief, Lorentz-invariance in itself is materially insufficient to characterize the vacuum for any free field (this remarkable fact is due to David Shale; it should perhaps be emphasized that this lack of uniqueness holds even in such a simple case as the conventional scalar meson field; in particular, there exist euclidean-invariant states satisfying the Haag-Coester cluster decomposition property which differ from the conventional vacuum, despite the existence of heuristic indications of uniqueness as presented in [2]), none of the Lorentz-invariant states other than the conventional vacuum is consistent with the postulate of the positivity of the energy, when suitably and simply formulated. More specifically, in physical terms, what is done here is first to investigate the possibility of characterizing the vacuum for free fields as the unique state invariant under *all* classical unitary transformations, or more exactly, their induced action on the quantum field. (Physically, this is similar to attempting to characterize the free-field representation as that admitting occupation-number operators with the usual formal properties, that of nonnegativity not being explicitly included however; see below.) Since conventionally it is assumed that Lorentz-invariance suffices to characterize the vacuum, the stronger invariance requirement described should a fortiori suffice to pick out the conventional vacuum state, but it is found that there exist actually a continuous infinity of states which are, let us say, "universally" invariant (in the sense just described), and so a fortiori, Lorentz-invariant. The basic states were discovered by David Shale, who defined them by means of a wave representation and integration in function space, and showed their distinctness from the conventional vacuum by explicit computation of their "generating functionals" (in the sense of [3]). After Shale's communication of this result to us and concurrent indication of a non sequitur in the proof of Corollary 3.1 of [3], whose conclusion was at variance with Shale's example (a similar indication being made independently by James Glimm), we developed more adequately the line of argument of this proof, which uses a particle representation, and arrived at a determination of the general universally invariant state as a mixture of the elements of a certain one-parameter family of such states. Explicit computation of the generating functions of these basic states shows they are in fact the same as those constructed by Shale. This shows incidentally that the latter may be described from a particle viewpoint, quite heuristically, as the states in which the various bare n-particle states (i.e., n-particle states in the conventional representation) occur with uniform probability proportional to the n^{th} power of a constant in the interval $[0, 1)$ (the bare vacuum corresponding to the value 0). Shale's construction is not required for the proofs of our present results, but it is interesting in relation to them as well as to a new representation for the canonical field variables, and an account of his construction is given below.

Taking any one of these universally invariant states as the physical vacuum fixes a physical theory, which at first glance would appear to satisfy the fundamental physical desiderata. In particular, the positivity of the energy might appear to be maintained, for the field has, from a conventional outlook, only positive-energy states. Now the states E considered here are not necessarily determined by normalizable state vectors in the usual representation, but a recent result of Glimm [5], pointed out by him in the present connection, shows that for all C^*-algebras of a certain type, including the present one, any state E is a limit of such conventional states, i.e., $E(A) = \lim_n(A\psi_n, \psi_n)$ for some net ψ_n of normalized state vectors in the conventional representation (n does not necessarily run through the positive integers, however). A similarly suggestive, though of course mathematically inconclusive, indication of the positivity of the energy, is the continuity of the dependence of the basic states on the constant c mentioned above; since the energy spectrum is positive for $c = 0$, it might be anticipated that it would remain positive for small values of c, at least in the case of fields of particles of nonvanishing mass, for which the field energy is essentially bounded from below by a positive constant when $c = 0$. A rigorous examination of the energy reveals however that it is partially negative in all cases except that of the conventional free field. This result does not depend on any special analytical form for the energy, and is equally true of the occupation numbers, which may be defined because of the universal invariance of the states in question. The "total number of particles" has e.g. a nonnegative spectrum for the conventional representation, while for all other universally invariant states the spectrum is partially negative, which, in view of the integral character of the spectrum of this observable, is a particularly clear example of a discontinuous shift in the spectrum of a field observable, if the Shale states are considered as $c \downarrow 0$. To summarize qualitatively, (1) even universal invariance, let alone Lorentz-invariance, does not imply positivity of the energy, even for the simplest fields; (2) in simple practice, as well as general theory, the spectra of relevant field operators (the energy, the total number of particles) may be quite discontinuous functions of the vacuum state. These inhibitory, if with hindsight quite understandable features—in the light of much heuristic work on the theoretical difficulties which arise in quantum field theory—are partially balanced by the reassuring result that there is a unique positive-energy free field.

The universally invariant vacuums are thus, to first order, say, rather fully explored in the present article. On the other hand. the postulate of universal invariance, while physically reasonable for free fields (being formally equivalent to the possibility of defining occupation numbers having the familiar properties described earlier), is not really physically conservative, in view of some ambiguity in the notion of physical particle, and especially the dubious applicability of this notion to interacting fields, where only Lorentz-invariance (or some similarly limited invariance) seems fairly certain. It is therefore quite desirable to have a characterization for the physical vacuum utilizing invariance requirements of the limited sort applicable in principle to interacting fields. The simplest relevant conjecture is that, although Lorentz-invariance itself is not sufficient to characterize the vacuum, Lorentz-invariance together with positivity of the energy for the resulting ("clothed") field is sufficient. This is the case; in fact only the action of the time-translation subgroup of the Lorentz group is relevant, so that ultimately one arrives at a precise and representation-independent version of the (somewhat oversimplified) heuristic principle that the vacuum of a positive-energy free field may be characterized as the state of lowest energy. The implications of this for the theory of interacting fields are briefly that, although Lorentz-invariance in itself is unlikely to suffice for unique characterization of the vacuum state expectation functional, this together with positivity of the energy and suitable smoothness of the vacuum may be expected to yield uniqueness.

Only Bose-Einstein fields are considered in this article. It seems probable that similar results are obtainable by suitable adaptation of the present methods in the case of Fermi-Dirac fields.

In a purely mathematical way what is done here basically is first to determine all states E of a certain C^*-algebra $\mathfrak{A}(\mathfrak{H})$ canonically associated with a complex Hilbert space $\mathfrak{H}$, which are invariant under all unitary operators U on $\mathfrak{H}$, in the sense that

$$E(A) = E(A^U) \qquad (A \in \mathfrak{A}),$$

where the map $A \rightarrow A^U$ is the automorphism of $\mathfrak{A}$ induced by the transformation $x \rightarrow Ux$ on $\mathfrak{H}$. $\mathfrak{A}(\mathfrak{H})$, which may reasonably be called the Weyl algebra over $\mathfrak{H}$, is connected with $\mathfrak{H}$ in somewhat the manner in which a Clifford algebra is related to a real Hilbert space, the essential formal difference being the substitution of certain minus signs for plus signs in the formulation of the algebra. By using an essentially familiar connection with holomorphic functions and a simple abstract Tauberian argument, it is found that only one of these states has the property that if T is a nonnegative self-adjoint operator on $\mathfrak{H}$, the corresponding operator T' on the representation space associated with E is likewise nonnegative. (T' corresponds to T in the following way: under the representation associated with E, the automorphisms $A \rightarrow A^{U_t}$, where $U_t = \exp(itT)$, are implementable by the one-parameter unitary group $[\exp(itT'); -\infty < t < \infty]$.) It develops that such a state has this property for any one T with nontrivial nullspace if and only if it has it for all nonnegative T. Following this, it is shown that the stated uniqueness holds if the requirement of invariance under all unitary U is replaced by that of invariance under all U of the form $U = \exp(itT)$, $-\infty < t < \infty$, for some nonnegative self-adjoint operator T with trivial nullspace (a clearly necessary condition), and if the mild and physically plausible condition of continuity of the generating functional as a function of the classical state is imposed. The proof depends on the cited connection with Fourier analysis in the complex domain.

1. Technical preliminaries

For the reader's convenience, the relevant definitions and results are repeated here, together with some technical background which may be helpful. For a detailed account of the background we must, however, due to limitations of space, refer the reader to the literature cited, and the further references it contains.

We shall deal throughout with a fixed complex Hilbert space $\mathfrak{H}$, an associated abstract C^*-algebra $\mathfrak{A}$, and a representation $U \rightarrow \gamma(U)$ of the group of all unitary transformations on $\mathfrak{H}$ by automorphisms of $\mathfrak{A}$. The algebra $\mathfrak{A}$ is most conveniently defined in terms of its concrete representations by operators on a Hilbert space, especially as a particular manner of labelling a set of generators of $\mathfrak{A}$, which is highly relevant, is involved in a simple way in this definition. A *Weyl* (canonical) *system over* $\mathfrak{H}$ (in bounded, or "Weyl" form) may be defined as a mapping $z \rightarrow W(z)$ from $\mathfrak{H}$ into the unitary operators in a complex Hilbert space $\mathfrak{K}$, satisfying the ("Weyl") relations

$$W(z)W(z') = \exp[i\,\mathrm{Im}(z, z')/2]W(z + z'),$$

together with the continuity condition: $W(tz)$ is for any fixed z a weakly continuous function of the real variable t. (It may be illuminating to note that the Weyl relation is formally equivalent to, and implies, the existence of self-adjoint operators $R(z)$ satisfying the "Heisenberg" relations

$$[R(z), R(z')] \subset i\,\mathrm{Im}(z, z')I;$$

specifically, $R(z)$ is the self-adjoint generator of the one-parameter group of unitary operators $[W(tz);\ -\infty < t < \infty]$.)

The "representation algebra of field observables associated with the system W"—or for short, let us say the concrete Weyl algebra for the system W ("concrete" since we are dealing with operators on a Hilbert space, and not with the elements of an abstract algebra introduced later; "Weyl" since such relations and their linear algebraic development were first explored by Hermann Weyl [6], for systems of a finite number of degrees of freedom)—is the uniform closure of the union over $\mathfrak{M}$ (set-theoretically) of the weak closures $\mathfrak{A}_{\mathfrak{M}}$ of the finite linear combinations of the $W(z)$, when z is restricted to the subspace $\mathfrak{M}$ of $\mathfrak{H}$, as $\mathfrak{M}$ varies over all *finite-dimensional* subspaces of $\mathfrak{H}$. Any two concrete Weyl algebras, defined by Weyl systems W and W'

say, are known to be algebraically isomorphic, via a unique isomorphism which makes $W(z)$ and $W'(z)$ correspond, for all z in $\mathfrak{H}$ (Theorem 1 of [4], inessentially modified). This makes it possible to define a unique abstract C^*-algebra $\mathfrak{A}$ with distinguished elements $W(z)$ satisfying the Weyl relations, as the equivalence class of all such concrete Weyl algebras, under the indicated isomorphisms. There are also abstract algebras $\mathfrak{A}_{\mathfrak{M}}$ similarly related to the concrete $\mathfrak{A}_{\mathfrak{M}}$'s defined above.

This algebra $\mathfrak{A}$ plays an important role in the following. Specifically, let the *Weyl algebra over* $\mathfrak{H}$ be defined as the essentially unique mathematical system composed of a C^*-algebra $\mathfrak{A}$, and a map $z \rightarrow W(z)$ of $\mathfrak{H}$ into $\mathfrak{A}$ satisfying the Weyl (and associated continuity) relations, such that the system $(\mathfrak{H}, W, \mathfrak{A})$ is algebraically isomorphic as indicated above to a concrete system $(\mathfrak{H}, W, \mathfrak{A}')$, where $\mathfrak{A}'$ is an algebra of operators. For clarity, we may refer on occasion to the *abstract* Weyl algebra.

Since the Weyl algebra is invariantly attached to the Hilbert space $\mathfrak{H}$, there is for any isomorphism of $\mathfrak{H}$ a corresponding automorphism of $\mathfrak{A}$. More specifically, if U is a unitary (resp. anti-unitary) transformation on $\mathfrak{H}$, there is a unique automorphism (resp. conjugate-linear automorphism) of $\mathfrak{A}$ carrying $W(z)$ into $W(Uz)$. The map $U \rightarrow \gamma(U)$ is evidently a representation of the group of all unitary and anti-unitary transformations on $\mathfrak{H}$ into the group of all linear and conjugate-linear automorphisms of $\mathfrak{A}$. It may be noted that in a conventional classical relativistic field, the action $L \rightarrow U(L)$ of the proper Lorentz group on the classical fields is unitary, when these are suitably normed to yield a Hilbert space; the automorphism $\gamma(U(L))$ is then the action of the Lorentz transformation L on the corresponding quantized field observables.

It remains only to introduce the appropriate notion of state. A *state* is a type of linear form on $\mathfrak{A}$; specifically, the term is used for a form E which is normalized ($E(I) = 1$ where I denotes the identity element of $\mathfrak{A}$) and positive ($E(A^2) \geqq 0$ if A is a self-adjoint element of $\mathfrak{A}$). For reasons treated elsewhere, it is appropriate to make an elementary regularity requirement on the states of the Weyl algebra (only these regular states are presumably capable of even idealized measurement). A regular state may be defined as one having certain continuity and approximability features which are physically plausible on a rather conservative basis; however, a more convenient definition for present purposes is as one which, relative to some concrete Weyl algebra, has the form

$$E(A) = (Az, z)$$

for some vector z in the representation space. We refer to [3] and [4] for further discussion of the notion of regularity and merely quote for future use the result that for any finite-dimensional subspace $\mathfrak{M}$, the restriction of E to the subalgebra $\mathfrak{A}_{\mathfrak{M}}$ of $\mathfrak{A}$ has the form

$$E(A) = \mathrm{tr}(AD),$$

where for simplicity $\mathfrak{A}_{\mathfrak{M}}$ is represented by the algebra of all bounded operators on a Hilbert space (to which it is unitarily equivalent, apart from multiplicity, in every concrete representation), and D is a nonnegative element of $\mathfrak{A}_{\mathfrak{M}}$ of absolutely convergent trace; "tr" is short for trace. More generally, the same equation is valid for E in any representation, with the use of the notion of relative trace introduced in the theory of rings of operators. We note also that a regular state E is uniquely determined by its generating functional $\phi(z)$, where

$$\phi(z) = E[W(z)]$$

(see [3]).

Now let $\mathfrak{M}$ be a Hilbert space of finite dimension. Then the Weyl algebra over $\mathfrak{M}$ has a certain well-known irreducible representation called the "Schrödinger" representation, which will be useful in the following. Specifically, let $\mathfrak{M}_0$ be any real-linear subspace of $\mathfrak{M}$ of dimension m equal to that of $\mathfrak{M}$; then every vector z in $\mathfrak{M}$ can be uniquely expressed as $z = x + iy$, with x and y in $\mathfrak{M}_0$ (equivalently, for some basis $e_1, \cdots, e_m$ of $\mathfrak{M}$, $\mathfrak{M}_0$ consists of all *real* linear combinations of $e_1, \cdots, e_m$). Let $\mathfrak{K}$ denote the Hilbert space of all square-integrable complex-valued functions over $\mathfrak{M}_0$ relative to the euclidean volume element, with the usual inner product, and define $U(z)$ as the following operation on $\mathfrak{K}$, where $z = x + iy$:

$$f(u) \rightarrow e^{i(x,u)} e^{(i/2)(x,y)} f(u + y).$$

It may be verified without difficulty that $U(\cdot)$ is a Weyl system over $\mathfrak{M}$.

We shall call $\mathfrak{K}$ the representation space for *the* Schrödinger representation, although it depends not only on $\mathfrak{M}$ but also on $\mathfrak{M}_0$, when it is immaterial, within obvious limits, what choice is made for $\mathfrak{M}_0$. Now if $\mathfrak{M} = \mathfrak{P} \oplus \mathfrak{Q}$, then $\mathfrak{M}_0$, $\mathfrak{P}_0$, and $\mathfrak{Q}_0$ may be chosen so that $\mathfrak{M}_0 = \mathfrak{P}_0 \oplus \mathfrak{Q}_0$, and it is not difficult to verify that $\mathfrak{K}_{\mathfrak{M}}$ is unitarily equivalent to the direct product $\mathfrak{K}_{\mathfrak{P}} \times \mathfrak{K}_{\mathfrak{Q}}$ by an equivalence which carries $U_{\mathfrak{M}}(z)$ into the direct product $U_{\mathfrak{P}}(z') \times U_{\mathfrak{Q}}(z'')$, where z' and z'' are the components of z' and z'' in $\mathfrak{P}$ and $\mathfrak{Q}$ respectively.

In the case of a Hilbert space $\mathfrak{H}$ there is a distinguished Weyl system W which we shall call the "conventional" system, which admits some additional relevant mathematical structure, including a distinguished vector v in the representation space $\mathfrak{K}$ which is cyclic under the $W(z)$, and a continuous unitary or anti-unitary representation Γ of the full combined unitary and anti-unitary group on $\mathfrak{H}$, on $\mathfrak{K}$, such that $\Gamma(U)W(z)\Gamma(U)^{-1} = W(Uz)$ for all unitary or anti-unitary U, and vectors z in $\mathfrak{H}$, with the property that $\Gamma(U)v = v$ for all U. This structure $(\mathfrak{H}, W, \mathfrak{K}, v, \Gamma)$ may be called the *conventional free-field system.* It can be described in terms of a particle representation as introduced heuristically by Fock, and in rigorous form by Cook [7]; in terms of a wave representation analogous to the Schrödinger representation, but based on integration in a real Hilbert space as in the work of Segal [8]; or in terms of another wave representation based on integration in a complex Hilbert space in a fashion equivalent to the restriction of the representation employed by Shale to the subspace spanned by the transforms of the vacuum state representative (cf. Section 7). Each of these representations is advantageous for certain special purposes; it will suffice here to describe any one of them. Specifically, we shall use Cook's construction [7] for $\mathfrak{K}$, i.e., $\mathfrak{K}$ is the direct sum of the spaces of covariant symmetric square-integrable n-tensors over $\mathfrak{H}$ ($n = 0, 1, 2, \cdots$), where for $n = 0$ the space is taken as one-dimensional. v is then defined as any element of this space $\mathfrak{H}_0$ of unit norm (so that the phase of v is ambiguous, in the present structure). $\Gamma(U)$ is the direct sum of the induced actions on the tensors over $\mathfrak{H}$ of the action on $\mathfrak{H}$ of U (this action being defined as leaving the 0-tensors invariant). $W(z)$ is determined by the condition that the self-adjoint generator of the one-parameter unitary group $[W(tz); -\infty < t < \infty]$ should be the closure of $2^{-(1/2)}(C(z) + C(z)^*)$, where $C(z)$ is the operation of "creation of a particle with wave function z", as defined in [7].

2. Statement of results

The structure of the general universally invariant regular state of the Weyl algebra (i.e., such as are invariant under the $\gamma(U)$ for all unitary operators U) is relatively uncomplicated. To describe the states in rather explicit form, it is useful however to introduce first certain operators whose somewhat heuristic correspondents in quantum field theory are known as the "projections onto the n-particle subspaces", and for which we shall use the same name. (These projections are also closely related to Wiener's polynomial chaos of order n [9], q.v.)

The subalgebra $\mathfrak{C}$ of the Weyl algebra $\mathfrak{A}$ over a finite-dimensional Hilbert space $\mathfrak{M}$ consisting of elements left invariant by all the automorphisms $\gamma(U)$ of $\mathfrak{A}$, where U is an arbitrary unitary operator on $\mathfrak{M}$, is known to be generated by a countable set P_0, P_1, $\cdots$, of mutually commutative projections. To describe these projections more fully we recall that, because of the finite-dimensionality of $\mathfrak{M}$, the automorphisms $\gamma(U)$ are implementable by unitary transformations $\Gamma(U)$,

$$\gamma(U): A \to \Gamma(U)A\Gamma(U)^{-1},$$

in any concrete representation of $\mathfrak{A}$; and the $\Gamma(U)$ may be chosen so as to give a strongly continuous representation of the group of all unitary transformations on $\mathfrak{M}$. The P_n ($n = 0, 1, 2, \cdots$) are then represented by the spectral manifolds of the one-parameter unitary group $[\Gamma(e^{it}I); -\infty < t < \infty]$, I being the identity operator on $\mathfrak{M}$, whose self-adjoint generator N is defined as the "total number of particles". N has the proper value n on the range of P_n, and P_n is called the "projection onto the n-particle subspace for the (concrete) Weyl algebra $\mathfrak{A}$".

THEOREM 1. *Any regular state E of the Weyl algebra $\mathfrak{A}$ over a Hilbert space $\mathfrak{H}$ which is invariant under the induced action of all unitary operators on $\mathfrak{H}$ has the form*

$$E = \int E_c \, dm(c),$$

where m is a regular probability measure on $[0, 1)$ and E_c is the invariant regular state whose value on any element A of $\mathfrak{A}_{\mathfrak{M}}$, $\mathfrak{M}$ being any finite-dimensional subspace of $\mathfrak{H}$, is given by the equation

$$E_c(A) = (1 - c)^k \sum_{n=0}^{\infty} c^n \operatorname{tr}(A D_{\mathfrak{M}}^{(n)}) \qquad (A \in \mathfrak{A}_{\mathfrak{M}}, \quad k = \dim \mathfrak{M}),$$

where $D_{\mathfrak{M}}^{(n)}$ is the projection onto the n-particle subspace for $\mathfrak{A}_{\mathfrak{M}}$, and tr *denotes the trace relative to $\mathfrak{A}_{\mathfrak{M}}$.*

It should perhaps be stated explicitly that the integration of the E_c is in the usual weak vector-valued sense; the above expression for E signifies precisely that

$$E(A) = \int E_c(A) \, dm(c) \qquad \text{for all } A \text{ in } \mathfrak{A}.$$

It should also be recalled that the set-theoretic union of the $\mathfrak{A}_{\mathfrak{M}}$ is dense in $\mathfrak{A}$, so that the specification of the values of E on the $\mathfrak{A}_{\mathfrak{M}}$ leads quite directly to a determination of all the values. The relative trace is meant in the sense of Murray and von Neumann's general theory of rings of operators, but may also be described quite simply in the present case as the ordinary trace when $\mathfrak{A}_{\mathfrak{M}}$ is irreducibly represented.

There is another way of stating this result which, while less succinct, exhibits them in a form which is explicitly independent of the formulation of the Weyl algebra, and is in closer relationship to the formulations of the "axiomatic" schools of quantum-field theorists. This formulation describes the structure of the general universally invariant free field.

THEOREM 1'. *Let W be a canonical system over the Hilbert space $\mathfrak{H}$, with representation space $\mathfrak{K}$; let Γ be a continuous representation of the group of all unitary operators U on $\mathfrak{H}$, on $\mathfrak{K}$, such that*

$$\Gamma(U) W(z) \Gamma(U)^{-1} = W(Uz) \qquad (z \in \mathfrak{H});$$

and suppose there exists a vector v in $\mathfrak{K}$ which is cyclic for the $W(z)$ and invariant under all the $\Gamma(U)$. Then the system (W, Γ, v) is unitarily equivalent to the (unique) representation system (see [4]) derived from one of the states described in Theorem 1; in particular,

$$(W(z)v, v) = \int_0^1 \exp\left[-(\tfrac{1}{4}) \| z \|^2 ((1 + c)/(1 - c))\right] dm(c).$$

The assumption of universal invariance implies, and in the presence of certain continuity restrictions is substantially equivalent to, the possibility of defining "occupation-number" operators for arbitrary submanifolds of $\mathfrak{H}$, which have all of the conventional properties of these operators in theoretical physics except that of being nonnegative (cf. [4]). There is actually only one universally invariant free field having the latter property; in fact the following stronger result is valid.

THEOREM 2. *Let* $(\mathfrak{H}, W, \Gamma, v)$ *be as in Theorem* 1′, *and suppose that for some nonnegative self-adjoint operator* A *on* $\mathfrak{H}$ *whose discrete spectrum, if any, does not contain* 0, $d\Gamma(A) \geqq 0$, *where* $d\Gamma(A)$ *denotes the infinitesimal generator of the one-parameter group* $[\Gamma(e^{itA});\ -\infty < t < \infty]$. *The system is then unitarily equivalent to the Fock-Cook (conventional) free-field system.*

This means that not only is the Fock-Cook system the only universally invariant free field admitting nonnegative occupation numbers, it is also the only one for which the energy spectrum of the field can possibly be nonnegative, starting from a physically reasonable nonnegative energy for the corresponding classical field.

The strength of the positive-energy assumption used in Theorem 2 is emphasized by the fact that it, together with the mild and physically plausible assumption of the continuity of the generating functional as a function of the classical field, makes it possible to replace the requirement of invariance under *all* unitary operators on $\mathfrak{H}$, in Theorem 2, by the much weaker one of invariance under a suitable one-parameter unitary group of operators on $\mathfrak{H}$. In a form which refers to representations rather than states of the Weyl algebra, this result is, specifically,

THEOREM 3. *Let* W *be a Weyl system on* $\mathfrak{K}$ *over the Hilbert space* $\mathfrak{H}$. *Let* $[U(t);\ -\infty < t < \infty]$ *be a continuous one-parameter group of unitary operators on* $\mathfrak{H}$ *whose self-adjoint generator is nonnegative and annihilates no nonzero vectors in* $\mathfrak{H}$. *Let* $[\Gamma(t);\ -\infty < t < \infty]$ *be a continuous one-parameter unitary group on* $\mathfrak{K}$ *whose self-adjoint generator is nonnegative, and is such that*

$$\Gamma(t)W(z)\Gamma(t)^{-1} = W(U(t)z).$$

Suppose that $\mathfrak{K}$ *contains a vector* v *which is cyclic in* $\mathfrak{K}$ *for the* $W(z)$, *and which is invariant under all the* $\Gamma(t)$. *Then if* $(W(z)v, v)$ *is continuous as a function of* z *on* $\mathfrak{H}$, *the system* $(\mathfrak{H}, W, \mathfrak{K}, v)$ *is unitarily equivalent to the conventional free-field system, with* $\Gamma(t)$ *the induced field action of* $U(t)$.

The significance of this result may be clarified by specializing it to the case of a standard relativistic field, say the scalar meson field. A conventional treatment usually involves the definition of a vacuum state vector as one annihilated by all annihilation operators (or all negative frequency components of the quantized field). It is assumed further that field operators may be successively applied to the vacuum state vector. This is materially stronger than the assumption of the finiteness of the vacuum expectation values of products of field operators, an assumption which itself is physically insecure because before renormalization such products appear as having apparently infinite vacuum expectation values; and after renormalization it has been merely a pure hypothesis that the positive-definiteness conditions on the vacuum expectation values (such as are obviously satisfied formally before renormalization) remain valid. One does not even have such purely formal reassurance as would be provided by the validity of this result in perturbation theory.

In addition to these assumptions, a number of relatively technical continuity assumptions are commonly made on the vacuum expectation values as functions of the weighting ("test") functions. It then follows that the representation is uniquely determined, being in fact unitarily equivalent to the particle representation of Fock, with corresponding vacuum. However, this result is not fully satisfactory as a characterization of the free field, being rather in large part an ad hoc description of it. It is gratifying therefore that the assumptions described can be eliminated, and a mathematically rigorous characterization given which avoids relatively unphysical hypotheses. Specifically, Theorem 3 implies

COROLLARY 3.1. *Let $f \to R(f)$ be a map from the infinitely differentiable functions of compact support on Minkowski space-time to the self-adjoint operators in a Hilbert space $\mathfrak{K}$, which satisfy the Weyl relations for a scalar meson field with weighting function f:*

$$W(f)W(f') = e^{iB(f,f')}W(f + f'),$$

where

$$W(f) = \exp(iR(f)); \qquad (2\pi)^{3/2}B(f,f') = \int_{k^2=m^2} \tilde{f}(k)\bar{\tilde{f}}'(k)\varepsilon(k)\frac{d_3 k}{|k_0|},$$

$\varepsilon(k) = \operatorname{sgn} k_0$, $\tilde{f}$ denotes the Fourier transform of f, and $R(sf) = sR(f)$ (s real, $\neq 0$). Suppose also that for all f, $R((\Box - m^2)f) = 0$.

Let $[\Gamma(t); -\infty < t < \infty]$ be a continuous one-parameter group of unitary operators on $\mathfrak{K}$ such that $\Gamma(t)R(f)\Gamma(t)^{-1} = R(f_t)$, where f_t denotes the function into which f is carried by translation in time through $-t$, which has also the property that its self-adjoint generator is nonnegative. Let v be a cyclic vector in $\mathfrak{K}$ for the $W(z)$, with the property that $\Gamma(t)v = v$ for all t. If $(W(f)v, v)$ is a continuous function of f in the invariant Hilbert space metric,[2] then there exists a unitary transformation from $\mathfrak{K}$ to the state space of the conventional (say, Fock-Cook particle) representation for a scalar meson field which carries $R(f)$ into the quantized field average with weight function f, $[\Gamma(t); -\infty < t < \infty]$ into the one-parameter unitary group generated by the conventional free-field Hamiltonian, and v into the vacuum state vector.

To paraphrase this in theoretical physical language, any field that (1) satisfies the canonical commutation relations and the Klein-Gordon equa-

[2] In which $\| f \|^2$ is $-2iB(f, Tf)$, where T denotes the Hilbert transform operation with respect to time.

tions, (2) has an energy operator that is positive, and a vacuum that is annihilated by the energy, and from which all states can be built up by the action of *bounded* functions of the field operators, (3) in which complex exponentials of weighted field averages depend in a mildly continuous way on the weight function, relative to the unique Lorentz-invariant inner product, is the conventional field in the conventional special representation. Thereby the following kinds of assumptions commonly made (implicitly or explicitly) in so-called axiomatic field theory are avoided: (a) the finiteness of vacuum expectation values of products of field operators; (b) relatively strong continuity restrictions on vacuum expectation values as functions of (microscopic) space-time.

Since Corollary 3.1 is included primarily for illustrative purposes, and will be of interest primarily to those familiar with conventional quantizations for the standard relativistic fields, it hardly seems appropriate to take the nonnegligable amount of space required to describe in detail the conventional representation, etc., from the special standpoint relevant here, especially since [3] includes an account of much of this material as well as references to background literature. The proof of the corollary is a matter of observing that the space of infinitely differentiable functions with compact support, modulo the subspace of functions of vanishing norm relative to the inner product

$$\langle f, f' \rangle = \int_{k^2=m^2} \tilde{f}(k)\bar{\tilde{f}}'(k) \frac{d_3\, k}{|k_0|}$$

is an incomplete Hilbert space relative to this norm; and that the proofs of the results quoted above depend not at all on the completeness of $\mathfrak{H}$.

3. Structure of the general universally invariant state

This section gives the proof of Theorem 1. To this end, let E be a regular state of the Weyl algebra $\mathfrak{A}$ over a Hilbert space $\mathfrak{H}$, which is invariant under all the automorphisms $\gamma(U)$ induced by unitary operators U on $\mathfrak{H}$. Let $\mathfrak{M}$ be an arbitrary subspace of $\mathfrak{H}$ of finite dimension n; then by the result cited above concerning arbitrary regular states, the restriction of E to $\mathfrak{A}_{\mathfrak{M}}$ has the form

$$E(A) = \operatorname{tr}(AD),$$

D being of absolutely convergent trace, relative to the obvious ring of operators. Now the invariance of E under the $\Gamma(U)$ together with the invariance of the trace implies the invariance of D under the $\Gamma(U)$. By the result cited above giving the structure of the subalgebra of the Weyl algebra consisting of elements invariant under the $\Gamma(U)$, in the case of a finite number of degrees of freedom, every invariant such D has the form

$$D = \sum_k a_k D_k ,$$

where D_k is the projection onto the n-particle subspace, and the a_k are nonnegative real constants. From the normalization condition on E, $E(I) = 1$, it results that $\sum_k a_k \operatorname{tr} D_k = 1$. From the invariance of E together with the possibility of connecting any two subspaces of $\mathfrak{H}$ of equal finite dimension by a unitary transformation on $\mathfrak{H}$, it follows that the a_k depend on $\mathfrak{M}$ (so that we write $a_k(\mathfrak{M})$) only through the dimension of $\mathfrak{M}$.

Now suppose that $\mathfrak{P}$ is a subspace of $\mathfrak{M}$. Then $\mathfrak{A}_{\mathfrak{P}} \subset \mathfrak{A}_{\mathfrak{M}}$, so that

$$\operatorname{tr}_{\mathfrak{P}}(AD(\mathfrak{P})) = \operatorname{tr}_{\mathfrak{M}}(AD(\mathfrak{M})), \qquad A \in \mathfrak{A}_{\mathfrak{P}}$$

(where the dependence of D on the relevant subspace is made expicit, and $\operatorname{tr}_{\mathfrak{M}}$ refers to the relative trace in $\mathfrak{A}_{\mathfrak{M}}$). Substituting $A = D_k(\mathfrak{P})$, we find that

$$a_k(\mathfrak{P}) \operatorname{tr}_{\mathfrak{P}} D_k(\mathfrak{P}) = \operatorname{tr}_{\mathfrak{M}} D_k(\mathfrak{P}) D(\mathfrak{M}) = \sum_j a_j(\mathfrak{M}) \operatorname{tr}_{\mathfrak{M}}(D_k(\mathfrak{P}) D_j(\mathfrak{M})),$$

so that

$$a_k(\mathfrak{P}) = \sum_j a_j(\mathfrak{M}) \frac{\operatorname{tr}_{\mathfrak{M}}(D_k(\mathfrak{P}) D_j(\mathfrak{M}))}{\operatorname{tr}_{\mathfrak{P}}(D_k(\mathfrak{P}))}.$$

To evaluate the coefficients explicitly in this representation of the $a_k(\mathfrak{P})$ as functions of the $a_j(\mathfrak{M})$, we require the following group-representation-theoretic result.

LEMMA. *If $\mathfrak{M} = \mathfrak{P} \oplus \mathfrak{Q}$ where $\dim \mathfrak{Q} = 1$ (so that as noted above, $\mathfrak{K}_{\mathfrak{M}} \cong \mathfrak{K}_{\mathfrak{P}} \times \mathfrak{K}_{\mathfrak{Q}}$ in such a fashion that $U(z + z') \cong U(z) \times U(z')$), then*

$$D_k(\mathfrak{M}) = \sum_{j=0}^{k} D_j(\mathfrak{P}) \times P_j,$$

where P_j is a projection of unit rank on $\mathfrak{K}_{\mathfrak{Q}}$.

For the proof we recall the result (cf. [8]) that the range of D_k is spanned by products of Hermite functions $h_{n_1}(x_1) \cdots h_{n_m}(x_m)$, where the n_i are non-negative integers with sum k, and $x_1, \cdots, x_m$ are orthonormal coordinates in m-dimensional euclidean space. It is no essential loss of generality to take a basis for $\mathfrak{M}$ which includes bases for $\mathfrak{P}$ and $\mathfrak{Q}$, so that the range of $D_k(\mathfrak{P})$ is spanned by the $h_{n_1}(x_1) \cdots h_{n_{m-1}}(x_{m-1})$ with $n_1 + n_2 + \cdots + n_{m-1} = k$, while the range of $D_k(\mathfrak{Q})$ is spanned by $h_k(x_m)$. From the obvious fact that $h_{n_1}(x_1) \cdots h_{n_m}(x_k) = [h_{n_1}(x_1) \cdots h_{n_{m-1}}(x_{m-1})] h_{n_m}(x_m)$, it results that every element of the range of $D_k(\mathfrak{M})$ has a unique expression of the form

$$\sum_r g_r(x_1, \cdots, x_{m-1}) h_{k-r}(x_m),$$

where g_r is in the range of $D_r(\mathfrak{P})$. Now by taking P_j as the projection onto the one-dimensional space spanned by $h_j(x_m)$, the conclusion of the lemma follows.

Now resuming the proof of the theorem, multiplying both sides of the equation in the lemma by $D_k(\mathfrak{P}) \times I_{\mathfrak{K}_{\mathfrak{Q}}}$ we find that

$$\begin{aligned} D_k(\mathfrak{P}) D_j(\mathfrak{M}) &= 0 && \text{if} \quad k > j, \\ &= D_k(\mathfrak{P}) \times P_{j-k} && \text{if} \quad k \leqq j. \end{aligned}$$

If we take $\text{tr}_{\mathfrak{M}}$ on both sides, and note that

$$\text{tr}_{\mathfrak{M}}(A \times B) = \text{tr}_{\mathfrak{P}}(A)\ \text{tr}_{\mathfrak{Q}}(B),$$

it results that

$$\begin{aligned} \text{tr}(D_k(P)D_j(\mathfrak{M})) &= 0 & \text{if} \quad k > j, \\ &= \text{tr}(D_k(\mathfrak{P})) & \text{if} \quad k \leqq j. \end{aligned}$$

Hence the simple formula

$$a_k(\mathfrak{P}) = \sum_{j \geqq k} a_j(\mathfrak{M})$$

is obtained for the case $\dim \mathfrak{M} = \dim \mathfrak{P} + 1$.

Conversely, the $a_k(\mathfrak{P})$ determine the $a_j(\mathfrak{M})$:

$$a_j(\mathfrak{M}) = a_j(\mathfrak{P}) - a_{j+1}(\mathfrak{P})$$

(with the convention that any a with a negative subscript is zero). Now set $a_j(\mathfrak{P}) = b_j$ for an arbitrary one-dimensional $\mathfrak{P}$ (which is legitimate since, as noted above, $a_j(\mathfrak{P})$ depends on $\mathfrak{P}$ only through its dimension), and consider the restrictions on the b_j imposed by the requirement that the $a_j(\mathfrak{M})$ determined from it for $\mathfrak{M}$'s of dimensions 2, 3, $\cdots$ be nonnegative. For $\dim \mathfrak{M} = 2$, the condition is that $b_j - b_{j+1} \geqq 0$, i.e., that the negatives of the first differences be nonnegative; and for $\dim \mathfrak{M} = n$, the requirement is that $(-1)^n$ times the n^{th} difference be nonnegative. Thus it is necessary that $\{b_j\}$ be a completely monotone sequence.

On the other hand, if $\{b_j\}$ is any sequence of nonnegative numbers whose sum is unity, and which has the property of being completely monotone, then a universally invariant regular state E exists for which the associated coefficients $a_j(\mathfrak{P}) = b_j$ for $\dim \mathfrak{P} = 1$. For then the successive $a_j(\mathfrak{M})$ defined by the equation above for $\mathfrak{M}$ of dimensions 2, 3, $\cdots$ will be nonnegative, and so determine a regular state $E_{\mathfrak{M}}$ of $\mathfrak{A}_{\mathfrak{M}}$ in such a fashion that if $\mathfrak{M} \subset \mathfrak{N}$, then $E_{\mathfrak{M}} \subset E_{\mathfrak{N}}$. Now by defining E_0 as the linear functional on the union $\mathfrak{A}_0$ of the $\mathfrak{A}_{\mathfrak{M}}$, as $\mathfrak{M}$ varies over all finite-dimensional subspaces of $\mathfrak{H}$, which on each $\mathfrak{A}_{\mathfrak{M}}$ coincides with $E_{\mathfrak{M}}$, it is clear that $| E_0(A) | \leqq \| A \|$ for arbitrary A in $\mathfrak{A}_0$, so that E_0 extends uniquely by continuity to a continuous linear functional E on $\mathfrak{A}$, which by continuity is positive, and therefore a state. Since regularity is a condition on the restriction of a state to the $\mathfrak{A}_{\mathfrak{M}}$, it is clearly satisfied by E. Further, if A is an element of an $\mathfrak{A}_{\mathfrak{M}}$ with finite-dimensional $\mathfrak{M}$, then since the transform A^U of A under the automorphism of the Weyl algebra induced by a unitary transformation U on $\mathfrak{H}$ is an element of $\mathfrak{A}_{U\mathfrak{M}}$, and since $E(A)$ and $E(A^U)$ are defined in the same way, it results that

$$E(A) = E(A^U), \qquad A \in \mathfrak{A}_0 .$$

There is no difficulty in deducing by a continuity argument that the same equation holds for all A in $\mathfrak{A}$, i.e., that E is universally invariant.

There is thus set up an explicit one-to-one correspondence between regular universally invariant states of the Weyl algebra and completely monotone sequences $\{b_j\}$ with $\sum_j b_j = 1$. It is well known (cf. [10]) that every such sequence has the form

$$b_j = \int c^j(1 - c)\, dm(c)$$

for some regular probability measure m on $[0, 1)$. If in particular, m is concentrated on the point $\{c\}$, it is readily computed that

$$a_j(\mathfrak{M}) = c^j(1 - c)^n, \qquad \text{where } n = \dim \mathfrak{M},$$

so that the state E_c described in Theorem 1 exists and has the stated properties. With a general $m(c)$, it is clear that the functional E_0 defined by the equation

$$E_0(A) = \int E_c(A)\, dm(c)$$

for A in an $\mathfrak{A}_{\mathfrak{M}}$ exists, $E_c(A)$ for such an A being easily seen to be a continuous function of c which is bounded by the bound of A; and it is evident that E_0 is normalized and nonnegative on the nonnegative elements of $\mathfrak{A}_0$, and so extends to a unique state of $\mathfrak{A}$, which can be described as in Theorem 1. That this state is regular is clear from the manner of its construction together with the observation that its restriction to any $\mathfrak{A}_{\mathfrak{M}}$ is weakly continuous relative to the unit sphere.

To derive Theorem 1′ from Theorem 1, we observe that the state of $\mathfrak{A}$,

$$E(A) = (A'v, v),$$

where A' is the operator on $\mathfrak{K}$ corresponding to the element A of the (abstract) Weyl algebra, and it is assumed, as is no essential loss of generality, that $|v| = 1$, is regular and universally invariant. It must accordingly be one of the states described in Theorem 1. Since the representation $A \to A'$ of the Weyl algebra is cyclic, it follows from the known mutual correspondence between cyclic representations and states of C^*-algebras [11] that the structure $(\mathfrak{H}, W, \mathfrak{K}, v)$ is unitarily equivalent to the stated system.

Now the Γ' of the representation system which is derived from E satisfies the equation

$$\Gamma'(U)W(z)\Gamma'(U)^{-1} = W(Uz)$$

for all unitary operators U on $\mathfrak{H}$ and z in $\mathfrak{H}$, and like the $\Gamma(U)$ the $\Gamma'(U)$ leave v invariant. This means that the $\Gamma(U)^{-1}\Gamma'(U)$ commute with all $W(z)$, and also leave invariant v. Since v is a cyclic vector for the $W(z)$, it follows that each of the $\Gamma(U)^{-1}\Gamma'(U)$ leaves invariant a dense subset of $\mathfrak{K}$, namely all finite linear combinations of the $W(z)v$, and so is the identity.

4. The positive-energy universally invariant state

Any state of the sort described in the preceding section will give rise to a clothed free quantum field having all of the qualitative physical features of the conventional free physical fields, with one notable exception. This is the positivity of the energy. Contrary to what might possibly be expected, this does not follow from all the other properties even in, say, the case of a free field of scalar mesons. In fact, in this section it is shown, by a theoretical rather than computational argument, that the (field) energy spectrum is partially negative in all cases except one, starting from an essentially arbitrary positive definite classical Hamiltonian. We thereby obtain a characterization of the conventional vacuum (or corresponding clothed field) as the unique positive-energy universally invariant field.

The universal invariance means precisely that occupation-number operators can be defined satisfying the usual formal physical requirements for a valid statistical interpretation: they have integral proper values, annihilate the vacuum, transform appropriately under Lorentz and similar transformations, and the total field momentum of a given character can be represented as the sum of the products of the values of the momentum with the occupation numbers for the single-particle states of that momentum value. However, it does not follow that the occupation numbers are nonnegative. Another way of characterizing the conventional representation is as that in which this is the case. The conventional representation is the only universally invariant one in which the total-number-of-particles operator is nonnegative, in fact, as follows from the observation that this latter operator is definable as $d\Gamma(I)$, in the notation of Theorem 2 (cf. [12]).

However, the positivity of the energy seems more basic than the nonnegativity of the occupation numbers, as it is unaffected by the possible existence of an interpretation of negative occupation numbers in terms of anti-particles. At first glance this poses a certain problem, for the energy, unlike the occupation numbers, can not be defined purely in terms of the Hilbert space structure of the underlying single-particle space; it depends on the action in this space of the unquantized Hamiltonian operator. But it turns out, nevertheless, that a strong general statement is possible; the mere existence of a nontrivial invariant development of the free system with nonnegative energy is sufficient to determine the vacuum uniquely as essentially the conventional one.

We turn now to the formal proof of Theorem 2, whose notation is employed in what follows. Set

$$F(\lambda) = (e^{-\lambda d\Gamma(A)}W(z)v, W(z)v),$$

where λ is an arbitrary complex number of nonnegative real part, and z is an arbitrary vector in $\mathfrak{H}$. From the positivity of $d\Gamma(A)$, it follows that $F(\lambda)$ is, as a function of λ, bounded and holomorphic, and is in fact an absolutely convergent Laplace-Stieltjes transform. Its boundary values have the form

$$\begin{aligned} F(it) &= (e^{-itd\Gamma(A)}W(z)v, W(z)v) \\ &= (W(e^{-itA}z)v, W(z)v) \\ &= (W(-z)W(e^{-itA}z)v, v) \\ &= e^{(i/2)\operatorname{Im}(z_t, z)}(W(z_t - z)v, v), \quad \text{where } z_t = e^{-itA}z. \end{aligned}$$

Now consider the function

$$f(\lambda) = \exp [(1/2)(e^{-\lambda A}z, z)]$$

for complex λ having nonnegative real part. Since, as a function of λ, $(e^{-\lambda A}z, z)$ is an absolutely convergent Laplace-Stieltjes transform, and the collection of all such is a Banach algebra relative to the total variation of the corresponding set function (cf. [13]), and since Banach algebras are closed under the application of entire functions, f is also an absolutely convergent Laplace-Stieltjes transform. The same argument shows that $g(\lambda) = (f(\lambda))^{-1}$ is also such. The product $F(\lambda)g(\lambda)$ hence is also such. On the other hand,

$$F(it) = \exp [(i/2)\mathrm{Im}(z_t, z)](W(z_t - z)v, v),$$

and by Theorem 1, $(W(z_t - z)v, v)$ is real, while

$$g(it) = \exp [-(i/2)\mathrm{Im}(z_t, z)] \exp [-(1/2)\mathrm{Re}(z_t, z)].$$

Thus $F(it)g(it)$ is real, for all real values of t. However, it follows readily from the fact that a regular set function on the reals is uniquely determined by its Fourier-Stieltjes transform, that a one-sided absolutely convergent Laplace-Stieltjes transform can have real boundary values only if it is a constant.

It results that

$$F(\lambda) = k(z)f(\lambda),$$

where $k(z)$ is a constant dependent on z, but independent of λ. Putting $\lambda = 0$ shows that $k(z) = \exp [-(1/2)(z, z)]$. Therefore $E(W(u))$ has the stated form provided u has the form $z_t - z$ for some z:

$$\begin{aligned} E(W(u)) &= \exp [-(1/2)(z, z)] \exp [(1/2)\mathrm{Re}(z_t, z)] \\ &= \exp [-(1/4)\| z_t - z \|^2]. \end{aligned}$$

If A annihilates no vector in $\mathfrak{H}$, the set of all such U's is dense. The proof may now be concluded by the observation that the formula of Theorem 1 shows that the generating functional $E(W(u))$ for a universally invariant regular state is a continuous function of u, in the usual Hilbert space topology on $\mathfrak{H}$. For apart from constant factors, the relevant integrand is

$$\sum_n c^n \,\mathrm{tr}_{\mathfrak{M}}(W(u)D_n),$$

taking $\mathfrak{M}$ as the one-dimensional subspace spanned by u, where D_n is the projection onto the one-dimensional space spanned by the n^{th} Hermite function; since this series is dominated by $\sum_n c^n$, it suffices to establish the continuity of $\mathrm{tr}_{\mathfrak{M}}(W(u)D_n)$, but by the unitary invariance of the trace, this depends only on $\| u \|$, and so has the form

$$\int \exp [i \| u \| 2^{-(1/2)} q](h_n(q))^2 \, dq,$$

which represents a continuous function.

5. Uniqueness of the positive-energy stationary free field

As noted above, it is desirable to replace the condition of universal invariance by the apparently much weaker one of Lorentz invariance. It seems, however, to be a rather difficult problem to determine *all* Lorentz-invariant regular states of the Weyl algebra. It is not even known whether there exist such states other than the universally invariant ones described in Theorem 1. However, what is physically more relevant is the replacement of Theorem 2 by a result requiring only Lorentz invariance, and it is possible, actually, to establish a much stronger result. As is physically not very surprising, full Lorentz invariance is not required for the characterization of the vacuum; invariance under the temporal development of the system suffices, provided the energy is nonnegative.

The proof is attained by a refinement of that of Theorem 2. As in that proof,

$$F(\lambda) = (e^{-\lambda d\Gamma(A)}W(z)v, W(z)v)$$

is a one-sided absolutely convergent Laplace-Stieltjes transform whose boundary values can be expressed as

$$F(it) = \exp[(1/2)i\,\mathrm{Im}((z_t, z))](W(z_t - z)v, v).$$

If we multiply $F(\lambda)$ by $g(\lambda)$, whose boundary values are

$$g(it) = \exp[-(1/2)(z_t, z)],$$

it follows that

$$(W(z_t - z)v, v)\exp[-(1/2)\mathrm{Re}((z_t, z))], \qquad = G(it) \text{ say,}$$

is, as a function of t, the boundary value function for an absolutely convergent one-sided Laplace-Stieltjes transform. Now this is true for any z; in particular, z may be replaced by $-z$, which shows that

$$(W(-z_t + z)v, v)\exp[-(1/2)\mathrm{Re}(z_t, z))], \qquad = H(it) \text{ say,}$$

is likewise the boundary value function of an absolutely convergent one-sided Laplace-Stieltjes transform. But it is evident that $H(it) = \overline{G(it)}$; and it is easily verified, from the uniqueness of the set function having a given Fourier-Stieltjes transform, that the only one-sided absolutely convergent Laplace-Stieltjes transform, the complex conjugate of whose boundary values are again a Laplace-Stieltjes transform of a regular set function on the positive semi-axis, is a constant.

By putting $t = 0$, the constant may be evaluated, to give the equation

$$(W(z_t - z)v, v) = \exp[-(1/4)\| z_t - z \|^2].$$

Thus the conclusion of Theorem 2 is valid for the dense set of z's in $\mathfrak{H}$ of the form $u_t - u$ for some u in $\mathfrak{H}$; and by continuity, it follows for all z.

6. The Shale representation for the extreme universally invariant states

As noted above, Shale originally set up certain universally invariant states through the use of integration in Hilbert space. In this section we describe this formulation, and show, through the use of the classical Mehler formula (cf. [14]; we are indebted to Professor Szegö for communicating to us a simple expression for the Fourier transform of the square of a Hermite function, which however has turned out to be more than is required in the present article) that Shale's states are precisely the extreme points of the convex set of all universally invariant states.

Let $\mathfrak{H}$ be a complex Hilbert space. It has then in particular also the structure of a real Hilbert space, the real inner product being the real part of the imaginary one; as a real Hilbert space, $\mathfrak{H}$ will be denoted $\mathfrak{H}'$. Now on $\mathfrak{H}'$ there exists an isotropic centered normal distribution n_c with variance parameter k, and corresponding theory of integration and analysis, as treated in [8]. Let $\mathfrak{K}_k$ denote the Hilbert space $L_2(\mathfrak{H}', n_k)$ of square-integrable functionals on $\mathfrak{H}'$ with respect to n_k. Now for any z in $\mathfrak{H}$ and tame function f on $\mathfrak{H}'$, Shale defines an operator $W_0(z)$ on the domain $\mathfrak{D}$ of all square-integrable tame functions, as follows:

$$W_0(z) : f(u) \rightarrow f(u + z)e^{(i/2)\mathrm{Im}(u,z)}\exp [(\| u \|^2 - \| u + z \|^2)/(4k)].$$

It is easily seen that $W_0(z)$ is isometric on $\mathfrak{D}$ as an operator in $\mathfrak{K}_k$ (with the usual inner product) and leaves this manifold invariant, and that $W_0(-z)$ is inverse to $W_0(z)$. Therefore there exists a unique unitary transformation $W(z)$ defined on all of $\mathfrak{K}_k$ and extending $W_0(z)$. It is straightforward to check that

$$W_0(z)W_0(z') = W_0(z + z') \exp [(i/2)\mathrm{Im}((z, z'))],$$

for arbitrary z and z' in $\mathfrak{H}$, which implies the same relation for W. There is no difficulty in verifying further that $(W_0(z)f, g)$ is a continuous function of z, relative to any finite-dimensional range of variation for z, from which it follows that $W(z)$ is a weakly continuous function of z, as z ranges over an arbitrary finite-dimensional subspace. Thus W is a concrete Weyl system over $\mathfrak{H}$.

Now if U is a unitary transformation on $\mathfrak{H}$, it acts as an orthogonal transformation on $\mathfrak{H}'$. The transformation $f(z) \rightarrow f(U^{-1}z)$ on $\mathfrak{D}$ is therefore isometric and extends to a unitary transformation $\Gamma(U)$ on $\mathfrak{K}_k$. It is straightforward to check that

$$\Gamma(U)W(z)\Gamma(U)^{-1} = W(Uz).$$

Since the unit functional v which is identically one on $\mathfrak{H}$ is invariant under all the $\Gamma(U)$, it follows that the regular state E ($= E^k$ through the dependence on the variance parameter k) is a universally invariant state of the concrete Weyl algebra for W, and so determines a universally invariant regular state of the abstract Weyl algebra over $\mathfrak{H}$.

Shale then shows that, apart from the identity of E^k with $E^{k^{-1}}$, the E^k are distinct, by computing the generating functional

$$g_k(z) = E^k(W(z)).$$

For this purpose let $\mathfrak{H}_r$ be a real-linear subspace of $\mathfrak{H}$ such that $\mathfrak{H}'$ is the real direct sum of $\mathfrak{H}_r$ and $i\mathfrak{H}_r$. By the unitary invariance of $g_k(z)$ it is no essential loss of generality to take z in $\mathfrak{H}_r$, and to make the computation in a one-dimensional Hilbert space. Thus, setting $z = x$ and $u = s + it$, we obtain

$$g_k(z) = \iint e^{(i/2)xt} \exp [(2xs + x^2)/4k] \exp [-(s^2 + t^2)/2k](2\pi k)^{-1}\, ds\, dt$$

$$= \exp [-(1/8) \| z \|^2 (k + k^{-1})].$$

Thus different k's give distinct g_k's, except that k and k^{-1} give the same generating function. The states E^k, or equivalently, the associated unitary equivalence classes of the concrete field representations, are therefore all distinct, except as noted.

The possibility of making the foregoing construction is in a way, modulo the theory of integration in Hilbert space, not as surprising as the fact that the resulting states are really distinct. At very first glance it might even appear that a change of scale of an inessential sort might suffice to transform all the states into a single one. How far this is from being so is underlined by the fact that for $k = 1$, the field is of positive energy, irrespective of the single-particle Hamiltonian (assumed definite), while for any other value of k the field is not of positive energy, again for all single-particle Hamiltonians (provided they annihilate no normalizable vectors).

To identify these states with those of Theorem 1, we must compute the generating functionals of the latter. Utilizing the Schrödinger representation, and unitary invariance to replace $W(z)$ by the operation of multiplication by $\exp [i \| z \| 2^{-(1/2)} t]$, acting on the square-integrable functions of the real variable t, and recalling that $D^{(n)}$ is then the projection onto the one-dimensional subspace spanned by the n^{th} Hermite function, we find that

$$E_c(W(z)) = (1 - c) \sum_{n=0}^{\infty} c^n \int \exp [i \| z \| 2^{-(1/2)} t](h_n(t))^2\, dt.$$

Applying Mehler's formula

$$\textstyle\sum_n c^n(h_n(t))^2 = (2\pi(1 - c^2))^{-(1/2)} \exp [-t^2(1 - c)/2(1 + c)],$$

and making an easily justified interchange of the order of the summation, we obtain

$$E_c(W(z)) = \exp [-(1/4) \| z \|^2 (1 + c)/(1 - c)].$$

Thus the two classes of generating functionals, and hence universally invariant states are the same.

7. The holomorphic functional representation for the free field

It is clear from the uniqueness of the determination of a field from its generating functional together with the fact that the representation of a concrete C^*-algebra associated with a state E of the form

$$E(A) = (Av, v),$$

can be canonically transferred to the closure of the transforms of v under the algebra, that an alternative representation for the free field is one utilizing the W and Γ defined in Section 6, with representation space limited to the closed linear span of the $W(z)v$. It is interesting in a general way, as well as relevant to [15], that this span may be explicitly and simply identified in terms of "holomorphic functionals" on a complex Hilbert space. This notion has not yet been investigated in the mathematical literature, but Theorem 4 may be regarded as a possible starting point for such an investigation.[3]

By a polynomial on the complex Hilbert space $\mathfrak{H}$ is meant a function $p(\cdot)$ on $\mathfrak{H}$ such that $p(z)$ is expressible as a polynomial in the elementary sense in a finite number of inner products $(z, e_1), \cdots, (z, e_n)$, the e_i being fixed vectors in $\mathfrak{H}$. An anti-polynomial is defined as the complex conjugate of a polynomial. The representation under consideration can be effected canonically either on the anti-holomorphic functionals on $\mathfrak{H}$ (the completion of the anti-polynomials), or on the holomorphic functionals on $\mathfrak{H}^*$ (the completion of the polynomials on $\mathfrak{H}^*$). The correspondence between these two sets of functionals is the unique unitary one extending the assignment to any polynomial p on $\mathfrak{H}^*$, the functional (which is, in fact, an anti-polynomial) p^* defined as follows: if $u \in \mathfrak{H}$, and if u^* is the linear functional given by the equation $u^*(w) = (w, u)$, w being a generic element of $\mathfrak{H}$, then $p^*(u) = p(u^*)$. It will suffice to describe one of the two representations, and we take in Theorem 4 that whose representation space consists of anti-holomorphic functionals on $\mathfrak{H}$.

THEOREM 4. *The closed linear span of the $W(z)v$ in the foregoing representation is identical with the closure in $L_2(\mathfrak{H}')$ of the set of all anti-polynomials on $\mathfrak{H}$.*

Any anti-polynomial is the sum of a finite number of anti-monomials, i.e., functions $p(\cdot)$ of the form

$$p(z) = (e_1, z)^{n_1}(e_2, z)^{n_2} \cdots (e_k, z)^{n_k},$$

where the n_i are nonnegative integers. To show that the anti-polynomials are contained in the span $\mathfrak{S}$ of the $W(z)v$, it suffices to show that any such monomial is in $\mathfrak{S}$. Now it is no essential loss of generality, in this connection, to suppose the e_i mutually orthogonal, by elementary linear algebra; let us do this. By defining a function on $\mathfrak{H}$ as "based on the submanifold $\mathfrak{M}$" when it

[3] The present representation was described in another connection at the Summer Seminar in Applied Mathematics, held in Boulder in 1960, following which Professor Bargmann informed us that in the case of systems of a finite number of degrees of freedom he had independently studied aspects of the representation.

depends only on the projection of the variable onto $\mathfrak{M}$, it is clear that the anti-monomial p above is based on the manifold $\mathfrak{M}$ spanned by $e_1, \cdots, e_n$.

Now if f is a measurable functional on $\mathfrak{H}$ which is based on a finite-dimensional submanifold $\mathfrak{M}$, the restriction f' of f to $\mathfrak{M}$ is likewise measurable and has norm in $L_2(\mathfrak{M}')$ equal to the norm of f in $L_2(\mathfrak{H}')$. That p is in the span of the $W(z)v$ is implied a fortiori by its being in the span of the $W(z)v$ with z restricted to $\mathfrak{M}$, but the norm in $L_2(\mathfrak{H}')$ of the difference between p and any linear combination of the $W(z)v$ with z's in $\mathfrak{M}$ may be evaluated by integration over $\mathfrak{M}$, by virtue of the observation just made. Thus, for the point under consideration, it is enough to show that if $\mathfrak{H}$ is finite-dimensional, any anti-monomial over $\mathfrak{H}$ is in the span $\mathfrak{S}$ of the $W(z)v$. Now by direct computation, $W(z)v$, as a functional of the variable u ranging over $\mathfrak{H}$, now assumed to be finite-dimensional, is $e^{-(1/4)(z,z)}e^{-(1/2)(z,u)}$. The finite linear combinations of such functionals on $\mathfrak{H}$ evidently form an algebra, and in particular include, for any $\varepsilon > 0$,

$$[\varepsilon^{-1}(e^{\varepsilon(z,u)} - 1)]^n.$$

It is straightforward to show, from the Lebesgue convergence theorem, that

$$\int_{\mathfrak{H}} | [\varepsilon^{-1}(e^{\varepsilon(z,u)} - 1)]^n - (z, u)^n |^2 \, dn(u) \to 0$$

as $\varepsilon \to 0$. This shows that the anti-monomial $(z, u)^n$ is in $\mathfrak{S}$, and it is clear that the same argument applies to arbitrary anti-monomials.

It remains to show, conversely, that the $W(z)v$ are in the closure in $L_2(\mathfrak{H}', n)$ ($\mathfrak{H}$ being of arbitrary dimension) of the anti-monomials. Now $W(z)v$ is based on the one-dimensional manifold spanned by z. It is enough to approximate to it by anti-monomials based on this manifold, relative to this manifold, as above. Therefore it suffices to show that the power series expansion of $W(z)v$ (clearly analytic as a function of (z, u)) is convergent in $L_2(\mathfrak{M}', n)$, $\mathfrak{M}$ being the complex line. This follows without difficulty from the Lebesgue convergence theorem.

The simplicity of the representation thereby obtained may perhaps be seen more readily through a restatement of the theorem in slightly less allusive form. At the same time the representation space will be expressed in terms of holomorphic rather than anti-holomorphic functionals.

COROLLARY 4.1. *Let $\mathfrak{H}$ be a complex Hilbert space, $\mathfrak{H}^*$ its dual, n the centered normal distribution of unit variance on $\mathfrak{H}$ as a real space, and $\mathfrak{R}$ the closure in $L_2(\mathfrak{H}^*, n)$ of the set of all polynomials on $\mathfrak{H}^*$. Let $W(z)$, for arbitrary z in $\mathfrak{H}$, be the unitary operator on $\mathfrak{R}$ uniquely determined by the property that it operates as follows on polynomials $p(u^*)$ $(u^* \in \mathfrak{H}^*)$:*

$$p(u^*) \to p(u^* + z^*) \exp[-(1/4)(z, z) - (1/2)u^*(z)],$$

where z^* is the element of $\mathfrak{H}^*$ defined by the equation $z^*(u) = (u, z)$. Let $\Gamma(U)$, for U a unitary operator on $\mathfrak{H}$, be the unitary operator on $\mathfrak{R}$ uniquely determined *by the property that it acts as follows on polynomials:*

$$p(u^*) \rightarrow p(U^* u^*),$$

where U^* *is the contragredient transformation to* U. *Let* v^* *be the functional identically one on* $\mathfrak{H}^*$. *Then* $(\mathfrak{H}, W, \Gamma, v^*)$ *is the conventional free-field system over* $\mathfrak{H}$, *within unitary equivalence.*

Furthermore, the creation and annihilation operators $C(z)$ and $C(z)^*$ have simple forms in the present representation. It is straightforward to compute the restriction of $R(z)$ to the domain of polynomials, and compute therefrom the actions of $C(z)$ and $C(z)^*$ on the polynomials. Defining differentiation in the direction z as the unique derivation on the algebra of polynomials on $\mathfrak{H}^*$ taking $u^*(w)$, for any w in $\mathfrak{H}$, into (w, z), we obtain

The creation operator for a particle with wave function z *acts on polynomials* $p(u^*)$ *as multiplication by* $(-2)^{-(1/2)}(z, u)$, *while the annihilation operator acts as differentiation in the direction* z, *multiplied by* $-(-2)^{(1/2)}$.

This result makes it possible to define in a natural manner exponentials of the creation and annihilation operators, in fact an extensive class of entire functions of such operators, as closed densely defined operators in the Hilbert space of square-integrable holomorphic functionals. This development is not specially relevant to the present paper, and we shall not enter into it on this occasion.

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