CONGRUENCES FOR THE PARTITION FUNCTION TO COMPOSITE MODULI[1]

BY

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Let $p(n)$ denote the number of unrestricted partitions of the integer n, so that

$$\sum_{n=0}^{\infty} p(n)x^n = \phi(x)^{-1}, \qquad \phi(x) = \prod_{n=1}^{\infty} (1 - x^n).$$

In a recent article [3] the conjecture was made that for all integers $m \geqq 2$ each of the m congruences

$$p(n) \equiv r \pmod{m}, \qquad 0 \leqq r \leqq m - 1,$$

has infinitely many solutions in positive integers n; and a proof of this conjecture was given for $m = 2, 5, 13$. The principal object of this note is to prove that the conjecture holds for $m = 65$ as well. Certain related congruences will also be proved.

The writer's interest in these matters (which have their origin in the famous Ramanujan congruences) was first awakened by H. Rademacher, and this note is dedicated to him.

It is convenient to introduce some notation and to reproduce some known material here. If n is a nonnegative integer, define $p_r(n)$ as the coefficient of x^n in $\phi(x)^r$; otherwise define $p_r(n)$ as 0. Thus $p(n) = p_{-1}(n)$. Then it is known (see [4], [7]) that

$$p(13n + 6) \equiv 11p_{11}(n) \pmod{13}. \tag{1}$$

The author has shown in [5] that if r is odd, $1 \leqq r \leqq 23$, and p is a prime such that

$$r\nu = r(p^2 - 1)/24 \quad \text{is an integer,}$$

then for all integral n

$$p_r(np^2 + r\nu) - \gamma_n\, p_r(n) + p^{r-2}p_r((n - r\nu)/p^2) = 0, \tag{2}$$

where

$$\gamma_n = c - \chi(r\nu - n)p^{(r-3)/2}(-1)^{(p-1)(p-1-2r)/8},$$
$$c = p_r(r\nu) + \chi(r\nu)p^{(r-3)/2}(-1)^{(p-1)(p-1-2r)/8},$$

and χ is the Legendre-Jacobi quadratic reciprocity symbol modulo p.

It is easy to deduce from (2) that if

$$a_n = r(p^{2n} - 1)/24, \qquad t_n = p_r(a_n),$$

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then $t_0 = 1$, $t_1 = p_r(r\nu)$,

(3) $$t_{n+1} - ct_n + p^{r-2}t_{n-1} = 0, \qquad n \geqq 1.$$

We make the choice $r = p = 11$. Then $c = t_1 = p_{11}(55)$. The number $p_{11}(55) = 29580$ is given in [6], and $29580 \equiv 5 \pmod{13}$, $11^9 \equiv 8 \pmod{13}$. Then (1) and (3) together imply

LEMMA 1. *Define*

(4) $$T_n = p((11^{2n-1}\cdot 13 + 1)/24).$$

Then $T_0 = 0$, $T_1 = 11$, *and*

(5) $$T_{n+1} \equiv 5(T_n + T_{n-1}) \pmod{13}, \qquad n \geqq 1.$$

The author has also shown in [3] for the numbers defined by (4)

LEMMA 2. *The numbers* T_n *satisfy*

(6) $$T_n \equiv n \pmod{5}.$$

We must examine the sequence $\{T_n\}$ modulo 65. Clearly, (6) implies that

(7) $$T_{n+5} \equiv T_n \pmod{5}.$$

From the general theory of linear recurring sequences it follows that the period of the sequence T_n modulo 13 is a divisor of 168. It is easy to show theoretically that the actual period is 56, but we prefer to do this numerically, in elementary fashion. Thus Table I shows that

(8) $$T_{n+56} \equiv T_n \pmod{13},$$

so that

(9) $$T_{n+280} \equiv T_n \pmod{65}.$$

Therefore, in order to prove that each residue modulo 65 occurs infinitely often in the sequence $\{T_n\}$ it is only necessary to verify that each residue modulo 65 occurs *at least once* among the numbers T_n, $0 \leqq n \leqq 279$. If

$$T_n \equiv \alpha \pmod{5}, \qquad T_n \equiv \beta \pmod{13},$$

TABLE I

$T_n \pmod{13}$, $0 \leqq n \leqq 57$

	0	1	2	3	4	5	6	7	8	9
0	*0*	*11*	3	5	1	4	12	2	5	9
1	5	5	11	2	0	10	11	1	8	6
2	5	3	1	7	1	1	10	3	0	2
3	10	8	12	9	1	11	8	4	8	8
4	2	11	0	3	2	12	5	7	8	10
5	12	6	12	12	3	10	*0*	*11*		

TABLE II

A value of n such that $T_n \equiv r \pmod{65}$, $0 \leqq r \leqq 64$

	0	1	2	3	4	5	6	7	8	9
0	70	81	7	43	149	10	51	23	18	9
1	15	1	32	28	4	40	21	37	3	19
2	135	31	177	138	124	45	56	17	13	54
3	5	11	187	23	39	65	26	12	53	14
4	25	96	2	93	59	75	191	87	33	49
5	35	6	42	73	29	110	61	67	163	79
6	95	121	82	68	109					

then

$$T_n \equiv 26\alpha + 40\beta \pmod{65}. \tag{10}$$

Table I, (6), and (10) together may be used to compute T_n modulo 65, $0 \leqq n \leqq 279$. Instead of reproducing this lengthy table here, we give for each residue r modulo 65 a value of n such that

$$T_n \equiv r \pmod{65}, \qquad 0 \leqq n \leqq 279.$$

The results are contained in Table II

Summarizing, we have proved

THEOREM 1. *Each of the* 65 *congruences*

$$p(n) \equiv r \pmod{65}, \qquad 0 \leqq r \leqq 64$$

has infinitely many solutions in positive integers n.

A noteworthy observation is that for $n \geqq 2$, $T_n \equiv 0 \pmod{11^3}$. This is implied by the Ramanujan congruence modulo 11^3 (proved by Lehner [2]), since

$$24((11^{2n-1}\cdot 13 + 1)/24) - 1 = 11^{2n-1}\cdot 13 \equiv 0 \pmod{11^3}.$$

We go on now to some related results. Ramanujan, Watson, Lehner, and others have shown that if α, β are arbitrary positive integers and n satisfies

$$24n \equiv 1 \pmod{5^\alpha 7^\beta 11^3},$$

then

$$p(n) \equiv 0 \pmod{5^\alpha 7^{1+[\beta/2]} 11^3}. \tag{11}$$

Furthermore the author has shown in [5] that the congruence

$$p(84n^2 - (n^2 - 1)/24) \equiv 0 \pmod{13}, \qquad (n, 6) = 1, \tag{12}$$

is valid. Now

$$24(84n^2 - (n^2 - 1)/24) - 1 = 2015n^2;$$

and choosing n so that

$$(n, 6) = 1, \qquad n^2 \equiv 0 \pmod{5^{\alpha-1}7^{2(\beta-1)}11^3},$$

we obtain from (11) and (12)

THEOREM 2. *Let α, β be arbitrary positive integers. Then there are infinitely many positive integers n such that*

$$p(n) \equiv 0 \pmod{5^\alpha 7^\beta 11^3 \cdot 13}.$$

We now state without proof some further congruences which can be derived in similar fashion.

THEOREM 3. *Each of the congruences*

$$p(n) \equiv 0, 7, \cdots, 28 \pmod{5 \cdot 7},$$
$$p(n) \equiv 0, 11, \cdots, 44 \pmod{5 \cdot 11},$$
$$p(n) \equiv 0, 7, \cdots, 84 \pmod{7 \cdot 13},$$
$$p(n) \equiv 0, 11, \cdots, 132 \pmod{11 \cdot 13}$$

has infinitely many solutions in positive integers n.

It is also possible to derive congruences with moduli $5 \cdot 7 \cdot 13$, $5 \cdot 11 \cdot 13$, $7 \cdot 11 \cdot 13$.

It is of some interest to consider the congruence

$$(13) \qquad p(n) \equiv 0 \pmod 5$$

in fuller detail. When $n \equiv 4 \pmod 5$, the Ramanujan congruence modulo 5 implies that (13) is satisfied. Kolberg has shown in [1] that

$$(14) \qquad p(n+1) \equiv p_{23}(n) \pmod 5, \qquad n \equiv 0, 1 \pmod 5;$$

and a special case of a theorem of the author given in [5] is that

$$(15) \qquad p_{23}(an^2 + 23(n^2-1)/24) \equiv 0 \pmod 5, \qquad (n, 6) = 1,$$

provided that $p_{23}(a) \equiv 0 \pmod 5$ and $24a + 23$ is square-free.

Now for $a = 6$ we find that

$$p_{23}(6) = -16445 \equiv 0 \pmod 5,$$
$$24 \cdot 6 + 23 = 167 \quad \text{is square-free.}$$

Furthermore if $(n, 5) = 1$, then

$$6n^2 + 23(n^2-1)/24 \equiv 0, 1 \pmod 5.$$

Thus we can conclude from (14) and (15)

THEOREM 4. *If $(n, 30) = 1$, then*

$$(16) \qquad p((167n^2+1)/24) \equiv 0 \pmod 5.$$

Hence we have shown that congruence (13) is satisfied for infinitely many $n \equiv 1 \pmod 5$ and infinitely many $n \equiv 2 \pmod 5$. Whether the same is true for $n \equiv 0, 3 \pmod 5$ seems difficult and remains an open question.

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