FIXED-POINT THEOREMS FOR COMPACT CONVEX SETS

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1. Historical remarks

L. E. J. Brouwer [1] proved the well-known *Brouwer fixed-point theorem*. Let K be an n-cell, that is, a homeomorphic image of an n-dimensional cube. Let f be a continuous function from K into itself. Then there is a point P of K such that $f(P) = P$.

Schauder [9] extended the domain of validity of this theorem by demonstrating the *Schauder fixed-point theorem*. If K is a compact convex subset of a normed linear space X, and if f is a continuous transformation which carries K into itself, then there is at least one point P of K left fixed by f; that is, $f(P) = P$.

This was generalized next by Tyhonov [10] when he showed that Schauder's proof could be adapted to prove the existence of a fixed point even if X is a locally convex linear topological space instead of a normed space.

It is clear that if P is fixed under f, then it is also a fixed point of every power of f, $f^2 = f \circ f$, $f^3 = f \circ f^2$, and so on; that is, P is fixed under the smallest semigroup of operators on K which includes f. In the same way, P is a fixed point of every one of the functions f of a family F of functions from K into K, if and only if P is also a fixed point of every finite product, $\bigcirc_{i \leqq n} f_i$, of functions from F.

It is not known, *even for the one-dimensional case*, whether every two commuting continuous functions from K into K share a common fixed point (see the research problem of Isbell [6]). This adds to the interest and value of the generalization of a special case of Tyhonov's theorem due to Kakutani [7] and A. A. Markov [8].

KAKUTANI-MARKOV THEOREM. *Let K be a compact convex set in a locally convex linear topological space, and let F be a commuting family of continuous, affine transformations, f, of K into itself. Then there is a common fixed point of the functions in F; that is, there is an x in K such that $f(x) = x$ for every f in F.*

As in the Tyhonov theorem we observe that it costs nothing in this theorem to replace F by $\Sigma(F)$, the smallest semigroup of continuous, affine transformations of K into itself which contains F. In this case the commutativity of the family F is carried to the semigroup $\Sigma(F)$, so the theorem above is equivalent to that obtained by replacing the word "family" by "semigroup".

The property of commutativity is not shared by all semigroups. We discuss here another property which all commutative semigroups have and some other

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semigroups have, and we give a new proof of the Kakutani-Markov Theorem which extends to this wider class of semigroups.

THEOREM 1. *Let K be a compact convex subset of a locally convex linear topological space X, and let Σ be a semigroup (under functional composition) of continuous affine transformations of K into itself. If Σ, when regarded as an abstract semigroup, is amenable, or even if it has a left-invariant mean, then there is in K a common fixed point of the family Σ.*

Some generalizations of Theorem 1 are indicated in §4; in particular, see Theorem 3. §4 also contains an example to show that left-amenable cannot be replaced by right-amenable and a converse of Theorem 3.

2. Semigroups and invariant means

An *abstract semigroup* is a set of elements with an associative binary law of multiplication; a *semigroup of operators* or *semigroup of transformations* is a semigroup of transformations of some set K into itself in which the binary operation used is functional composition—that is, $[f \circ g](x) = f(g(x))$ for each x in K. $m(\Sigma)$ is the Banach space of all bounded, real-valued functions on Σ, with the least upper bound norm. Let e be that element of $m(\Sigma)$ for which $e(g) = 1$ for every g in Σ. A *mean on* Σ is an element μ of $m(\Sigma)^*$ such that

$$\| \mu \| = 1 = \mu(e).$$

(For general information and bibliography on means see my paper [3]; in particular, if μ is a mean and $f \in m(\Sigma)$, then $\text{glb} f(\Sigma) \leqq \mu(f) \leqq \text{lub} f(\Sigma)$.)

The *right* [or *left*] *regular representation of* Σ *over* $m(\Sigma)$ is the homomorphism ρ [or antihomomorphism λ] defined from Σ into the multiplicative semigroup of the algebra of bounded linear operators from $m(\Sigma)$ into itself by: For each h in Σ, ρ_h [or λ_h] is that linear operator defined by: For each f in $m(\Sigma)$ and each g in Σ

$$[\rho_h f](g) = f(gh) \qquad [\text{or } [\lambda_h f](g) = f(hg)].$$

A mean μ on Σ is called *right* [or *left*] *invariant* if for each f in $m(\Sigma)$ and each h in Σ

$$\mu(\rho_h f) = \mu(f) \qquad [\text{or } \mu(\lambda_h f) = \mu(f)].$$

μ is *invariant* if it is both right and left invariant. Σ is called *amenable* if there exists an invariant mean on Σ. If we express this in terms of adjoint operators of the linear operators ρ_h or λ_h, a mean μ is a right, or left, or two-sided, invariant mean if and only if μ is a fixed point of every ρ_g^*, or every λ_g^*, or both, respectively.

Note that the amenable *groups* are the groups called "messbar" in von Neumann's paper [11] on the theory of measure. Some recent Scandinavian writers (Følner [5]) call them "groups with full Banach mean value".

It is well known that the Kakutani-Markov theorem implies a general theorem about means on abelian semigroups; Banach proved it for some special abelian groups by an application of the Hahn-Banach theorem; von Neumann's paper referred to above has it for abelian and solvable groups; my 1942 paper [2] on ergodicity of abelian semigroups has a proof for the general case by means of the Hahn-Banach theorem, essentially using Banach's proof.

THEOREM 2. *Every Abelian semigroup is amenable.*

The proof from the Kakutani-Markov theorem is very brief.

3. Proof of the general fixed-point theorem, Theorem 1

Let y be any element of K, and define a linear mapping T from X^* into $m(\Sigma)$ by attaching to each φ in X^* the function $T\varphi$ on Σ such that $[T\varphi](g) = \varphi(g(y))$ for each g in Σ. Because φ is continuous, it is bounded on the compact set K, so $T\varphi$ is in $m(\Sigma)$ for each φ in X^*. It is clear that T is linear from X^* into $m(\Sigma)$; hence the function $T^\#$ dual to T can be constructed[1] to carry each element μ of $m(\Sigma)^*$ to an element of $X^{*\#}$: $[T^\#\mu](\varphi) = \mu(T\varphi)$ for each φ in X^*.

Let K' be the image of K under the canonical mapping Q of X into $X^{*\#}$: $[Qx](\varphi) = \varphi(x)$ for all φ in X^*. Let M be the set of means on Σ, that is, the positive face of the unit ball in $m(\Sigma)^*$.

LEMMA 1. *If $\mu \in M$, then $T^\#\mu \in K'$.*

Proof. If $\varphi \in X^*$, then

$$\begin{aligned}[T^\#\mu](\varphi) &= \mu(T\varphi) \leqq \sup\{T\varphi(g) : g \in \Sigma\}\\ &= \sup\{\varphi(gy) : g \in \Sigma\} \leqq \sup\{\varphi(x) : x \in K\} = \sup\{Qx(\varphi) : x \in K\}\\ &= \sup\{\tilde{x}(\varphi) : \tilde{x} \in K'\}.\end{aligned}$$

This says that $T^\#\mu$ is in every half space of $X^{*\#}$ which is determined by a φ in X^* and which is closed. By Mazur's theorem applied to $X^{*\#}$ in its weak* topology, K' is itself that intersection of half spaces; hence $T^\#\mu \in K'$.

From this lemma we know that if μ is in M, then $T^\#\mu$ is in QK, so $Q^{-1}T^\#\mu$ is in K. Let j be the mapping $Q^{-1}T^\#$ of M into K.

LEMMA 2. *If μ is in M and h is in Σ, then $j(\lambda_h{}^*(\mu)) = h(j(\mu))$.*

Proof. If $g \in \Sigma$, let $\bar{g}$ be the element of M defined by $\bar{g}(f) = f(g)$ for each f in $m(\Sigma)$. Then for each h in Σ

$$[\lambda_h{}^*\bar{g}](f) = \bar{g}(\lambda_h f) = [\lambda_h f](g) = f(hg) = [\overline{hg}](f);$$

that is,

$$\lambda_h{}^*\bar{g} = \overline{hg}.$$

Also for each φ in X^*

$$\varphi(j(\bar{g})) = \varphi(Q^{-1}T^\#\bar{g}) = [T^\#\bar{g}](\varphi) = \bar{g}(T\varphi) = [T\varphi](g) = \varphi(gy).$$

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[1] See my book [4] for notation and for basic results on linear spaces.

Hence

$$j\bar{g} = gy \qquad \text{if } g \text{ is in } \Sigma.$$

Because j is affine, for each finite mean $\tilde{\alpha} = \sum_g \alpha(g)\bar{g}$—that is to say, all $\alpha(g)$ are nonnegative, all but finitely many are zero, and $\sum_g \alpha(g) = 1$—it follows that

$$j\tilde{\alpha} = j(\textstyle\sum_g \alpha(g)\bar{g}) = \sum_g \alpha(g)j(\bar{g}) = \sum_g \alpha(g)g(y).$$

But

$$\begin{aligned} j(\lambda_h^*\tilde{\alpha}) &= j(\textstyle\sum_g \alpha(g)\lambda_h^*\bar{g}) = j(\sum_g \alpha(g)\overline{hg}) = \sum_g \alpha(g)j(\overline{hg}) \\ &= \textstyle\sum_g \alpha(g)h(g(y)) = \sum_g \alpha(g)h(j(\bar{g})) \\ &= h(\textstyle\sum_g \alpha(g)j(\bar{g})) \qquad \text{(because } h \text{ is affine)} \\ &= h(j\tilde{\alpha}). \end{aligned}$$

But each of the functions λ_h^*, h, and j is continuous in an appropriate sense; also the finite means are w^*-dense in M. Hence for each μ in M

$$\begin{aligned} j(\lambda_h^*(\mu)) &= j(\lambda_h^*(w^*\text{-lim}_\sigma \tilde{\alpha}_\sigma)) = j(w^*\text{-lim}_\sigma (\lambda_h^*\tilde{\alpha}_\sigma)) \\ &= \text{lim}_\sigma j(\lambda_h^*(\tilde{\alpha}_\sigma)) = \text{lim}_\sigma h(j(\tilde{\alpha}_\sigma)) \\ &= h(j(w^*\text{-lim}_\sigma \tilde{\alpha}_\sigma)) = h(j(\mu)). \end{aligned}$$

From the lemma we see at once that if μ is a left-invariant mean on Σ, then $j\mu$ is a common fixed point of all h in Σ. This is the desired conclusion for our theorem.

4. Generalizations: An example

(a) The proofs of the preceding section can be carried through in some cases when there is not a left-invariant mean on the whole of $m(\Sigma)$, because there are many situations in which a small subspace of $m(\Sigma)$ has appropriate properties and contains all of the functions $T\varphi : \varphi \in X^*$. Call a closed, linear subspace Y of $m(\Sigma)$ *left-invariant* if for each h in Σ and each y in Y, $\lambda_h y \in Y$; call a left-invariant Y *left-amenable* if $e \in Y$ and there is a left-invariant mean μ on Y: that is, there is an element μ of Y^* such that for each y in Y, glb $y(\Sigma) \leqq \mu(y) \leqq$ lub $y(\Sigma)$ and $\mu(\lambda_h y) = \mu(y)$ for each h in Σ.

With these assumptions on Y the proofs of Lemmas 1 and 2 go through unchanged. (The fact that finite means are w^*-dense in the means in Y^* depends on the fact that each mean in Y^* has an extension which is a mean in $m(\Sigma)^*$; this is a consequence of the Kreĭn monotone extension theorem; see [4, Chap. I, Sec. 6].)

(b) Suppose, for example, that Σ, which is set of functions from K into K, is given the topology of pointwise convergence. Then composition, the multiplication in Σ, is continuous in each variable (but not both at once), and each $T\varphi$ is continuous on Σ. Clearly $C(\Sigma)$, the space of bounded, continuous real-valued functions on Σ, is left-invariant. If $C(\Sigma)$ is also left-amenable, then there is in K a common fixed point of the elements of Σ.

(c) With this in mind let $A(K)$ be the semigroup (under composition) of all affine, continuous mappings of K into K, and topologize $A(K)$ with the topology of pointwise convergence. Let S be any semigroup with a topology in which multiplication is continuous in each variable, and let τ be a continuous homomorphism of S into $A(K)$. Applying the remarks above to $\Sigma = \tau(S)$ we obtain the following generalization of Theorem 1.

THEOREM 3. *If there is a left-invariant mean on $C(S)$, then for each compact convex set K in each locally convex space X and for each continuous homomorphism τ of S into $A(K)$, there is in K a common fixed point p of all the transformations in the set $\tau(S)$.*

It should be recalled that Haar measure defines a left-invariant mean on any compact group, so this theorem includes the cases where K is a discrete abelian semigroup or a compact group.

The condition of the theorem is sufficient as well as necessary. The adjoint representation λ_h^* of the left regular representation over $C(S)$ has a fixed point if and only if $C(S)$ is left-amenable. For S a discrete group this was first observed by Granirer.

(d) An example shows that left-amenable can not be replaced here by right-amenable. This is a consequence of the order in which composition is carried out. Let K be the unit interval $0 \leqq t \leqq 1$, and let $h_t(s) = t$ for $0 \leqq s \leqq 1$, $t = 0$ or 1. Setting $\Sigma = \{h_t : t = 0 \text{ or } 1\}$, we get a semigroup in which the product of each ordered pair of elements is the first element of the pair. Every ρ_h is the identity in $m(\Sigma)$; no mean is left-invariant on $m(\Sigma)$. Of course, there is no common fixed point of the transformations of the set Σ, because the ranges of the two transformations are disjoint.

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