PICARD BUNDLES

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On a nonsingular algebraic variety V of dimension n, a large maximal family of positive $(n - 1)$-cycles is naturally fibered by the linear systems into which these divisors may be grouped. The algebraic variety parametrizing these "fibers" is the Picard variety of V; it is a complete group variety. This is the construction of the Picard variety given by Matsusaka, following ideas of Chow; it is still the only construction which gives it *a priori* as a projective variety.

The relation between these maximal families and the Picard variety may be expected to shed light on both. For example, using the results here proved, we have shown that when V is a curve C, one is led to structural information about the rational equivalence ring of high symmetric products of C as well as to certain relations in the rational equivalence ring of its Jacobian. Again, in the classical case, Kodaira [2] has in this way studied the characteristic series of maximal families of divisors, while the projective character of Matsusaka's construction holds out some hope of studying the behavior of Picard varieties under specialization. We devote therefore the first part of this paper to showing that under the obvious geometric hypotheses, a complete maximal family of positive divisors is actually an algebraic projective bundle over the Picard variety. The essential point here is to prove the algebraic local triviality; that it is a bundle follows automatically, since we are dealing with projective bundles.

Once one has a projective bundle, one significant question is whether or not it has cross-sections. We show in the second part that these indeed exist, if the family of divisors is large enough. In this way one gets algebraic families of divisors parametrizing the Picard variety in a one-one way. Such families were constructed by Weil in the classical case [9], in order to show that analytic sets of divisors were mapped analytically into the Picard variety. We follow his ideas, which are basically algebraic, only taking some care to perform the construction as efficiently as possible.

For the special case when V is a curve C, the Picard bundle is the n-fold symmetric product $C(n)$, $n > 2g - 2$, and one can then obtain from the preceding proof the explicit estimation that $C(n) \to J$ has cross-sections if $n > 4g$. Thus high symmetric products contain Jacobians as subvarieties. Just how good the estimation is seems hard to say; in truth we would be happier with $n > 3g$, but do not seem to be able to get it. In any event, we prove in the last section that at least it does not always have cross-sections for all n; namely, if $n = 2g - 1$ and $g > 1$, it has in general none. This follows from a general criterion for nonexistence of cross-sections of projective

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bundles, together with (of all things) the fact that the exponential polynomial of degree greater than one has no rational roots.

PART I

1. Introduction and statement of result

Consider on a nonsingular projective variety V, defined over an algebraically closed ground field k, an open irreducible algebraic family $\{X\}$ of positive divisors, also over k. That is, the Chow variety C of $\{X\}$ should be a variety over k, open in its projective closure. By choosing an arbitrary but fixed k-rational divisor X_0 from $\{X\}$, we get a map $\pi: C \to P$, where P is the Picard variety of V. Namely, if $x \in C$ represents $X \in \{X\}$, define $\pi(x)$ to be the point u of P representing $\mathrm{Cl}(X - X_0)$. Then π is a single-valued, rational map over k, since if x is K-rational, $K \supset k$, so in turn are the divisor X, $\mathrm{Cl}(X - X_0)$, and u, by a property of the Picard variety. Moreover, the image W under π is a locally closed set in P: a closed set minus a closed subset.

We assume now that (C, W, π) is a sort of crude fiber space. Specifically, we assume

(1) W is a nonsingular variety.

(2) For each $u \in W$, the algebraic set $\pi^{-1}(u)$ represents a linear system of divisors, whose dimension r is independent of u.

(3) If u is a generic point of W, the variety $\pi^{-1}(u)$ is defined over $k(u)$.

Alternatively, since it is defined at worst over an inseparable extension of $k(u)$, we can suppose equivalently

(3′) The map π is separable.

Let now C' be the normalization of C, and $\pi': C' \to W$ the corresponding map. Then since π' is single-valued, it is regular by Zariski's Main Theorem.

THEOREM. *With the above assumptions, (C', W, π') is an algebraic projective bundle, with fiber the projective r-space.*

The most important case is $W = P$, and $\{X\}$ is all the divisors of a complete family, which by definition is a maximal family for which (1) and (2) hold, with $W = P$. For instance, if V is a curve, then $\{X\}$ would be all divisors of a fixed degree $n > 2g - 2$, and J the Jacobian. More generally, one can start with a maximal regular family of divisors, (that is, P is covered, the fibers being complete linear systems, but not necessarily all of the same dimension), and remove from P the locus over which the fibers are too large. In these cases hypothesis (3) is satisfied because, according to the Chow-Matsusaka construction, P is essentially the Chow variety of the linear systems of a complete family, so that u is actually the Chow point of the variety $\pi^{-1}(u)$, and one knows that a variety is defined over the field of its Chow point. Thus to drop hypothesis (3), one should presumably substitute the

Chow variety parametrizing the fibers for W. To show that it cannot in general be dropped, take V to be an elliptic curve over a field of characteristic p for which, in Weil's notation, $\nu(p\delta)_i = p^2$, and take $\{X\}$ to be the family of divisors of the form $p(u)$, as u runs over the points on V. Then for u generic, the divisor $p(u)$ (or its Chow point) is not rational over $k(\pi(p(u))) = k(pu)$ since $[k(u):k(pu)]_i = p^2$.

In the proof of this theorem, C' enters only at the last moment. We first construct enough local cross-sections to prove that our space C is essentially locally trivial (locally a product). The normalization C' is invoked only at the end to make it really a product.

(*Added in proof.* J. P. Murre has proved [Amer. J. Math., vol. 83 (1961), pp. 99–110] that if $\{X\}$ is a maximal, complete, and regular family of positive divisors on V, and if the embedding of V in its ambient projective space is sufficiently well-behaved, then C is actually nonsingular. Thus in this case, $C = C'$, and we can say then that (C, P, π) is a projective bundle. This was a significant point left unsettled here.)

2. Preliminary adjustments

Since the theorem has biregular character, we may assume that V is a nonsingular and projectively normal variety contained in a projective space S^N whose dimension N is large compared to the dimension r of the fibers, and that V is not properly contained in any hyperplane of S^N. We may also assume that any member of the algebraic family $\{X\}$ is contained in a hyperplane section of V. In fact, to realize these conditions, one first makes V into a projectively normal variety V' by a biregular transformation; $\{X\}$ carries over to V', and the corresponding Chow varieties are one-one birationally equivalent, so that their normalizations are biregularly equivalent. A generic divisor $X \in \{X\}$ is contained in a hypersurface section of sufficiently high degree m; by specialization so is every divisor of $\{X\}$. Making now the biregular transformation of V' into V'' which turns a maximal linearly independent set of hypersurface sections of degree m into the coordinate hyperplane sections, we see that V'' will still be nonsingular and projectively normal, and our conditions are satisfied. (Here m is chosen large enough so that $N(m) \gg r$, this being possible since $N(m)$ is the Hilbert characteristic function which goes to infinity with m.)

For a crucial step of the proof, we will also need the following information. The linear system $\mathfrak{L}_m$ of hypersurface sections on V' is complete, and so therefore are the residual systems $\mathfrak{L}_m - X$, for all $X \in \{X\}$. What is more, if m has been chosen large enough, they all have the same dimension; thus on V'', the systems $\mathfrak{L}_1 - X$ all have the same dimension.

This follows more or less from a result of Matsusaka [4]. It is easy to see that the union of the systems $\mathfrak{L}_m - X$ on V' is an irreducible algebraic family; Matsusaka proves then approximately the result we want, in the form:

If $\{Y\}$ is an algebraic family of positive divisors which is maximal and regular (covers P), then for sufficiently large n all the divisors $Y + nH$ have the same dimension (H is a hyperplane section). Now the assumption of maximality is trivially irrelevant; the regularity is not really used essentially in the proof (the hypothesis does not even carry over inductionwise in the induction proof Matsusaka gives).

A similar proof would start from the Riemann-Zariski theorem [10]: for all $m > m_0(X)$, we have

$$\dim | mH - X | = (-1)^n[p_a(V) + p_a(X - mH)] - 1.$$

Since algebraically equivalent divisors have the same arithmetic genus on a nonsingular variety (Matsusaka, [4]; this is the only place in the proof which requires V to be nonsingular, and not just normal), it is enough to show: If X is a generic member of $\{X\}$, then $m_0(X') = m_0(X)$ for all X' in some open set of the Chow variety. By analyzing Zariski's proof, this would follow from: If $\{Y\}$ is an irreducible family of positive cycles on V, then the Hilbert characteristic function $\chi(Y, m)$ gives the dimension of the linear system of hypersurface sections of degree m on Y for all $m \geqq m_1$, and all Y in some open subfamily—that is, m_1 does not depend on Y as long as Y stays in an open set. This however is an easy consequence of the inductive proof of the existence of the characteristic function; the point is that each of the hyperplanes used step-by-step in the induction to give, by intersection with a given Y, lower-dimensional cycles will also work for all Y in an open subfamily.

3. Construction of local cross-sections

For $u \in W$ we denote by $\mathfrak{L}_u$ the r-dimensional linear system of divisors represented by the points of $\pi^{-1}(u)$; then $\mathfrak{L}_u$ is naturally a projective space, and we refer to the space $\mathfrak{L}_u$ or the system $\mathfrak{L}_u$ according to the context.

Here is an easy way to construct approximate cross-sections of the crude fiber space (C, W, π): Choose r points $p_1, \cdots, p_r$ in general position on V, and put $\mathfrak{a} = p_1 + \cdots + p_r$. Then whenever for some given $u \in W$, there is only one divisor in the system $\mathfrak{L}_u$ passing through $\mathfrak{a}$, we will denote this divisor by $X[\mathfrak{a}, u]$; the totality of them (as u varies) defines a sort of partial cross-section, which we shall denote by $\chi[\mathfrak{a}]$, and we shall say that $\chi[\mathfrak{a}]$ is defined at $u \in W$ if $X[\mathfrak{a}, u]$ exists. We emphasize that these "cross-sections" are purely set-theoretic so far.

LEMMA. *Let u' be an arbitrary k-rational point of W. Then for suitable positive k-rational zero-cycles $\mathfrak{a}_0, \cdots, \mathfrak{a}_r$, the cross-sections $\chi[\mathfrak{a}_0], \cdots, \chi[\mathfrak{a}_r]$ will be defined over an open set containing u', and the particular divisors $X[\mathfrak{a}_0, u'], \cdots, X[\mathfrak{a}_r, u']$ will span the space $\mathfrak{L}_{u'}$.*

Proof. Since from an open set in projective space a basis can always be selected, it is enough to show that there is an open set in the space $\mathfrak{L}_{u'}$ through all the k-rational divisors of which a cross-section of the above type passes.

This breaks up into two parts: to find an open set each point of which represents a divisor of the form $X[\mathfrak{a}, u']$ for some k-rational $\mathfrak{a}$, and to show that the cross-section $\chi[\mathfrak{a}]$ through $X[\mathfrak{a}, u']$ is defined over some open set containing u'.

We observe first that if $p_1, \cdots, p_r$ are independent generic points of V, then $X[p_1 + \cdots + p_r, u']$ exists: there is one and only one divisor of the system $\mathfrak{L}_{u'}$ through the r points. For if one thinks of the system $\mathfrak{L}_{u'}$ as cut out on V (apart from a residual divisor) by an r-dimensional projective family of hyperplanes—and this is possible because V is projectively normal—there will be only one hyperplane through all the p_i: if for example the point p_{s+1} does not impose on these hyperplanes a condition independent of the conditions imposed by the preceding p_i, then p_{s+1} belongs to the $(N - (r - s))$-space defined over $k(p_1, \cdots, p_s)$ in which the hyperplanes through $p_1, \cdots, p_s$ intersect; thus V is also in this space, contradicting one of our assumptions about V.

Consider now the correspondence Z between $V \times V \times \cdots \times V$ (r factors) and the space $\mathfrak{L}_{u'}$ which associates with each divisor of the system $\mathfrak{L}_{u'}$ all ordered r-tuples of points on it. Thinking of the system $\mathfrak{L}_{u'}$ once more as cut out by hyperplanes, it is easy to see this correspondence is irreducible and thus $(p_1 \times \cdots \times p_r, X[p_1 + \cdots + p_r, u'])$ is its generic pair. Since for an arbitrary divisor X of the system $\mathfrak{L}_{u'}$, $Z^{-1}(X)$ is of dimension $r(\dim V - 1)$, by counting constants it follows that $Z[V \times \cdots \times V]$ is all of $\mathfrak{L}_{u'}$. It follows that $Z[U]$ is an open set on $\mathfrak{L}_{u'}$, where U is the open set on $V \times \cdots \times V$ consisting of all $q_1 \times \cdots \times q_r$ for which $Z(q_1 \times \cdots \times q_r)$ has dimension zero, that is, for which $X[q_1 + \cdots + q_r]$ exists.

We show now that any divisor uniquely determined as $X[q_1 + \cdots + q_r, u']$ for some k-rational points q_i actually is part of a cross-section $\chi[\sum q_i]$ defined over an open set containing u'. Let $\{X\}$ be the original algebraic family of divisors, represented by points on C, and $\{X[q_1 + \cdots + q_r]\}$ the closed subfamily consisting of all divisors through the zero-cycle $q_1 + \cdots + q_r$. Then $\mathfrak{L}_{u'} \cap \{X[q_1 + \cdots + q_r]\}$ consists of a single point on C; since $\mathfrak{L}_u \cap \{X[q_1 + \cdots + q_r]\}$ is evidently never empty, and since all the $\mathfrak{L}_u$ are represented by complete subvarieties of C, by one form of the dimension theorem [7, p. 36], it follows that for all u in some open set around u', $\mathfrak{L}_u \cap \{X[\sum q_i]\}$ will be of dimension zero, and since it is a linear system, it will then necessarily consist of a single point representing the divisor $X[q_1 + \cdots + q_r, u]$.

4. Local triviality

Let u be a generic point of W; then the cross-sections of the preceding lemma give us unique divisors $X_i = X[\mathfrak{a}_i, u]$ over u, for $i = 0, \cdots, r$. To prove the local triviality, we have to cut out these divisors by hyperplanes H_i in such a way that as $u \to u'$, our hyperplanes will be constrained to move along nicely to hyperplanes H'_i which will cut out $X'_i = X[\mathfrak{a}_i, u']$.

Consider therefore the $2r + 2$ linear systems of all hyperplanes through X_i, $i = 0, \cdots r$, and also X'_i. As noted at the end of Section 2, their dimension s is independent of i. Select a k-rational positive zero-cycle $\mathfrak{b}$ of degree s in such a way that, in each of these $2r + 2$ systems, there is only one hyperplane H_i or H'_i passing through $\mathfrak{b}$; this is clearly possible, for one has only to choose the s points one by one so as to avoid the base loci of a certain finite number of linear systems which we need not specify. This fixes H_i and H'_i therefore; let h_i, h'_i be the linear forms determined up to a constant factor defining H_i and H'_i, and $[h_i]$, $[h'_i]$ the hyperplane sections $H_i \cdot V$, $H'_i \cdot V$.

Now $[h_i] = X_i + Y_i$ clearly; we claim however that the residual divisor $Y = Y_i$ does not depend on i, and is in fact rational over $k(u)$. We have $X_0 + Y_i \sim X_0 + Y_0 = [h_0]$. But since the system of hyperplane sections is complete, $X_0 + Y_i = [h''_i]$, for some hyperplane section $[h''_i]$. By construction, Y_i passes through the zero-cycle $\mathfrak{b}$; therefore h''_i defines a hyperplane H''_i through X_0 and $\mathfrak{b}$; such a hyperplane is uniquely determined however as H_0, which shows that $Y_i = Y_0$.

As for the $k(u)$-rationality of Y, the family $\mathfrak{L}_u$ is defined by hypothesis as a variety over $k(u)$, and a generic divisor X_u on it with Chow point x is then rational over $k(x) = k(x, u)$. It is then easy to see from the last theorem in Weil's "Foundations" that since $\mathfrak{a}_0$ is k-rational, X_i will be $k(x)$-rational. Changing to another generic divisor X'_u with Chow point x' shows that X_i is also $k(x')$-rational; hence it is $k(x) \cap k(x') = k(u)$-rational. Thus the h_i, which are determined uniquely by X_i and $\mathfrak{b}$, are $k(u)$-rational; therefore Y is also $k(u)$-rational.

The same reasoning shows that $[h'_i] = X'_i + Y'$, where Y' is independent of i and of course k-rational.

Now when we specialize $u \to u'$, everything moves along in a well-controlled fashion. The specialization extends uniquely to $X_i \to X'_i$, since X_i must go into a divisor of $\mathfrak{L}_{u'}$ which passes through $\mathfrak{a}_i$, but there is only one. From there it extends uniquely to $H_i \to H'_i$, since H_i must go into a hyperplane through X'_i and through $\mathfrak{b}$. And finally, it extends uniquely to $Y \to Y'$ by virtue of the above relations.

Our object now is to attach an $(r + 1)$-dimensional vector space to each point of W which moves around "regularly" as the point moves on W. The basis for this space now at $u \in W$ will be a set of $k(u)$-rational linear forms $h_0, \cdots, h_{r+1}$ which define $H_0, \cdots, H_r$. Each of these of course is determined only up to a constant factor in $k(u)$, which we may choose so that the coefficients are all regular at the given initial point $u' \in W$. To show this, let $c_0(u)Z_0 + \cdots + c_N(u)Z_N$ be one of the forms, say defining H_i. Then we can divide by a suitable $c_j(u)$, so that the resulting form h_i has a finite specialization h'_i over $u \to u'$ which therefore must be the form associated with H'_i; since $H_i \to H'_i$ is the unique extension of $u \to u'$, so is $h_i \to h'_i$ unique. The normality of $u' \in W$ shows that the coefficients of h_i must be regular at u', by Zariski's Main Theorem.

We can now prove that the map of our theorem is locally trivial. The forms $h_0, \cdots, h_r$ that we have just described have coefficients which are regular in some neighborhood U' of u'; moreover since their specializations $h_0', \cdots h_r'$ are linearly independent (the divisors X_i' forming a projective basis for $\mathfrak{L}_{u'}$), it is clear there is a neighborhood, which we may suppose is U', in which they continue to be independent. Let now U be the intersection of U' with the neighborhoods of u' over which the cross-sections $\chi[\mathfrak{a}_i]$ were all defined.

We define a regular, one-one, fiber-preserving map

$$f: U \times S^r \rightarrow \pi^{-1}(U).$$

Let $t_0, \cdots, t_r$ be independent transcendentals over $k(u)$, u generic, homogeneous coordinates for S^r. Let $H_{t,u}$ be the hyperplane $t_0 h_0 + \cdots + t_r h_r = 0$, rational over $k((t), u)$, where (t) denotes the ratios t_i/t_j. Then

$$H_{t,u} \cdot V = X_{t,u} + Y, \qquad X_{t,u} \in \mathfrak{L}_u,$$

and $X_{t,u}$ is $k((t), u)$-rational (since $H_{t,u}$ and Y are), hence is represented by a point $f(u, t)$ of C rational over $k((t), u)$. This defines the map f.

First, f is regular: any specialization $(u, t) \rightarrow (u'', t'')$ where $(u'', t'') \in U \times S^r$ extends uniquely to $X_{t,u}$ we claim. Clearly the given specialization extends uniquely to $H_{u,t} \rightarrow H_{u'',t''}$, defined by

$$t_0'' h_0(u'') + \cdots + t_r'' h_r(u'') = 0$$

(the latter not being identically zero since the $h_i(u)$ are independent all over U), and it is also extends uniquely to $Y \rightarrow Y''$ since $H_0 \rightarrow H_0''$, $H_0 \cdot V \rightarrow H_0'' \cdot V$, $X_0 = X[\mathfrak{a}_0, u] \rightarrow X[\mathfrak{a}_0, u'']$ are all well-determined and $H_0 \cdot V = X_0 + Y$, $H_0'' \cdot V = X[\mathfrak{a}_0, u''] + Y''$. Therefore by subtraction it extends uniquely to $X_{t,u}$; we call the image $X_{t'',u''}$. This shows the map f is single-valued on $U \times S^r$, so that by Zariski's Main Theorem it is regular.

Next, f is a fiber-preserving map. For $\pi f(u, t) = \pi(X_{u,t}) = u$, since $X_{u,t} \sim X_0$. And f is one-one on each fiber $u'' \times S$: for if $H_{u'',t'} \neq H_{u'',t''}$, then $H_{u'',t'} \cdot V \neq H_{u'',t''} \cdot V$ since V is not contained in any hyperplane; $u'' \times S$ and $\pi^{-1}(u'')$ are complete varieties of the same dimension, so f is onto.

Finally, f^{-1} is a rational map. Let the field of definition for the point representing $X_{t,u}$ be $K \supset k$. Then $u = \pi(X_{t,u})$ is K-rational, so $K \supset k(u)$. Also, $X_{t,u}$ is K-rational [7, p. 104], so that $X_{t,u} + Y = H_{t,u} \cdot V$ is K-rational. But $X_{t,u} \sim X_0$ which is $k(u)$-rational, so by the last theorem in Weil's "Foundations," $X_{t,u} - X_0 = (\phi)$, where ϕ is a function on V rational over K. However, the vector-space of functions $> -X_0 - Y$ has as basis $1, h_1/h_0, \cdots, h_r/h_0$ which are all $k(u)$-rational. Therefore $\phi = t_0' + \cdots + t_r' h_r/h_0$ where the t_i' are K-rational. But $X_{t,u} - X_0 = [t_0 + \cdots + t_r h_r/h_0]$ we know, so that $t_i = t_i'$, and thus $H_{t,u}$ is K-rational.

Since now its minimal field of definition containing $k(u)$ is clearly $k(u, (t))$, this shows $K \supset k(u, (t))$, so that f^{-1} is a rational map.

To complete the proof of local triviality, we remark that since f is a regular map of a normal variety into $\pi^{-1}(U)$, it factors through the normalization of the image, and this is just $\pi'^{-1}(U)$. The new map

$$f': U \times S^r \to \pi'^{-1}(U)$$

is evidently regular, birational, one-one, and fiber-preserving. By Zariski's Main Theorem, it is then also biregular.

5. Completion of the proof: Projective bundles

To finish now, we have the general result:

THEOREM. *If $\pi: Y \to X$ is a locally trivial map, with X nonsingular and fiber the projective space S^n, then (Y, X, π) is an algebraic projective bundle.*

Proof. We choose homogeneous coordinates $(t_0, \cdots, t_n)$ in S^n, and let $e_i = (0, \cdots, 1_i, \cdots, 0)$ as usual, $e_{n+1} = (1, \cdots, 1)$, be the $n + 2$ reference points. For any choice of points $d_0, \cdots, d_{n+1}$ such that the first $n + 1$ points span the whole space, there is one and only one $\sigma \in PGL(n)$ for which $\sigma(e_i) = d_i$, $i = 0, \cdots, n + 1$. By representing the e_i as row vectors, then in coordinates σ is represented by the unique (up to a constant multiple) matrix whose i^{th} row represents $\sigma(e_i)$, and the sum of whose rows represents $\sigma(e_{n+1})$. If σ is defined over a field K, then since the $\sigma(e_i)$ are all K-rational points, it is immediately seen from the theory of linear equations that the matrix coordinates for σ will be K-rational (in the projective sense, that is, after adjustment by a suitable constant factor, the entries will be K-rational). When we refer to $PGL(n)$, it will be to this matrix representation.

Let now U_1 and U_2 be two small open sets on X, $U_{12} = U_1 \cap U_2$, and $(\pi, \phi_i): \pi^{-1}(U_i) \to U_i \times S$ the biregular fiber-preserving maps giving the local triviality of π over U_1 and U_2. Then the map

$$f = (\pi, \phi_1)(\pi, \phi_2)^{-1}: U_{12} \times S \to U_{12} \times S$$

s biregular and fiber-preserving; thus it is biregular on each fiber and so can be written $f(x, y) = (x, s(x)y)$ where $s(x)$ is some element of $PGL(n)$ for each $x \in U_{12}$. Here we have used that the elements of $PGL(n)$ are the only biregular maps of projective space onto itself. The compatibility conditions for the maps $s(x)$ derived from different U_{12} being automatically fulfilled, we have to show simply that s is a regular map of U_{12} into $PGL(n)$.

Let x be a generic point of U_{12} over some field of definition k for everything. Then $s(x)$ is easily seen to be a $k(x)$-rational automorphism and is therefore by the above represented by a $k(x)$-rational point of $PGL(n)$. By Zariski's Main Theorem it will thus be enough to show that as $x \to x'$, $x' \in U_{12}$, the specialization extends uniquely to $(\alpha) \to (\alpha')$, (α) and (α') being the matrices representing $s(x)$ and $s(x')$ respectively. First of all, since f is regular, for any (x, y) in $U_{12} \times S$, the specialization $(x, y) \to (x', y)$ extends uniquely to $f(x, y) \to f(x', y)$, that is, to $s(x)y \to s(x')y$, in particular therefore uniquely to $s(x)e_i \to s(x')e_i$. Now after adjusting the matrix (α) by a constant factor, the specialization extends to $(\alpha) \to (\alpha'')$. But the rows of (α''), being specializations of the corresponding rows of (α), must represent then the points $s(x')e_i$, and similarly the sum of the rows must be $s(x')e_{n+1}$. It follows that indeed (α'') is the matrix representing $s(x')$, that is, $(\alpha'') = (\alpha')$. This completes the proof.

Remark. The theorem is also valid for affine and vector bundles: if the fiber is affine space V^n, then local triviality gives an algebraic bundle with the affine group as structural group; if further the local triviality maps (π, ϕ_i) match up so as to give a cross-section, that is, if $f(x, 0) = (x, 0)$, i.e., if $s(x) \in GL(n)$, then one gets a vector bundle with $GL(n)$ as structural group. The proof is the same, the essential point being that these are the full group of automorphisms of the fibers and that K-rational automorphisms are represented in them by K-rational points.

PART II. CROSS-SECTIONS

6. Poincaré divisors

For the convenience of the reader, we recall that if V is a normal variety and P is its Picard variety, then a Poincaré divisor D on $P \times V$ is one for which the associated family of divisors $\{D(u)\}$ on V runs over in an approximately one-one way the linear equivalence classes into which the divisors algebraically equivalent to one of the $D(u)$ fall. Precisely [3, p. 114]

1. For every $u' \in P$ and point $u \in P$ generic over $k(u')$, the map $\phi: u' \to \mathrm{Cl}[D(u + u') - D(u)]$ is an isomorphism of P onto the Picard group of divisor classes algebraically equivalent to zero on V, the map being independent of u by the theorem of the square.
2. If $\phi(u')$ is a class rational over $K \supset k$, then u' is K-rational also.

If $D_1 \sim D + X_1 \times V + P \times X_2$ for some divisors X_1 and X_2, then D_1 is also a Poincaré divisor.

If u_0 is a fixed rational point of P, the map $\phi_0: u' \to \mathrm{Cl}[D(u' + u_0) - D(u_0)]$ is almost an isomorphism onto the Picard group; but in general it is not defined everywhere. This is what we shall now fix up, by a simple method due to A. Weil [9].

THEOREM. *On $P \times V$ there exists a Poincaré divisor $D > 0$ such that $D(u)$ is defined for all $u \in P$.*

Proof. Assume that P and V are imbedded respectively in projective spaces S^M and S^N, that neither is contained in any hyperplane of the space, and assume also that $N \geqq q = \dim P$.

By means of the Segre imbedding [7, p. 21], we view $P \times V$ as a variety in $S^{(M+1)(N+1)-1}$, given by the homogeneous generic point $(\cdots, u_i x_j, \cdots)$, where (u) and (x) are homogeneous generic points of P and V respectively. The linear system $\mathfrak{L}$ of hyperplane sections on $P \times V$ contains divisors of the form $H_1 \times V + P \times H_2$ where H_i are hyperplane sections on the respective varieties: for say $u_0 x_0 = 0$ only if either $u_0 = 0$ or $x_0 = 0$. The dimension of the system $\mathfrak{L}$ is moreover $(M+1)(N+1) - 1$ since it is easily checked that no hyperplane of the ambient space contains $P \times V$, neither V nor P being contained in a hyperplane.

Fix now a point $u' \in P$; then the dimension of the linear subsystem of $\mathfrak{L}$ consisting of the hyperplane sections containing $u' \times V$ is immediately seen to be $M(N+1) - 1$. It follows that for each $u' \in P$ there is a complementary linear subsystem $\mathfrak{L}[u']$ of $\mathfrak{L}$, all the hyperplanes of which intersect $u' \times V$ properly, and this system $\mathfrak{L}[u']$ has dimension N.

Now let D_0 be a Poincaré divisor on $P \times V$. From what we have said about a hyperplane section $H \in \mathfrak{L}$ of $P \times V$, it follows that for any n, any divisor of $| D_0 + nH |$ is also a Poincaré divisor. *Assume now that D_1 is a Poincaré divisor whose linear system $| D_1 |$ contains for any $u' \in P$ a divisor $D_{u'}$ intersecting $u' \times V$ properly.* For example, take the system $| D_0 + nH |$ above: If n is large, this system is effective, and its base locus is contained in $P \times W$, where W is the singular locus of V, so such $D_{u'}$ certainly exist in it. Then under these assumptions, we claim that *the system $| D_1 + H |$ contains divisors D for which $D(u')$ is always defined*, that is, which intersect all the $u' \times V$ properly.

Namely, for each u', $| D_1 + H |$ contains the N-dimensional subsystem $D_{u'} + \mathfrak{L}[u']$, no divisor of which contains $u' \times V$. By hypothesis, $N \geqq q = \dim P$. Thus the subsystem $\mathfrak{L}'[u']$ consisting of all divisors of $| D_1 + H |$ which *do* contain $u' \times V$ is of codimension $\geqq q + 1$. The assigning of $\mathfrak{L}'[u']$ to the point u' gives an algebraic correspondence between P and $| D_1 + H |$ whose image in $| D_1 + H |$ must have codimension at least one, by the preceding, and is therefore not all of $| D_1 + H |$. This image is the set-theoretic union of all the $\mathfrak{L}'[u']$; thus any $D \in | D_1 + H |$ not in this union intersects all $u' \times V$ properly.

7. Cross-sections of Picard bundles

Once again we recall that a regular family of divisors $\{X\}$ is one mapping onto the Picard variety; by [4], the maximal family $\{X + nH\}$ will be complete, so that the normalization of its Chow variety will be a bundle over the Picard variety.

THEOREM. *Let $\{X\}$ be a regular family of divisors, and H_1 a hyperplane section of the nonsingular variety V. For large n, the Picard bundle associated with the complete family $\{X + nH_1\}$ has regular cross-sections.*

Proof. By hypothesis, the Chow variety of $\{X\}$ is mapped onto P; for u generic, the linear system $\pi^{-1}(u)$ of dimension r say is rational over $k(u)$, or at least will be if one adds nH_1 to X, which we can suppose has been done. Then there is only one divisor X_u of the system through r generally chosen rational points on V, and it is $k(u)$-rational. There is then a unique minimal positive rational divisor D' on $P \times V$ such that $D'(u) = X_u$, and D' is in fact easily seen to be a Poincaré divisor (most easily by the "generic" form of the conditions stated in Section 6; see [3, p. 116]).

According to the proof of the preceding theorem, if n is large, we can find a positive divisor $D \sim D' + nH$, also a Poincaré divisor, such that all the $D(u')$ are defined, $u' \epsilon P$. The divisors $D(u')$ all belong to the maximal family $\{X + nH_1\}$, since $H \sim H_2 \times V + P \times H_1$ on the Segre product $P \times V$. If now n is also large enough to make $\{X + nH_1\}$ a complete family, we claim that the family $D(u)$, as u varies, defines a regular cross-section of the Picard bundle associated with $\{X + nH_1\}$.

Let therefore, with the notations of Part I, $\pi':C' \to P$ be this Picard bundle, with $D(0)$ being chosen as reference divisor, so that $\pi'(D(0)) = 0$. We define $f:P \to C$ by setting $f(u)$ equal to the Chow point $c(D(u))$ of the divisor $D(u)$; since $D(u)$ is $k(u)$-rational, f is a rational map. Since $D(u) \to D(u')$ is the unique cycle specialization extending $u \to u'$ (compatibility of specialization with intersection), $c(D(u)) \to c(D(u'))$ is the unique extension of $u \to u'$, which shows that f is single-valued. Thus the associated map $f':P \to C'$ is finite-valued, hence regular by Zariski's Main Theorem. Finally, f' is a cross-section, since for all $u \epsilon P$, $\pi'f'(u) = \pi f(u) = \pi(c(D(u))) = u$.

8. Cross-sections of Jacobian bundles

When the ambient variety is a nonsingular curve C, the Picard bundle is nothing but the map $\pi:C(n) \to J$, where $C(n)$ is the n-fold symmetric product, and $n > 2g - 2$. In this case, we obtain the following estimate on the existence of a cross-section:

THEOREM. *If $g > 1$, then $\pi:C(n) \to J$ has regular cross-sections if $n > 4g$ (if $g = 1$, trivially when $n \geqq 1$).*

Proof. Let $\Theta = W_1$ be the usual divisor on J, and think of C as imbedded in its Jacobian. Then there is a unique minimal positive divisor D_a on $J \times J$ defined by $D_a(u) = \Theta_{u+a}$, $a \epsilon J$; it is a Poincaré divisor, and it exhibits J as its own Picard variety since a given divisor of J algebraically equivalent to zero is linearly equivalent to one and only one divisor of the form $\Theta_{u'} - \Theta$. It follows therefore that if $(i \times \phi):J \times C \to J \times J$ is the injection map (thought of here as the identity inclusion), then the divisor $Z_a = (i \times \phi)^{-1}(D_a)$ on $J \times C$ defined by $Z_a(u) = \Theta_{u+a}\cdot C$ is a Poincaré divisor exhibiting J as the Picard variety of C.

We note that if x is a generic point of C, then

$$^tZ_a(x) = (\Theta^-)_{x-a}\,.$$

For this, it is enough to prove that $^tD_a(v) = (\Theta^-)_{v-a}$ for any $v \epsilon J$ where it is defined, since by intersection theory, $^tZ_a(x) = {}^tD_a(x)$. But $D_a = f^{-1}(\Theta)$, where f is the map of $J \times J \to J$ defined by $f(u, v) = -u + v - a$. Therefore by the projection formula,

$$f({}^tD_a(v) \times v) = f(D_a \cdot (J \times v)) = \Theta \cdot f(J \times v) = \Theta,$$

and taking account of what f is when restricted to $J \times v$, this proves that ${}^tD_a(v) = (\Theta^-)_{v-a}$ as asserted.

We claim now that on $J \times C$, the linear system of Poincaré divisors $| Z_0 + \Theta^- \times C + J \times \mathfrak{a} |$, where $\mathfrak{a}$ is a positive divisor on C of degree g, is large enough so that for every $u' \in J$, the system will contain a positive divisor intersecting $u' \times C$ properly.

Namely, for generic x on C, we have by our preceding formula,

$${}^t(Z_0 + \Theta^- \times C)(x) = (\Theta^-)_x + \Theta^-,$$

$${}^t(Z_a + (\Theta^-)_a \times C)(x) = (\Theta^-)_{x-a} + (\Theta^-)_a .$$

By the theorem of the square, we have on J

$$(\Theta^-)_x + \Theta^- \sim (\Theta^-)_{x-a} + (\Theta^-)_a ,$$

so that by the see-saw principle, for some divisor $\mathfrak{b}_a$ on C, we have

$$Z_0 + \Theta^- \times C \sim Z_a + (\Theta^-)_a \times C + J \times \mathfrak{b}_a .$$

Now if we intersect both sides with $u \times C$, u generic on J, we get on C, $\Theta_u \cdot C \sim \Theta_{u+a} \cdot C + \mathfrak{b}_a$, which shows that $\mathfrak{b}_a$ is a divisor of degree zero on C. Therefore $\mathfrak{b} + \mathfrak{a}$ is of degree g, and hence $\mathfrak{b}_a + \mathfrak{a} \sim \mathfrak{a}_a$ where $\mathfrak{a}_a$ is also of degree g and positive. In sum, adding $J \times \mathfrak{a}$ to both sides of the above gives

$$Z_0 + \Theta^- \times C + J \times \mathfrak{a} \sim Z_a + (\Theta^-)_a \times C + J \times \mathfrak{a}_a .$$

But this proves our contention because the divisor on the right is positive and intersects $u' \times C$ properly if a is chosen reasonably—namely if a is chosen so that $C \not\subset \Theta_{u'+a}$, and so that $u' \notin (\Theta^-)_a$; any a in a certain open set will thus do.

This being established, according to the italicized portion of the proof of the theorem of Section 6, by adding a hyperplane section H of $J \times C$ to this linear system of Poincaré divisors, we obtain another system containing a Poincaré divisor Z' for which all the $Z'(u')$ are defined, and which therefore gives a cross-section. We therefore now need only calculate the degree of the positive divisors $Z'(u')$ giving this cross-section.

$$\begin{aligned} \deg Z'(u) &= \deg (Z_0 + J \times \mathfrak{a} + H) \cdot (u \times C) \\ &= \deg (\Theta_u \cdot C) + \deg \mathfrak{a} + \deg H_1 \\ &= 2g + \deg H_1 \end{aligned}$$

by a known intersection formula [3, pp. 39, 93], and where H_1 is a hyperplane section of C (since $H \sim J \times H_1 + H_2 \times C$ by a previous remark). There was a catch however; C must be nonsingularly imbedded in projective r-space, where $r \geqq g$, and not contained in any hyperplane. With these restrictions, how small can we make the order of C, which is just deg H_1 ? Any complete linear system of degree $2g + 1$ gives a nonsingular biregular imbedding of C into projective $(g + 1)$-space; this we have proved elsewhere [6]. For genus 2, this is the best one can do; for genus three and higher, a more careful analysis should presumably produce a linear system of degree $2g$ giving an imbedding into g-space. In any event, this gives $\deg Z(u) = 4g + 1$, which is our theorem.

9. A criterion for nonexistence of cross-sections of projective bundles

In these two last sections, we add a little to the preceding (and amuse ourselves) by showing that since the exponential polynomial has no rational roots, in general the smallest bundle over the whole Jacobian, namely $\pi: C(2g - 1) \to J$, has no cross-sections.

Let X be quasi-projective, nonsingular, and let (E, X, π') be an algebraic vector bundle of rank p; from it we derive canonically a projective bundle $(P(E), X, \pi)$ whose fibers $\pi^{-1}(x)$ are the $(p - 1)$-dimensional projective spaces whose points represent the lines through the origin in the p-dimensional vector spaces $\pi'^{-1}(x)$. Conversely, given such an algebraic projective bundle, it is known that it is so derived from a vector bundle, and if E_0 is one such bundle, then it is also derived exactly from the bundles $E_0 \otimes L$, where L is any line bundle over X. [1, 8].

We give now the projective form of a result of Grothendieck [1].

Cross-section Criterion. *Let (F, X, π) be an algebraic projective bundle, with fibers of dimension $p - 1$. A necessary condition for it to have a cross-section is that for some divisor class α in X, we have*

$$\alpha^p + c_1 \alpha^{p-1} + \cdots + c_p = 0,$$

where $1 + c_1 + \cdots + c_p$ is the total Chern class of some arbitrarily chosen rank p vector bundle E from which F is derived.

Proof. We let S^{p-1} be the projective space, V^p the vector space. Let F be given by an open covering $\{U_i\}$ of X and transition functions $g_{ij}: U_i \cap U_j \to PGL(p - 1)$ which are regular maps obeying the usual compatibility condition. We may then suppose that the same open covering will do for E; in other words, E is defined by regular transition maps $g'_{ij}: U_i \cap U_j \to GL(p)$ into the general linear group; to say then that F is derived from E means that, after suitably adjusting the functions, we can suppose that $g_{ij} = \phi g'_{ij}$ where $\phi: GL(p) \to PGL(p - 1)$ is the canonical homomorphism.

Let $\{f_i\}$ be a cross-section of F, that is, a collection of regular maps $x \to (x, f_i(x))$ of $U_i \to U_i \times S$ such that $f_i = g_{ij} f_j$. If the U_i are small enough, which we can assume, we can lift each of these maps f_i by finding maps $f'_i: U_i \to V - \{0\}$ such that $f_i = \psi f'_i$, where $\psi: V - \{0\} \to S$ is the canonical map. The $\{f'_i\}$ form a never-zero cross-section of some vector bundle F_0 having E as derived projective bundle; namely, for $x \in U_i \cap U_j$, since $f_i(x) = g_{ij}(x)f_j(x)$, it follows that $f'_i = \alpha_{ij}\, g'_{ij}\, f'_j$ where α_{ij} is a regular map of $U_i \cap U_j \to V^1 - \{0\}$, that is, a nonvanishing regular function on $U_i \cap U_j$. Since the α_{ij} commute with the elements of $GL(p)$, it follows immediately that they satisfy the compatibility condition, hence define a line bundle L over X, and $E \otimes L$ has $\alpha_{ij}\, g'_{ij}$ as transition functions, $\{f'_i\}$ as nonvanishing regular cross-section, and F as derived bundle.

Let $1 + d_1 + \cdots + d_p$ in the rational equivalence ring of X be the total Chern class of $E \otimes L$. By a result of Grothendieck, if a vector bundle of rank p has a regular cross-section which intersects the zero cross-section transversally, then the projection of this cross-section onto X is just the p^{th} Chern class of the bundle. It follows in our case that $d_p = 0$, since the section $\{f_i\}$ is never zero. Now if $1 + \alpha$ is the total Chern class of L, and $1 + c_1 + \cdots + c_p$ the Chern class of E, we have explicitly

$$0 = d_p = \alpha^p + c_1\, \alpha^{p-1} + \cdots + c_p$$

by the usual formula for the Chern class of a tensor product.

10. An application

It is "undoubtedly" true that in general the Picard number of a Jacobian is one, at least in the classical case. Since there is an injection of the Picard group of J into the group of endomorphisms of J, it would suffice to show that in general a Jacobian has only the trivial complex multiplications. In this form, a proof might be extractable from say the work of Baily which furnishes a variety of moduli for Jacobians in the classical case. In characteristic p the group of endomorphisms is larger, due to the presence of the Frobenius endomorphism, but the result could still be true. At any rate,

THEOREM. *If the Picard number of J is one, then the bundle $\pi: C(2g-1) \to J$ has no cross-section* $(g > 1)$.

Proof. We know that J always has the divisor $W_1 = \Theta$ not algebraically equivalent to zero; if the Picard number of J is one, this means that any divisor X on J must be numerically equivalent to zW_1, where z is some rational number.

We have shown elsewhere [6] that there is a vector bundle (E, X, π') from which $(C(2g-1), J, \pi)$ is derived, and its Chern classes are $(-1)^i W_i^*$; here $W_i = W^{g-i}$ is as usual the subvariety of J obtained as the locus of $x_1 + \cdots + x_{g-i}$ on J, where the x_i are independent generic points of C, and * is the result of applying the map $u \to -u + c$ to J (c = canonical point). Let w_i and w_i^* denote the rational equivalence classes of W_i and W_i^*. Then by an intersection formula of Matsusaka-Weil [5],

$$W_1 \cdot \cdots \cdot W_1 \text{ (m factors)} \equiv m!\, W_m \quad \text{(mod numerical equivalence)}.$$

Thus since the biregular map * preserves intersection multiplicities, if we apply it to both sides and observe that $W_1 = W_1^*$, we get that $W_m \equiv W_m^*$.

From the criterion of Section 9 therefore, if the bundle has a cross-section, then certainly

$$(zw_1)^g - (zw_1)^{g-1}w_1 + (zw_1)^{g-2}w_2 - \cdots + (-1)^g w_g \equiv 0$$

for some rational number z. From the above formulas, this is equivalent to finding a rational root of

$$z^g - z^{g-1} + z^{g-2}/2! - z^{g-3}/3! + \cdots + (-1)^g/g! = 0.$$

Putting $z = -1/y$ and multiplying by $\pm y^g$, this becomes the exponential polynomial of degree g: $1 + y + y^2/2! + y^3/3! + \cdots + y^g/g!$ which we shall show has no rational roots for $g > 1$.

LEMMA. *$y^n + ny^{n-1} + n(n-1)y^{n-2} + \cdots + n!\,y + n! = 0$ has no rational roots, if $n > 1$.*

Proof. Any rational root α must in fact be integral by Gauss's Lemma. Now α cannot be divisible by any prime p, for if it were, writing the $n!$ on the right side, we obtain a contradiction by showing that for $0 \leqq r \leqq n-1$, $\operatorname{ord}_p(n!/(n-r)!)\alpha^{n-r} > \operatorname{ord}_p n!$, in other words,

$$\operatorname{ord}_p \alpha^m > \operatorname{ord}_p m!, \qquad m \geqq 1.$$

Namely, the left side is by hypothesis at least m, while the right side by a classical result is $(m - \sum a_i)/(p-1)$, the a_i being the coefficients in the p-adic expansion of m, and so it is less than m.

Thus no prime divides α; there remains only the possibility that $\alpha = -1$, which if $n > 1$ may be discarded because the sum is then obviously congruent to ± 1, modulo n, hence not zero.

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