ON THE INTERPOLATION OF L^p FUNCTIONS BY JACKSON POLYNOMIALS

BY

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1. Through the work of Marcinkiewicz and Zygmund [2], it is known that there is an analogy between the behaviour of the Fourier series and certain trigonometric polynomials corresponding to a given function f of class L^p, $p > 1$. The polynomials which they consider are of two types, the ordinary interpolating polynomials and the Jackson polynomials. However, the usual points of interpolation are translated by an arbitrary real u. Thus we write

$$(1)\quad \begin{aligned} I_{n,u}(x;f) &= \frac{2}{2n+1}\sum_{j=0}^{2n} f\left(u + \frac{2\pi j}{2n+1}\right) D_n\left(x - u - \frac{2\pi j}{2n+1}\right), \\ J_{n,u}(x;f) &= \frac{2}{n+1}\sum_{j=0}^{n} f\left(u + \frac{2\pi j}{n+1}\right) K_n\left(x - u - \frac{2\pi j}{n+1}\right) \end{aligned}$$

for the ordinary and Jackson polynomials of order n respectively. D_n and K_n are the Dirichlet and Fejer kernels respectively. Each interpolates the periodic function f at the corresponding interpolation points. The Jackson polynomials have a certain smoothness, and the fact that the kernel is positive makes them easier to treat. Since our own results are much more precise in this case, we emphasize their treatment and reduce to the status of corollaries our results about the sequence $I_{n,u}(x;f)$.

It is known [1] that for every $p > 0$ there are functions f in L^p such that the sequence $J_{n,u}(x;f)$ diverges for almost every (x, u): i.e., x and u are real variables, and the exceptional set is of two-dimensional zero measure. In the next section, we make this result exact by proving an order condition for Jackson polynomials which is shown to be best possible by examples. The construction of the examples involves the sharpening of a known technique [1]. In the following section, the positive result is generalized to certain sequences of linear operators; and this general result is then applied to the ordinary interpolating polynomials $I_{n,u}(x;f)$. Finally we apply our general result to the case of Riemann sums; and from this follows a result on localization theory for Jackson polynomials.

2. THEOREM 1 (i). *Let $p > 1$. Given f in L^p, then for almost every (x, u)*

$$\lim_n | J_{n,u}(x;f)|^p/n = 0.$$

(ii). *Given any positive sequence $\omega(n) = o(n)$ and $p > 0$, there exists a function f of class L^p such that for almost every (x, u)*

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$$\limsup_n | J_{n,u}(x;f)|^p/\omega(n) = \infty.$$

We may assume that f is real and positive. Let $m(y)$ be the distribution function of f, i.e., the measure of the set on which f exceeds y. Then (cf. [4, p. 112]) if $p \geqq 1$,

$$(2) \qquad -\int_0^\infty y^p\,dm(y) = p\int_0^\infty y^{p-1}m(y)\,dy = \int_0^{2\pi} |f(x)|^p\,dx.$$

Let ε be a small positive number. It is sufficient to prove that almost everywhere

$$\limsup_n | J_{n,u}(x;f)|/n^{1/p} \leqq \varepsilon.$$

We define the function $f_n(x)$ to equal $f(x)$ on the set where $f(x) \geqq \varepsilon n^{1/p}$ and to equal 0 on the complement of this set, so that $0 \leqq f(x) - f_n(x) \leqq \varepsilon n^{1/p}$. Thus for every (x, u) and every positive integer n

$$| J_{n,u}(x;f - f_n)|/n^{1/p} \leqq \varepsilon,$$

as is clear from (1) and the fact that K_n is positive. Also

$$\limsup_n | J_{n,u}(x;f)|/n^{1/p} \leqq \varepsilon + \limsup_n J_{n,u}(x;f_n)/n^{1/p}.$$

It remains to show that for almost every (x, u) the second term on the right is 0. This is true if

$$\sum_{n=1}^\infty | J_{n,u}(x;f_n)|/n^{1/p} < \infty,$$

and the last inequality will follow if we prove

$$(3) \qquad \sum_{n=1}^\infty \frac{1}{n^{1/p}} \int_0^{2\pi}\int_0^{2\pi} | J_{n,u}(x;f_n) |\,du\,dx \leqq A_p\,\varepsilon^{1-p} \int_0^{2\pi} |f(x)|^p\,dx,$$

where A_p is a constant depending only on p. It follows easily from (1) that

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} J_{n,u}(x;f_n)\,du\,dx = \int_0^{2\pi} f_n(x)\,dx$$

by integrating first with respect to x. Both integrands here are positive. The distribution function corresponding to the function f_n is constant if $0 \leqq y < \varepsilon n^{1/p}$, and equals $m(y)$, the distribution function of f, if $y \geqq \varepsilon n^{1/p}$. We set $a(n) = \varepsilon n^{1/p}$. Thus from (2)

$$\int_0^{2\pi} f_n(x)\,dx = -\int_{a(n)}^\infty y\,dm(y).$$

By an application of Abel's transformation, it follows that

$$(4) \qquad \begin{aligned} \frac{1}{2\pi}\sum_{n=1}^\infty \frac{1}{n^{1/p}}\int_0^{2\pi}\int_0^{2\pi} J_{n,u}(x;f_n)\,du\,dx &= -\sum_{n=1}^\infty \frac{1}{n^{1/p}}\int_{a(n)}^\infty y\,dm(y) \\ &\leqq -A_p\sum_{n=1}^\infty n^{1/q}\int_{a(n)}^{a(n+1)} y\,dm(y), \end{aligned}$$

where q is the conjugate of p, i.e., $1/p + 1/q = 1$. The validity of the transformation involves the fact that

$$-n^{1/q} \int_{a(n)}^{\infty} y\, dm(y) = o(1).$$

This is proved by noting that since $a(n) \leqq y$, then $n^{1/q} \leqq (y/\varepsilon)^{p-1}$. For the same reason

$$-\sum_{n=1}^{\infty} n^{1/q} \int_{a(n)}^{a(n+1)} y\, dm(y) \leqq -\varepsilon^{1-p} \sum_{n=1}^{\infty} \int_{a(n)}^{a(n+1)} y^p\, dm(y)$$

$$\leqq \varepsilon^{1-p} \int_0^{2\pi} | f(x) |^p\, dx.$$

Combining this with (4) establishes (3) and so part (i) of the theorem.

Since part (ii) of the theorem is a refinement of a known one [1], we shall be concise in the proof. It may be assumed that the sequence $\omega(n)$ is monotonically increasing. If it is not, it may be replaced by $\max_{m \leqq n} \omega(m)$ without affecting the order condition adversely. Let $\omega(n) = n\delta_n^4$ where, by hypothesis, $\delta_n = o(1)$. The following sequences of reals are defined:

$$A_m = (m\delta_m^2)^{1/p}, \qquad \varepsilon_m = 2\pi/\delta_m\, m^2.$$

It follows from the assumptions on $\omega(m)$ that $\lim m\delta_m = \infty$, so that $\varepsilon_m = o(1/m)$. We also note that

$$(5) \qquad 2mA_m^p\, \varepsilon_m = 4\pi\delta_m\, ; \qquad A_m^p/\omega(n) \geqq A_m^p/\omega(m) = 1/\delta_m^2 \quad \text{if } n \leqq m.$$

The periodic function $f_m(x)$ is defined to be A_m if $| x - 2\pi j/m | \leqq \varepsilon_m$ for some $j = 0, 1, \cdots, m - 1$ and to be 0 otherwise.

Let E_m be the subset of $(0, 2\pi)$ such that for u in E_m there is a rational $k/(n + 1)$ with $m\delta_m \leqq n + 1 \leqq m$ for which $| u - 2\pi k/(n + 1)| \leqq 2\pi/m^2\delta_m$. It is known [1] that $| E_m |$, the measure of E_m, exceeds $2\pi - C\delta_m$ for some constant C. Let $E_{m,j}$ be the set E_m translated by $-2\pi j/m$, and let I_j be the interval $| x - 2\pi j/m | \leqq \pi/m, j = 0, 1, \cdots, m - 1$. Given $-u$ modulo 2π in $E_{m,j}$ and x in I_j, there exist integers k and $n + 1$, $m\delta_m \leqq n + 1 \leqq m$, such that

$$| u - 2\pi j/m + 2\pi k/(n + 1)| < 2\pi/m^2\delta_m\, ;$$

$$| x - u - 2\pi k/(n + 1)| \leqq | x - 2\pi j/m | + 2\pi/m^2\delta_m \leqq 3\pi/2m.$$

The last inequality on the right holds for m large enough since $\lim m\delta_m = \infty$. The first inequality above implies that $f_m(u + 2\pi k/(n + 1)) = A_m$, and the second that $K_n(x - u - 2\pi k/(n + 1)) \geqq C(n + 1)$ for some constant C. Hence if $(x, -u)$ belongs to the product set $I_j \times E_{m,j}$ of the xu-plane, there exist integers k and $n + 1$, $m\delta_m \leqq n + 1 \leqq m$, such that

$$(6) \qquad J_{n,u}(x; f_m) \geqq (1/(n+1)) f_m(u + 2\pi k/(n+1)) \cdot K_n(x - u - 2\pi k/(n+1)) \geqq CA_m .$$

Let $H_m = \bigcup_{j=0}^{m-1} I_j \times E_{m,j}$. Since the sets $I_j \times E_{m,j}$ are mutually disjoint and have the same measure, $2\pi \mid E_m \mid/m$, then $\mid H_m \mid = 2\pi \mid E_m \mid \geqq 4\pi^2 - C\delta_m$.

By (5), $\| f_m \|_p^p = 4\pi\delta_m$, and f_m is 0 outside a set of measure $4\pi/m\delta_m$. Thus we may let $f(x) = \sum_{i=1}^{\infty} f_{m(i)}(x)$ with the integers $m(i)$ increasing rapidly enough so that f belongs to L^p and so that certain other conditions are met (cf. [1] for details): in particular, so that $H = \lim\sup H_{m(i)}$ has measure $4\pi^2$. By $\lim\sup H_{m(i)}$, we mean, as usual, the set of points which belong to infinitely many $H_{m(i)}$. Let $(x, -u)$ belong to H and to a particular H_m. Then by (5) and (6)

$$\mid J_{n,u}(x; f) \mid^p/\omega(n) \geqq \mid J_{n,u}(x; f_m) \mid^p/\omega(n) \geqq C^p A_m^p/\omega(n) \geqq C^p/\delta_m^2$$

for appropriate n. Since the inequality occurs for infinitely many n, the theorem follows.

3. Part (i) of Theorem 1 is capable of a good deal of generalization. Let T_n be a sequence of linear operators transforming certain functions defined on a measure space R_1 into functions defined on a second measure space R_2. Let μ and ν be measures defined on R_1 and R_2 respectively. For the sake of simplicity, we may think of R_1 and R_2 as compact subsets of Euclidean spaces, not necessarily of the same dimension, and μ and ν as Lebesgue measures. We assume each T_n is of type (1, 1) by which we mean the following (cf. [4, p. 95]). Let f be an integrable function defined on R_1: i.e., $\int_{R_1} \mid f \mid d\mu < \infty$. Then $T_n f = h$ is defined as an integrable function on R_2, and

$$\int_{R_2} \mid h \mid d\nu \leqq M_n \int_{R_1} \mid f \mid d\mu$$

for some number M_n independent of f. We also write $M_n = \| T_n \|_1$, the L^1 norm of T_n. It is further assumed that each T_n is of type (∞, ∞), by which is meant the following: if f is an essentially bounded function (with respect to the μ measure) defined on R_1, then $T_n f = h$ is defined as an essentially bounded function (with respect to the ν measure) on R_2, and

$$\text{ess sup} \mid h \mid \leqq N_n \text{ ess sup} \mid f \mid$$

for some number N_n independent of f. We also write $N_n = \| T_n \|_\infty$, the L^∞ norm of T_n. It follows from the Riesz-Thorin interpolation theorem that each T_n carries L^p functions on R_1 into L^p functions on R_2, $p \geqq 1$. Let $T_n(y; f)$ designate the function $T_n f$ evaluated at the point y of R_2.

THEOREM 2. *Let $p > 1$. Let T_n be a sequence of linear operators as defined above such that $\| T_n \|_1 + \| T_n \|_\infty \leqq M$, a number independent of n. Then for any function f in $L^p(R_1)$*

$$\lim_n | T_n(y; f)|^p/n = 0$$

for almost every point y of R_2 .

There is only one minor modification we have to make in the proof of Theorem 1 (i). Let us assume that f is real and positive. Let $f_n(x)$ equal $f(x)$ on the set where $f(x) \geqq \varepsilon n^{1/p}$, and let it equal 0 otherwise. The function $f(x) - f_n(x)$ is positive and strictly bounded by $\varepsilon n^{1/p}$ so that outside a subset E_n of R_2 of measure 0, $| T_n(y; f - f_n)| \leqq M\varepsilon n^{1/p}$. This is true for every n outside the union of the E_n's, a set of measure 0. The rest of the proof is precisely that of Theorem 1 (i).

In the proof, we used only the fact that strictly bounded functions are transformed into essentially bounded functions; but it is convenient, in order to use the Riesz-Thorin theorem, to have this true for essentially bounded functions. To see this is so for the sequence J_n of Theorem 1, we argue as follows. Let f be bounded outside a set E of measure 0. Let G be the union of E with all its translates by $2\pi r$ for r rational. G is of measure 0. Let G' be the complement of G. If (x, u) belongs to $I \times G'$, where I is the interval $(0, 2\pi)$, then $J_{n,u}(x; f)$ has the same bound as $f(x)$ in E', the complement of E. It is clear that the other hypotheses of Theorem 2 are true for the sequence J_n , so that Theorem 2 is a real generalization of Theorem 1 (i). Furthermore Theorem 1 (ii) shows that no improvement is possible in the general case.

By a renumbering process, we may consider subsequences of the original sequence. For example, let n_k be a lacunary sequence of integers, i.e., $n_{k+1}/n_k \geqq \lambda > 1$. Since $k \leqq C$ $(\log n_k)/\log \lambda$ for some constant C and $n_k > 1$,

$$\lim_k | J_{n_k,u}(x; f)|^p/\log n_k = 0$$

for almost every (x, u) if f belongs to L^p, $p > 1$.

4. It is well known and easy to verify from (1) that the sequence $I_{n,u}(x; f)/\log(n + 1)$ satisfies the hypotheses of Theorem 2. Hence if f belongs to L^p, $p > 1$, then for almost every (x, u)

$$\lim_n | I_{n,u}(x; f)|^p/n(\log n)^p = 0.$$

Although it is known [1] that for every $p > 0$ there are functions in L^p for which $\lim | I_{n,u}(x; f)| = \infty$ almost everywhere, the examples do not show that the above result is best possible. The difficulty is a familiar one: the Dirichlet kernel is not always positive.

Another example of interest is that of Riemann sums (cf. [2], [3])

$$R_n(x; f) = (1/n) \sum_{j=0}^{n-1} f(x + 2\pi j/n).$$

Since the hypotheses of Theorem 2 are obviously satisfied, we may say that if f belongs to L^p, then $R_n(x; f) = o(n^{1/p})$ for almost every x. There are examples ([2], [3]) to show that for f in L^p, $p < 2$, the order of $R_n(x; f)$ may be close to $n^{2/p-1}$, and again this falls short of our criterion. However the misbehaviour of the examples is due to the value of f at only one interpolating point. Our next theorem shows that no essential improvement is possible along these lines.

THEOREM 3. *Let $p \geqq 1$, and let f belong to L^p. For almost every x*

$$\lim_n \max_j |f(x + 2\pi j/n)|^p/n^2 = 0.$$

The result in the case $p = 2$ is due to Ursell [3]. Let f be positive, and let $E(y)$ be the set on which $f(x)$ exceeds εy. Let the distribution function of f be $m(y)$. For $n = 1, 2, \cdots$, let $G(n, y)$ be the set of x values such that there exists an integer j for which $x + 2\pi j/n$ belongs to $E(y)$. Since $G(n, y)$ is the union of n translates of $E(y)$, $|G(n, y)| \leqq nm(\varepsilon y)$. Thus

$$\sum_{n=1}^{\infty} |G(n, n^{2/p})| \leqq \sum_{n=1}^{\infty} nm(\varepsilon n^{2/p}) \leqq C \int_0^{\infty} um(\varepsilon u^{2/p})\, du$$
$$= \frac{C}{\varepsilon^p} \int_0^{\infty} y^{p-1} m(y)\, dy,$$

which is finite by (2). This implies that for almost every x,

$$\limsup_n \max_j |f(x + 2\pi j/n)|^p/n^2 < \varepsilon,$$

which is sufficient for the proof of the theorem.

Let $s_n(x; f)$ be the n^{th} partial sum of the Fourier series of f. It is known [4, p. 166] that almost everywhere $|s_n(x; f)|^p = o(\log n)$, if f belongs to L^p, $1 < p \leqq 2$. Comparison of this result with Theorem 1 (ii) shows how much better behaved in the matter of order are the partial sums of Fourier series than Jackson polynomials. On the other hand, Jackson polynomials behave quite satisfactorily with regard to localization, not without, however, restrictions on the uniformity, as indicated by our last theorem.

THEOREM 4. *Let $p > 1$, and let f belong to L^p. Let $f(x) = 0$ when $|x| \leqq \Delta$. There is a set E of measure 2π such that for fixed u in E,*

$$J_{n,u}(x; f) = o(n^{1/p-1}) \quad \textit{uniformly in } x, \qquad |x| \leqq \Delta_1 < \Delta.$$

If $f(u + 2\pi j/(n + 1)) \neq 0$, then $|x - u - 2\pi j/(n + 1| \geqq \Delta - \Delta_1$, so that $K_n(x - u - 2\pi j/(n + 1)) \leqq C/(n + 1)$ for some constant C. Thus

$$|J_{n,u}(x; f)| \leqq (C/(n + 1)^2) \textstyle\sum_{j=0}^{n} |f(u + 2\pi j/(n + 1))|$$
$$= (C/(n + 1))R_{n+1}(u; |f|).$$

Since $R_{n+1}(u; |f|) = o(n^{1/p})$ almost everywhere according to Theorem 2, the result follows. The argument does not hold either for $p = 1$ or for the ordinary polynomials $I_{n,u}(x; f)$. Any improvement in the order of $R_n(x; f)$ results in a corresponding improvement in the localization theorem.

References

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