FUNCTIONALS RELATED TO MIXED VOLUMES

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We denote by R^n a fixed euclidean space of dimension n. A coset of a sub-vector-space of R^n, of dimension m, will be termed m-flat. $\mathcal{C}$ stands for the family of al' ˏmpact, *convex* subsets of R^n. An $A \in \mathcal{C}$ will be called a convex body; ˈ.ˬ will be termed proper if it has inner points in R^n. $\mathcal{C}$ is a locally compact, separable, metric space with the topology introduced by Minkowski and Blaschke. A real valued, *continuous* function $\varphi: \mathcal{C} \to R$ will be called a *functional* (of convex bodies). We will deal only with φ's having the following properties:

$$\varphi(tA) = \varphi(A) \quad (t: R^n \to R^n;\ t(x) = x + x_0) \tag{1}$$

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \quad (A, B, A \cup B \in \mathcal{C})\cdot \tag{2}$$

We choose now a proper convex body U, to be fixed in the rest of this note. Theorem 1 below could be formulated in terms of Minkowski integral geometry [3], [4], the convex body U being either the indicatrix, or the isoperimetrix. However, in the present note, we do not want to pursue this direction. It suffices to say, in order to suggest the role of U in the present context, that, if we substitute the unit ball, $B^n = \{x : x \in R^n,\ \| x \| \leqq 1\}$, in place of U in Theorem 1 below, the statement is a well known and useful theorem of euclidean geometry.

The mixed volumes [2; p. 40]

$$\varphi_i(A) = V_i(A, U) \qquad (i = 0, \cdots, n), \tag{3}$$

considered as functions of the first argument, are particular functionals having properties (1), (2). If $U = B^n$, φ_0, φ_1 are proportional to the volume and surface area, respectively. Furthermore, it is well known [7; p. 221], that the functionals

$$W_i(A) = V_i(A, B^n) \qquad (i = 0, \cdots, n) \tag{4}$$

form a basis in the vector space of the functionals ψ, which are additive, in the sense of (2), and are invariant under isometries, that is to say, such that

$$\psi(gA) = \psi(A) \qquad (g : R^n \to R^n;\ g \textit{ isometry}) \tag{5}$$

holds true for every isometry g.

A weaker form of this statement is the following. A functional ψ is of the form

$$\psi(A) = \textstyle\sum_{i=0}^{n} \alpha_i W_i(A) \qquad (\alpha_i \in R) \tag{6}$$

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if and only if (1), (2), and the following condition hold true:

If $W_i(A) = W_i(B)$, $i = 0, \cdots, n$, *then* $\psi(A) = \psi(B)$.

We will generalize this statement in the present note. Specifically, the proper, convex body U being given, we consider functionals φ such that (1), (2), and the following condition:

$$(7) \qquad \text{If } V_i(A, U) = V_i(B, U),\ i = 0, \cdots, n, \text{ then } \varphi(A) = \varphi(B)$$

hold true, and we characterize these functionals.

THEOREM 1. *Let U be a given proper convex body. Then the mixed volumes $V_i(A, U)$, $i = 0, \cdots, n$, form a basis in the vector space of functionals φ satisfying* (1), (2), (7). *In other words, if φ is a* (*continuous*) *functional of convex bodies, which is translation invariant, additive, and such that* (7) *holds true, then*

$$(8) \qquad \varphi(A) = \textstyle\sum_{i=0}^{n} \alpha_i V_i(A, U) \qquad (\alpha_i \in R)$$

where the α_i's are well determined constants. Clearly, all these conditions are also necessary in order that φ be of the form (8).

Proof. We will first state and prove some facts on mixed volumes, which are needed in the proof of Theorem 1.

For every $A \in \mathfrak{C}$, we have $V_0(A, U) = V(A)$, i.e., the volume of A, and $V_n(A, U) = V(U)$. It is known, and easy to prove,[1] that

$$(9) \qquad \text{If } \dim A \leqq k, \text{ then } V_i(A, U) = 0, \quad i = 0, \cdots, n-k-1.$$

Let us denote by $| C |$ the volume of $C \in \mathfrak{C}$ in the smallest flat containing C. Thus $| C | = 0$, $| D | = V(D)$ amounts to saying that C is empty and D is proper. Given a direction u, i.e., a unit vector u, we denote by A_u the projection of A into an $(n-1)$-flat perpendicular to u. In fact, we will use the well determined $| A_u |$ only. Then we have

$$(10) \qquad V_i(A + \mu[u], U) = cV_i^*(A_u, U_u)\mu + V_i(A, U),$$

where $\mu \in R$, $\mu > 0$, $[u]$ denotes the segment joining the origin to u, c is a constant, and $V_i^*(A_u, U_u)$ stands for the i^{th} mixed volume in the $(n-1)$-flat containing A_u and U_u. Equation (10) can easily be proved by computing

$$\frac{\partial}{\partial \mu} V_i(A + \mu[u], U);$$

see [6] for a similar result.

The last auxiliary result needed concerns $V_i(C_k, U)$, where C_k is a k-dimensional box. Let there be given k linearly independent unit vectors

[1] See [5] for complete proofs. ([5] is a Technical Report, which can be obtained from the University of California, Berkeley, Department of Mathematics.) The present note is but a short version of [5], which gives detailed proofs as well as some discussion of the relevant part of the theory of convex bodies.

$u_1, \cdots, u_k$, and k real numbers $\mu_i > 0$, $i = 1, \cdots, k$. If C_k denotes the box spanned by $\mu_1 u_1, \cdots, \mu_k u_k$, or, equivalently, the Minkowski sum $\mu_1[u_1] + \cdots + \mu_k[u_k]$, we have

$$V_i(C_k, U) = 0, \qquad i = 0, \cdots, n-k-1; \tag{11}$$

$$V_{n-l}(C_k, U) = \sum c^{i_1 \cdots i_l} \mu_{i_1} \cdots \mu_{i_l} \qquad (l = 0, \cdots, k). \tag{12}$$

In (12) we have a *full* homogeneous polynomial of degree l; moreover,

$$c^{i_1 \cdots i_l} > 0 \tag{13}$$

for every choice of l integers, repetitions allowed, from the set $\{1, \cdots, k\}$; we may agree that the c's are symmetric in the superscripts.

Of course, (11) is just a special case of (9). We prove (12) by an induction on k, using the decomposition $C_k = C_{k-1} + \mu_k[u_k] = A + \mu I$ in (10). In this manner the c's can also be computed, but it is enough to know (13) in the sequel.

Let there be given $\{u_1, \cdots, u_k\}$ and $\{v_i, \cdots, v_k\}$, two sets of k linearly independent vectors each. Given $\mu = (\mu_1, \cdots, \mu_k)$ and $\nu = (\nu_1, \cdots, \nu_k)$, we denote by C_k and D_k the boxes spanned by $\{\mu_1 u_1, \cdots, \mu_k u_k\}$ and $\{\nu_1 u_1, \cdots, \nu_k u_k\}$, respectively. We define a map $F\colon R_u^k \to R_\xi^k$ by

$$\begin{aligned} F(\mu) &= (V_{n-1}(C_k, U), \cdots, V_{n-k}(C_k, U)) \\ &= (\xi_1, \cdots, \xi_k); \end{aligned}$$

the map $G\colon R_\nu^k \to R_\xi^k$, with the same R_ξ^k, is defined similarly, using D_k in place of C_k. It is not difficult to prove then, that there are numbers $\mu_i > 0$, $\nu_i > 0$, $i = 1, \cdots, k$, such that $F(\mu) = G(\nu)$ holds true. This statement can be proved by an induction on k, using (12), (13), and the fact that by an appropriate choice of $\mu_i > 0$, $\nu_i > 0$, the functional determinants of the maps F, G are $\neq 0$.

Reformulating this result in terms of mixed volumes, we have the following. Given two sets of k independent vectors each, there are k-dimensional boxes C_k, D_k, spanned by proportional vectors, and such that

$$V_i(C_k, U) = V_i(D_k, U), \qquad i = 0, \cdots, n, \tag{14}$$

holds true.

We come now to the proof proper of Theorem 1. Given φ, satisfying the conditions of the theorem, we will define functionals

$$\psi_k(A) = \varphi(A) - \textstyle\sum_{i=n-k}^{n} \alpha_i V_i(A, U) \qquad (\alpha_i \in R) \tag{15}$$

by induction on k. Our construction will be such that (1), (2), (7) hold true; furthermore ψ_k will be such that

$$\textit{If } \dim A \leqq k, \textit{ then } \psi_k(A) = 0. \tag{16}$$

By the last condition, ψ_n will be identically zero; thus we will have (8).

We set $\alpha_n = \varphi(x_0)/V(U)$, where x_0 is a fixed point of R^n; here we use the hypothesis that U is a *proper* convex body; thus $V(U) \neq 0$. Then $\psi_0(A) = \varphi(A) - \varphi(x_0)$; thus ψ_0 is zero for every point $x \in R^n$, because φ is translation invariant. This function also has the other required properties.

We suppose now that ψ_{k-1} has been defined and satisfies (1), (2), (7), (15), (16). Let L be a given k-flat in R^n. We consider the restriction of ψ_{k-1} to the family of convex bodies contained in L. This restriction is then simply additive, i.e.,

$$\psi_{k-1}(A \cup B) = \psi_{k-1}(A) + \psi_{k-1}(B) \quad (A, B \subset L, A \cup B \in \mathfrak{C}),$$

if $\dim (A \cap B) \leqq k - 1$, in view of (2) and (16). From this, and from the other properties of ψ_{k-1} follows easily [7] that

$$\psi_{k-1}(A) = \beta V_k(A) \qquad (A \subset L; \beta \in R), \tag{17}$$

where V_k is the volume in the k-flat L. Now, the mixed volume $V_{n-k}(A, U)$, as a function of $A \subset L$, has also the properties (1), (2), (16) for $k - 1$ in place of k; thus

$$V_{n-k}(A, U) = \gamma V_k(A) \qquad (A \subset L; \gamma \in R). \tag{18}$$

From (17) and (18) follows:

$$\textit{If } A \subset L, \ \psi_{k-1}(A) = \alpha_{n-k} V_{n-k}(A, U), \tag{19}$$

with a well determined constant α_{n-k}.

Let M be another, given k-flat in R^n. There exists then another constant α'_{n-k}, such that

$$\textit{If } A \subset M, \textit{ then } \psi_{k-1}(A) = \alpha'_{n-k} V_{n-k}(A, U) \tag{20}$$

holds true. We will prove now that, in fact, the two constants are equal:

$$\alpha'_{n-k} = \alpha_{n-k}. \tag{21}$$

We choose boxes $C_k \subset L$ and $D_k \subset M$, whose k-volumes are nonzero, and such that (14) holds true. Then, by hypothesis (7), we have $\psi_{k-1}(C_k) = \psi_{k-1}(D_k)$, and, by (19), this implies (21). As now in (19) the constant α_{n-k} is independent of the k-flat L, we can formulate our result as follows:

$$\textit{If } \dim A \leqq k, \textit{ then } \psi_{k-1}(A) = \alpha_{n-k} V_{n-k}(A, U), \tag{22}$$

where α_{n-k} is a constant. We now define ψ_k by

$$\psi_k(A) = \psi_{k-1}(A) - \alpha_{n-k} V_{n-k}(A, U),$$

using this constant α_{n-k}. Then ψ_k has properties (1), (2), (7), (15), (16) which completes the induction step. As we remarked at the beginning of the proof, this implies the statement of Theorem 1.

The mixed volumes (3), and thus, in general, the functionals (8), are not invariant under rotations. As to the average value of $V_\nu(gA, U)$, as a function of the element g of the orthogonal group O_n, we have the following result.

THEOREM 2. *We consider convex bodies in a given euclidean space R^n. There are constants α_ν^{ij}, $\nu, i, j = 0, \cdots, n$, such that, if dA is the kinematic density of the convex body A (i.e., the volume element in the rotation group O_n), then*

$$\int V_\nu(A, U)\, dA = \sum_{i,j=0}^{n} \alpha_\nu^{ij} W_i(A) W_j(U) \tag{23}$$

where W_i denotes the functional defined in (4).

Proof. We fix an integer ν, $0 \leqq \nu \leqq n$, and set

$$f(U) = \int V_\nu(A, U)\, dA. \tag{24}$$

Then $f(U)$ is continuous in U and has properties (1), (2), (5); thus, for a given, fixed A, we have

$$f(U) = \textstyle\sum_{i=0}^{n} \alpha_i(A) W_i(U). \tag{25}$$

Simple transformations of the integral (24) and uniqueness of the representation of an additive and isometry-invariant functional in terms of (4) gives for the coefficients $\alpha_i(A)$ in (25) that $\alpha_i(gA) = \alpha_i(A)$, $g: R^n \to R^n$, g isometry, and

$$\alpha_i(A \cup B) + \alpha_i(A \cap B) = \alpha_i(A) + \alpha_i(B) \quad (A, B, A \cup B \in \mathfrak{C})$$

hold true. Hence, using again [7; p. 221], we have:

$$\alpha_i(A) = \textstyle\sum_{j=0}^{n} \alpha_\nu^{ij} W_j(U).$$

This completes the proof of (23).

Remark. I was unable to compute effectively the constants α_ν^{ij}, but I conjecture that this should be possible, by evaluating the integrals

$$\int V_\nu(C_k, C_l)\, dC_k,$$

where C_m denotes an m-dimensional box. So far as I know, it is possible that the α_ν^{ij}'s are all zero, except for the diagonal ones.

The result formulated below follows immediately from the two previous theorems.

THEOREM 3. *Let φ be a (continuous) functional of convex bodies which is translation invariant, additive, and which satisfies* (7); *thus φ is as in Theorem* 1. *If dA denotes the kinematic density of the convex body A, we have*

$$\int \varphi(A)\, dA = \sum \beta^{ij} W_i(A) W_j(U)$$

where the β^{ij} are constants determined by φ.

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