THE POWER SERIES COEFFICIENTS OF FUNCTIONS DEFINED BY DIRICHLET SERIES

BY

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If the Dirichlet series $f(s) = \sum_{n=1}^{\infty} h(n)n^{-s}$ has abscissa of convergence $\operatorname{Re} s = a$ and a simple pole at $s = a$, then $f(s)$ has the Laurent expansion

$$f(s) = \frac{C}{s-a} + \sum_{r=0}^{\infty} \frac{(-1)^r C_r}{r!} (s-a)^r.$$

The purpose of this paper is to derive expressions for the C_r and to list the results for various number-theoretic functions $h(n)$, thus generalizing the special case of $h(n) = 1$ found in [1]. It is assumed throughout that $f(s)$ is of the above form with C, a, $h(n)$, and C_r referring to this relation, and that $E(x) = \sum_{n \leqq x} h(n) - Ca^{-1}x^a = O(x^b)$ where $0 \leqq b < a$.

Two lemmas are stated without proof.

Lemma 1. *If $b_1, b_2, b_3, \cdots$ is a sequence of complex numbers and $v(x)$ has a continuous derivative for $x > 1$, then*

$$\sum_{n \leqq x} b_n v(n) = (\sum_{n \leqq x} b_n) v(x) - \int_1^x (\sum_{n \leqq t} b_n) v'(t)\, dt.$$

Lemma 2.

$$f(s) = s \int_1^{\infty} x^{-s-1} [\sum_{n \leqq x} h(n)]\, dx, \qquad \operatorname{Re} s > a$$

Lemma 3. *If* $\operatorname{Re} s > b$, *then*

$$f_1(s) \equiv s \int_1^{\infty} x^{-s-1} E(x)\, dx = -\frac{C}{a} + \sum_{r=0}^{\infty} \frac{(-1)^r C_r}{r!} (s-a)^r.$$

Proof. The integral is an analytic function for $\operatorname{Re} s > b$ and equals $f(s) - C/a - C/(s-a)$ for $\operatorname{Re} s > a$.

Theorem 1. *If $u < -b$, then*

$$\sum_{n \leqq x} n^u h(n) \log^r n = C \int_1^x t^{u+a-1} \log^r t\, dt + D_r + (-1)^r f_1^{(r)}(-u) + o(1),$$

where $D_r = C/a$ if $r = 0$ and $D_r = 0$ otherwise.

Proof. From Lemma 1

$$S = \sum_{n \leqq x} n^u h(n) \log^r n = \sum_{n \leqq x} h(n) x^u \log^r x - \int_1^x [\sum_{n \leqq t} h(n)] \frac{d}{dt} (t^u \log^r t)\, dt.$$

But $(d/dt)(t^u \log^r t) = (d^r/du^r)(ut^{u-1})$. Hence

$$S = Ca^{-1}x^{u+a} \log^r x + O(x^{u+b} \log^r x)$$
$$+ \frac{d^r}{du^r}\left\{-u \int_1^{\infty} t^{u-1} E(t)\, dt - u \int_1^x Ca^{-1} t^{u+a-1}\, dt + u \int_x^{\infty} t^{u-1} E(t)\, dt\right\}.$$

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But by Lemma 3, $-u\int_1^\infty t^{u-1}E(t)dt = f_1(-u)$, and

$$x^{u+a}\log^r x - \frac{d^r}{du^r}u\int_1^x t^{u+a-1}\,dt$$
$$= \frac{d^r}{du^r}\left[x^{u+a} - 1 - u\int_1^x t^{u+a-1}\,dt\right] + \frac{d^r}{du^r}1$$
$$= \frac{d^r}{du^r}\left[a\int_1^x t^{u+a-1}\,dt\right] + \frac{d^r}{du^r}1 = a\int_1^x t^{u+a-1}\log^r t\,dt + \begin{cases}0 \text{ if } r > 0\\ 1 \text{ if } r = 0.\end{cases}$$

The r^{th} derivative of the third integral appearing in S approaches zero as $x \to \infty$, which completes the proof.

THEOREM 2.

$$C_r = \lim_{x\to\infty}\left(\sum_{n\leqq x} n^{-a}h(n)\log^r n - \frac{C}{r+1}\log^{r+1} x\right).$$

Proof. Lemma 3 gives $f_1(a) = -C/a + C_0$ and $f_1^{(r)}(a) = (-1)^r C_r$ for $r > 0$. The proof is completed by setting $u = -a$ in Theorem 1 and letting $x \to \infty$.

It should be noted that the same method can be used when a pole of second order appears. For instance if $\zeta(s)$ is the Riemann zeta function, then

$$\zeta^2(s) = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + \sum_{r=0}^{\infty}\frac{(-1)^r A_r}{r!}(s-1)^r,$$

and

$$A_r = \lim_{x\to\infty}\left(\sum_{n\leqq x} n^{-1}\,d(n)\log^r n - \frac{1}{r+2}\log^{r+2} x - \frac{2\gamma}{r+1}\log^{r+1} x\right).$$

The following table lists various cases to which Theorem 2 may be applied. The notation of [2] is used.

$h(n)$	$f(s)$	a	C
1	$\zeta(s)$	1	1
$\chi_k(n)$, principal character modulo k	$L_k(s)$	1	$\phi(k)/k$
$\chi_k(n)$, nonprincipal character modulo k	$L_k(s)$	1	0
$\phi(n)$	$\zeta(s-1)/\zeta(s)$	2	$1/\zeta(2)$
$\sigma_k(n)$, $k > 0$	$\zeta(s)\zeta(s-k)$	$1+k$	$\zeta(1+k)$
$q_k(n)$, $k \geqq 2$ (integer)	$\zeta(s)/\zeta(ks)$	1	$1/\zeta(k)$
$r(n)$	$4L(s)\zeta(s)$	1	π

REFERENCES

1. W. E. BRIGGS AND S. CHOWLA, *The power series coefficients of $\zeta(s)$*, Amer. Math. Monthly, vol. 62 (1955), pp. 323–325.
2. G. H. HARDY AND E. M. WRIGHT, *An introduction to the theory of numbers*, Oxford, 1960, particularly Chapter 17.

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