ON THE DENSITY OF SETS OF INTEGERS POSSESSING ADDITIVE BASES

BY

Basil Gordon and L. A. Rubel

Let $K = \{k_0, k_1, k_2, \cdots\}$ be an infinite set of positive integers with $k_0 < k_1 < k_2 < \cdots$. Let S be the set of all integers which can be expressed as the sum of distinct elements of K. It is convenient to regard 0 (the empty sum) as belonging to S. In the special case where every element of S has a unique representation as the sum of distinct elements of K, Wintner [2] has called S a π-set with basis K. For example, the set of all nonnegative integers forms a π-set whose basis consists of the powers of 2.

The relationship between a π-set and its basis can be expressed analytically by the formula

$$\sum_{n=0}^{\infty} c_n x^n = \prod_{n=0}^{\infty} (1 + x^{k_n}) \qquad (|x| < 1),$$

where c_n is the characteristic function of S, i.e., $c_n = 1$ or 0 according as n is or is not in S.

Wintner in [2] investigated the question of when a π-set has a density, i.e., when

$$\lim_{n\to\infty} (c_0 + c_1 + \cdots + c_n)/n = \theta \tag{1}$$

exists. He proved that if

$$\lim_{n\to\infty} 2^n/k_n = \theta \tag{2}$$

exists, then S has density θ. In the present paper it is shown that (2) is necessary, as well as sufficient, for the existence of a density except possibly in the special case $\theta = 0$. Wintner's question of whether or not every π-set has a density can then easily be answered in the negative.

We remark that our methods apply to the more general case where the k_n are positive real numbers not assumed to be integers and where S is not assumed to be a π-set, provided that multiplicities are counted properly.

Theorem. *Suppose* (1) *holds with* $\theta > 0$. *Then* (2) *follows.*

Proof. Any element $m \in S$ with $m < k_n$ can only involve $k_0, k_1, \cdots, k_{n-1}$ in its representation as a sum of basis elements. There are only 2^n possible sums that can be formed from $k_0, k_1, \cdots, k_{n-1}$. Hence

$$c_0 + c_1 + \cdots + c_{k_n} \leqq 2^n + 1.$$

Dividing by k_n and letting $n \to \infty$, we find that $\liminf_{n\to\infty} 2^n/k_n \geqq \theta$. Hence, for any $\theta' < \theta$, we have $k_n \leqq 2^n/\theta' = \rho' 2^n$ for $n > n(\theta')$. Write, for

Received May 14, 1959.

$m > n(\theta')$,

$$\prod_{n=0}^{\infty}(1 + x^{k_n}) = (1 + x^{k_m})\prod_{n=0}^{n(\theta')}(1 + x^{k_n})\prod_{n>n(\theta'),n\neq m}(1 + x^{k_n})$$

$$\geqq (1 + x^{k_m})\prod_{n=0}^{n(\theta')}(1 + x^{k_n})\prod_{n>n(\theta'),n\neq m}(1 + x^{\rho' 2^n})$$

$$= \frac{1 + x^{k_m}}{1 + x^{\rho' 2^m}}\frac{\prod_{n=0}^{n(\theta')}(1 + x^{k_n})}{\prod_{n=0}^{n(\theta')}(1 + x^{\rho' 2^n})}\prod_{n=0}^{\infty}(1 + x^{\rho' 2^n}).$$

Applying Euler's famous identity,

$$\textstyle\prod_{n=0}^{\infty}(1 + y^{2^n}) = 1/(1 - y),$$

we get

$$(1 - x)\prod_{n=0}^{\infty}(1 + x^{k_n}) \geqq \frac{1 - x}{1 - x^{\rho'}}\frac{1 + x^{k_m}}{1 + x^{\rho' 2^m}}\prod_{n=0}^{n(\theta')}\left(\frac{1 + x^{k_n}}{1 + x^{\rho' 2^n}}\right),$$

and finally,

$$(1 - x)\sum_{n=0}^{\infty} c_n x^n \geqq \theta' \frac{1 + x^{k_m}}{1 + x^{\rho' 2^m}}\prod_{n=0}^{n(\theta')}\left(\frac{1 + x^{k_n}}{1 + x^{\rho' 2^n}}\right).$$

If now there are infinitely many m for which $k^m < 2^m/\theta'' = \rho'' 2^m$ where $\theta'' > \theta$, i.e., $\rho'' < \rho = 1/\theta$, then for these values of m,

$$(1 - x)\sum_{n=0}^{\infty} c_n x^n \geqq \theta' \frac{1 + x^{\rho'' 2^m}}{1 + x^{\rho' 2^m}}\prod_{n=0}^{n(\theta')}\left(\frac{1 + x^{k_n}}{1 + x^{\rho' 2^n}}\right). \tag{3}$$

As is well known [1, §7.5], (1) implies that the left-hand side of (3) tends to θ as $x \to 1$. By putting $x = x_m = 2^{-1/2^m}$ in (3) and letting $m \to \infty$, it follows that

$$\theta \geqq \theta' \frac{1 + (\frac{1}{2})^{\rho''}}{1 + (\frac{1}{2})^{\rho'}}.$$

Since $\theta' < \theta$ is arbitrary, and $\theta \neq 0$, we have

$$1 \geqq \frac{1 + (\frac{1}{2})^{\rho''}}{1 + (\frac{1}{2})^{\rho}},$$

which is impossible for $\rho'' < \rho$. This completes the proof.

Whether or not the theorem is true without the hypothesis $\theta > 0$ is an open question.

We now turn to the construction of some π-sets, including some which do not have a density. Let $k_0, k_1, \cdots, k_{r-1}$ be positive integers with

$$k_0 < k_1 < \cdots < k_{r-1}.$$

Suppose that the 2^r sums which can be formed by adding these integers are all distinct, and denote them by $\sigma_1, \cdots, \sigma_{2^r}$. Let M be an integer such that $\sigma_i \not\equiv \sigma_j \pmod{M}$ for $i \neq j$, and such that $k_0 M > k_{r-1}$. Then define

k_n for all n by the formula $k_n = M^q k_r$, where $n = q\tau + r$, and $0 \leqq r < \tau$. It is easily verified that the resulting set $K = \{k_n\}$ is the basis of a π-set S.

For example, let $k_0 = 2$, $k_1 = 3$. Then $\sigma_1 = 0$, $\sigma_2 = 2$, $\sigma_3 = 3$, $\sigma_4 = 5$. The σ_i are incongruent (mod 4), and so M can be taken equal to 4. The resulting basis is $K = \{2, 3, 8, 12, 32, 48, \cdots\}$, where in general $k_{2i} = 2\cdot 4^i$ and $k_{2i+1} = 3\cdot 4^i$. In this case

$$\tfrac{1}{2} = \liminf_{n\to\infty} 2^n/k_n < \limsup_{n\to\infty} 2^n/k_n = \tfrac{2}{3}$$

and by our theorem, S cannot have a density unless it has density zero, since $\lim 2^n/k_n$ fails to exist. It is easy to prove that S does not have density zero, hence any density whatever, either directly or in the following manner. By a simple modification of the proof of our theorem, it can be proved that if

$$\theta_1 = \liminf 2^n/k_n \quad \text{and} \quad \theta_2 = \limsup 2^n/k_n,$$

then

$$\limsup_{x\to 1-} (1 - x) \sum_{n=0}^{\infty} c_n x^n \geqq \theta_1 \frac{1 + r^{1/\theta_2}}{1 + r^{1/\theta_1}}$$

for any r in the open interval $(0, 1)$. If S had density zero, we would have $\lim\, (1 - x) \sum c_n x^n = 0$, which is impossible since $\theta_1 = \frac{1}{2}$ and $\theta_2 = \frac{2}{3}$.

More generally, for the π-sets of this special type,

$$2^n/k_n = 2^{q\tau+r}/M^q k_r.$$

It is impossible that $M < 2^\tau$, since $\sigma_i \equiv \sigma_j \pmod{M}$ would then hold for some $i \neq j$ by Dirichlet's box principle. If $M > 2^\tau$, then $2^n/k_n \to 0$, and so S has density 0 by Wintner's theorem. If $M = 2^\tau$, then $2^n/k_n = 2^r/k_r$, and hence $2^n/k_n \to \theta$ if and only if $k_r = 2^r/\theta$ for $r = 0, 1, \cdots, \tau - 1$. In this case S consists of all multiples of $1/\theta$, a particularly simple example. All other such π-sets with $M = 2^\tau$ fail to have a density.

References

1. E. C. Titchmarsh, *The theory of functions*, 2nd ed., London, Oxford University Press, 1939.
2. A. Wintner, *On restricted partitions with a basis of uniqueness*, Revista de la Unión Matemática Argentina, vol. 13 (1948), pp. 99–105.

University of California
Los Angeles, California
Redstone Arsenal
Huntsville, Alabama

University of Illinois
Urbana, Illinois