HOMOLOGY OPERATIONS AND LOOP SPACES

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I. Introduction

Kudo and Araki [4] have computed the homology ring with coefficients integers mod 2, of the iterated loop spaces of an n-sphere. Their technique involved the definition of homology operations in so-called H_n-spaces. We give here a new treatment of these homology operations which leads us to a new operation of two variables, defined for all coefficient domains. This is done in Sections II and III. In Section IV we apply results of II and III to calculate the homology ring of the iterated loop spaces of iterated suspensions of a space mod 2, in terms of the homology of the original space mod 2, with the number of loop spaces less than or equal to the number of suspensions. The cohomology ring is also computed mod 2, if the number of loop spaces is less than the number of suspensions.

Some of the results of II and III may be applied to coefficients other than the integers mod 2. This will be done elsewhere [8].

The definition of loop space employed will be that of Moore (see [2], 22).

I would like to express my warm appreciation to Professor J. C. Moore. This paper is part of a dissertation written under his direction, presented to Princeton University.

II. H_n-spaces

In Sections II and III we reformulate the results of Kudo and Araki [4] in such a way that the techniques of Steenrod for defining cohomology operations [6] can be applied to obtain homology operations in H_n-spaces. In the course of this, besides the operations of one variable mod 2 of Kudo and Araki, we get a new operation of two variables which is defined for any coefficient domain.

Let π = the symmetric group on two letters. Then if X is any space, π acts on $X \times X$ by permuting the two coordinates. The group π also acts on the n-sphere S^n by the antipodal map. If π acts on two spaces M and N, let π act on $M \times N$ by $T(x, y) = (Tx, Ty)$, for $T \in \pi$.

DEFINITION. A space X is called an H_n-space if there exists an *equivariant* map

$$(1) \qquad \phi: S^n \times (X \times X) \to X,$$

where π acts trivially on the right, such that there is an element $e \in X$ such

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that for any $t \in S^n$, $x \in X$

$$\phi(t, (e, x)) = \phi(t, (x, e)) = x.$$

We will call ϕ the structure map of X.

Examples. 1. An H-space X is an H_0-space where $\phi: S^0 \times X \times X \to X$ is defined by $\phi(1, (x, y)) = xy$ and $\phi(-1, (x, y)) = yx$.

2. Let X be a homotopy-commutative H-space, i.e., $h: I \times X \times X \to X$ such that $h(0, x, y) = xy$ and $h(1, x, y) = yx$ and $h(t, e, x) = h(t, x, e) = x$. Then X is an H_1-space where (if $S^1 = \{e^{i\theta\pi} \mid 0 \leqq \theta \leqq 2\}$) $\phi: S^1 \times X \times X \to X$ is defined by

$$\phi(\theta, x, y) = h(\theta, x, y) \qquad \text{for} \quad 0 \leqq \theta \leqq 1,$$

$$\phi(\theta, x, y) = h(\theta - 1, y, x) \quad \text{for} \quad 1 \leqq \theta \leqq 2.$$

3. If X is a *commutative* H-space, then it is trivially an H_n-space for every n, where ϕ is defined by the diagram

$$\begin{tikzcd} S^n \times (X \times X) \arrow[d, "p_2"'] \arrow[rd, "\phi"] & \\ X \times X \arrow[r, "\mu"] & X, \end{tikzcd}$$

where p_2 is projection on the second factor and μ is multiplication.

The space of paths P of a space X is the set of all pairs (f, r) where f is a map of the positive real numbers R^+ into X such that $f(t) = f(r)$ if $t \geqq r$. We define a map $h: P \to X^I \times R^+$ (I = the unit interval) by $h(f, r) = (f', r)$ where $f': I \to X$ is defined by $f'(t) = f(tr)$, $0 \leqq t \leqq 1$. Topologize P so that h is a homeomorphism. Define two maps, p_1 and $p_2: P \to X$, by $p_1(f, r) = f(0)$ and $p_2(f, r) = f(r)$. Then p_1 and p_2 are fibre maps, and we define E = the space of paths beginning at $x_0 \in X$ to be $p_1^{-1}(x_0)$. Then $p = p_2 \mid E: E \to X$ is a fibre map, and we define the space of loops Ω of X based at x_0 to be the fibre of p over x_0, i.e., $\Omega = p^{-1}(x_0)$.

The importance of H_n-spaces arises from the following theorem due to Kudo and Araki.

THEOREM 1. *Let X be an H_n-space. Then E, the space of paths over X beginning at the base point e is also an H_n-space such that $p: E \to X$, the projection is a map of H_n-spaces, and there is a map $\bar{\phi}: E^{n+1} \times \Omega \times E \to E$ such that*

(1) $\bar{\phi} \mid S^n = \phi'$, *where ϕ' is the structure map of E,*

(2) $\bar{\phi}(\eta, \bar{e}, x) = \bar{\phi}(\eta, x, \bar{e}) = x$

where $\bar{e}$ is the path stationary at e, $\eta \in E^{n+1}$,

(3) $p\bar{\phi}(\eta, x, y) = p(y)$.

Finally it follows from (3) *that Ω is an H_{n+1}-space.*

(*Note.* For $n = 0$, this is the familiar theorem that the loops of an H-space are homotopy-commutative.)

Proof. The space of paths E is made into an H_n-space in the obvious way. Let (f, r), $(g, s) \in E$. Then for each $t \in S^n$ define $k(t): R^+ \to X$ by $k(t)(\tau) = \phi(t, f(\tau), g(\tau))$, where ϕ is the structure map for the H_n-space X. Then $k(t)$ is a continuous map, $k(t)(0) = e$ the base point. If $\tau > u = \max(r, s)$, then $k(t)(\tau) = k(t)(u)$. So $(k(t), u) \in E$. Further we have

$$\begin{aligned} p(k(t), u) &= k(t)(u) = \phi(t, f(u), g(u)) \\ &= \phi(t, p(f, r), p(g, s)) \quad \text{since } u = \max(r, s). \end{aligned}$$

So we define $\phi': S^n \times E \times E \to E$ by

$$\phi'(t, (f, r), (g, s)) = (k(t), u).$$

It is easily verified that ϕ' is continuous, and E is an H_n-space under ϕ' such that p is a map of H_n-spaces, i.e., $p\phi' = \phi(1 \times p \times p)$.

We will denote the upper hemisphere of S^n by E^n_+, the lower by E^n_-. We construct a map $\mu: I \times E^n \times \Omega \times E \to E$.

Define $P: I \times E \times E \to E \times E$ as follows:

$$P(t, (f, r), (g, s)) = ((f, r), (h, s + tr))$$

where

$$\begin{aligned} h(\tau) &= g(\tau - tr) \quad \text{if} \quad \tau \geqq tr, \\ &= e \quad \text{if} \quad \tau \leqq tr, \end{aligned}$$

where

$$\tau \in R^+, \quad (f, r), (g, s) \in E, \quad t \in I.$$

Consider E^n as the flat disk bounding the equator of S^n. Let p^+ be the projection of E^n up onto E^n_+, p^- that down on E^n_-.

Define $\mu: I \times E^n \times \Omega \times E \to E$ by

$$\begin{aligned} \mu(t, \xi, x, y) &= \phi'(p^+\xi, P(2t, x, y)) \quad \text{if} \quad 0 \leqq t \leqq \tfrac{1}{2}, \\ &= \phi'(p^-\xi, P(2 - 2t, x, y)) \quad \text{if} \quad \tfrac{1}{2} \leqq t \leqq 1, \end{aligned}$$

where $x \in \Omega$, $y \in E$, $t \in I$, $\xi \in E^n$. For $t = \frac{1}{2}$ we have $\phi'(p^+\xi, P(1, x, y))$ from the first definition, $\phi'(p^-\xi, P(1, x, y))$ from the second. Let $x = (f, r)$, $y = (g, s)$. Then

$$\begin{aligned} \phi'(p^+\xi, P(1, (f, r), (g, s)))(\tau) &= \phi'(p^+\xi, f(\tau), e) = f(\tau) \quad \text{if} \quad \tau \leqq r, \\ &= \phi'(p^+\xi, e, g(\tau)) = g(\tau) \quad \text{if} \quad \tau \geqq r, \end{aligned}$$

and similarly for $\phi'(p^-\xi, P(1, x, y))$. Thus the two definitions coincide for $t = \frac{1}{2}$. Since each definition is continuous, μ is continuous in its whole domain.

It is clear also that

$$\mu(t, \xi, \bar{e}, y) = y \quad \text{and} \quad \mu(t, \xi, x, \bar{e}) = x,$$

where $\bar{e}$ is the constant map $(e, 0)$. On $1 \times E^n$ and $0 \times E^n$, μ coincides with ϕ' on E^n_+ and E^n_-, respectively.

If C is a cell, define bC = boundary of C.

We now define a homotopy of μ on $b(I \times E^n) \times \Omega \times E$, i.e., a map

$$\bar{\mu}: I \times b(I \times E^n) \times \Omega \times E \to E$$

such that $\bar{\mu}(0, \cdots) = \mu$ and

$$\begin{aligned} \bar{\mu}(1, t, \xi, x, y) &= \mu(t, \xi, x, y) \quad \text{if} \quad t = 0, 1, \\ &= \mu(1, \xi, x, y) \quad \text{if} \quad \xi \in S^n. \end{aligned}$$

Then we have a map

$$\mu' = \bar{\mu} \cup \mu: (I \times b(I \times E^n) \times \Omega \times E) \cup (I \times E^n \times \Omega \times E) \to E,$$

but since $I \times b(I \times E^n) \cup (I \times E^n)$ is an $(n + 1)$-cell,

$$\mu': E^{n+1} \times \Omega \times E \to E.$$

Further one can divide S^n, the boundary of E^{n+1}, into three parts: two n-cells $\bar{E}^n_+$ and $\bar{E}^n_-$ such that $\mu' \mid \bar{E}^n_+ = \phi' \mid E^n_+$ and $\mu' \mid \bar{E}^n_- = \phi' \mid E^n_-$, and a set homeomorphic to $I \times S^n$ such that $\mu'(t, \xi) = \mu'(\xi) = \phi'(\xi)$. Then since μ' is constant on each segment $I \times \xi$, μ' defines a map $\bar{\phi}: E^{n+1}_{\#} \times \Omega \times E \to E$ where $E^{n+1}_{\#}$ is the $(n + 1)$-cell gotten by identifying each segment $I \times \xi$ to $1 \times \xi$ in E^{n+1}. Then $\bar{\phi} \mid S^n_{\#} = \phi'$ where $S^n_{\#}$ is the boundary of $E^{n+1}_{\#}$.

Define $\bar{\mu}: I \times b(I \times E^n) \times \Omega \times E \to E$ as follows:

$$\begin{aligned} \bar{\mu}(\alpha, t, \xi, x, y) &= \phi'(p^+\xi, P((1 - \alpha)2t, x, y)) \qquad \text{if} \quad t \leqq \tfrac{1}{2}, \\ &= \phi'(p^-\xi, P((1 - \alpha)(2 - 2t)x, y)) \quad \text{if} \quad t \geqq \tfrac{1}{2}. \end{aligned}$$

It is clear that $\bar{\mu}$ fulfills the conditions above, and that $\bar{e}$ is an identity for $\bar{\mu}$ and thus for $\bar{\phi}$. Hence $\bar{\phi}$ is constructed, and we have only to verify condition (3), that $p\bar{\phi}(t, x, y) = p(y)$. But $p\phi'(\eta, x, y) = \phi(\eta, px, py)$ and

$$(p \times p)P(t, x, y) = (px, py).$$

Hence $p\bar{\phi}(\eta, x, y) = p(y)$ since $p(x) = e$.

Now Ω is an H_n-space under ϕ' since $\phi'(S^n \times \Omega \times \Omega) \subseteqq \Omega$. It follows from (3) that $\bar{\phi}(E^{n+1} \times \Omega \times \Omega) \subseteqq \Omega$. Since $\bar{\phi} \mid S^n = \phi'$ which is equivariant, we define $\tilde{\phi}: S^{n+1} \times \Omega \times \Omega \to \Omega$ by identifying E^{n+1}_+ with E^{n+1} and defining $\tilde{\phi}(\xi, x, y) = \bar{\phi}(\xi, x, y)$, $\xi \in E^{n+1}_+$, and $\tilde{\phi}(T\xi, x, y) = \bar{\phi}(\xi, y, x)$. Hence Ω is an H_{n+1}-space, and the theorem is proved.

The above proof is a modification of a standard proof that the loop space of an H-space (H_0-space) is homotopy-commutative (an H_1-space). The concept of H_n-space is a generalization of homotopy-commutative H-space, and the index n is a measure of how homotopy-commutative the space is. Thus the theorem states in a sense that the loop space of an H-space is one degree more homotopy-commutative than the H-space is.

III. Homology operations in H_n-spaces

Let X be an H_n-space, that is, let there be given a map $\phi: S^n \times (X \times X) \to X$ which is equivariant with respect to the action of π

(the symmetric group on two letters). Let ∇ be the natural map of normalized singular chains

$$\nabla : C(S^n) \otimes C(X) \otimes C(X) \to C(S^n \times X \times X).$$

If $\phi_\natural$ is the chain map induced by ϕ, then $\phi_\natural \circ \nabla = \phi_\#$ is an equivariant chain map $\phi_\# : C(S^n) \otimes C(X) \otimes C(X) \to C(X)$. This induces a map

$$\phi_* : H_*(S^n) \otimes H_*(X; A) \otimes H_*(X; A) \to H_*(X; A),$$

where A is any coefficient domain. Now we can define a homology operation ψ_n of two variables over any coefficient domain. Choose a generator γ of $H_n(S^n)$.

DEFINITION. Let $x \in H_p(X; A)$, $y \in H_y(X; A)$. Define $\psi_n(x, y) \in H_{p+q+n}(X; A)$ by

$$\psi_n(x, y) = \phi_*(\gamma \otimes x \otimes y).$$

If X is an H_0-space (see Example 1, Section II), then

$$\psi_0(x, y) = x * y - (-1)^{pq} y * x,$$

where $*$ is the Pontrjagin product (for one choice of γ).

We now define the operations of Kudo and Araki. Since the action of π on the right side of (1) is trivial, we can factor the map $\phi_\#$ through the collapsed module, i.e., $\phi_\# = \phi_\flat \circ \eta$;

$$C(S^n) \otimes C(X) \otimes C(X) \xrightarrow{\eta} C(S^n) \otimes_\pi (C(X) \otimes C(X)) \xrightarrow{\phi_\flat} C(X).$$

We have here a situation very similar to Steenrod's method of defining cohomology operations [6]. Following Steenrod, if $\bar{u} \in H_q(X; Z_\theta)$ $(Z_0 = Z)$, define an elementary chain complex $M(\theta, q)$ as follows. The chain group $C_r(M) = 0$ if $r \neq q$ or $q - 1$, $C_q(M)$ is infinite cyclic with generator u, $C_{q-1}(M) = 0$ if $\theta = 0$, and $C_{q-1}(M)$ is infinite cyclic with generator v if $\theta \neq 0$. Define $\partial u = \theta v$. Then every chain map $f : M \to C(X)$ defines a homology class $\bar{u} = \{f(u)\} \in H_q(X; Z_\theta)$, and conversely, for every $\bar{u} \in H_q(X; Z_\theta)$, one can choose a chain representative of $\bar{u}$ which gives rise to a map $f : M(q, \theta) \to C(X)$. Hence we have a map

$$f_\# : C(S^n) \otimes M \otimes M \to C(S^n) \otimes C(X) \otimes C(X)$$

which is equivariant and thus leads to a map

$$\bar{f} : C(S^n) \otimes_\pi (M \otimes M) \to C(S^n) \otimes_\pi (C(X) \otimes C(X)),$$

$$\phi_\flat \circ \bar{f} : C(S^n) \otimes_\pi (M \otimes M) \to C(X),$$

which induces $\Phi : H_*(C(S^n) \otimes_\pi (M \otimes M)) \to H_*(X)$. Since $C(S^n)$ is a π-free complex, we can apply the techniques of Steenrod (see [6], Theorem 3.1) to show that any two chain representations of a cycle $\bar{u}$ lead to the same homomorphism Φ.

Now the group $H_*(C(S^n) \otimes_\pi (M \otimes M))$ is simply the homology of P^n = real n-dimensional projective space, with local coefficients in $H_*(M \otimes M)$.

Assume S^n to be subdivided so as to have two cells in each dimension i for $i \leqq n$, compatible with the antipodal map, i.e., e_i and Te_i = the antipodal cell to e_i.

DEFINITION. The m^{th} operation of Kudo and Araki

$$Q_m(\bar{u}) = \{\phi_\flat \bar{f}(e_m \otimes u \otimes u)\} = \Phi(\xi_m),$$

where ξ_m is the generator of $H_m(P^n; A)$, where P^n is the n-dimensional real projective space, and $A = u \otimes u \otimes Z_2$, u as above.

If $\bar{u}$ is even-dimensional and $\theta = 0$, (i.e., $\bar{u} \in H_{2q}(X; Z)$), we are dealing with the homology of P^n with ordinary coefficients, and then $Q_m(\bar{u}) \in H_{4q+m}(X; Z)$ for m odd, while if $\bar{u}$ is odd-dimensional and $\theta = 0$ (the case of twisted coefficients), then $Q_m(\bar{u})$ is an integral cycle for m even.

If X is an H_n-space, the following proposition describes the properties of the operation Q_m for $m \leqq n$ and the relation of Q_n to ψ_n.

PROPOSITION. (1) $Q_0(x) = x^2$, $x \in H_*(X; Z_2)$.

(2) $\beta_2 Q_m(x) = Q_{m-1}(x)$ *if* $m + \dim x \equiv 0 \pmod 2$ *for* $m < n$,

$$\beta_2 Q_n(x) = ((m + \dim x + 1)Q_{n-1}(x)) + \psi_n(\beta_2 x, x) \pmod 2,$$

where β_2 is the Bockstein boundary operator associated with the coefficient sequence $0 \to Z_2 \to Z_4 \to Z_2 \to 0$.[1]

(3) $Q_m(x + y) = Q_m(x) + Q_m(y)$ *if* $m < n$,

$$Q_n(x + y) = Q_n(x) + Q_n(y) + \psi_n(x, y), \qquad x, y \in H_*(X; Z_2).$$

The proof of (1) is obvious.

To prove (2), we assume x is represented by a chain c with $\partial c = 2b$ ($x \in H_q(X; Z_2)$) so that b represents $\beta_2 x$. Then $Q_m(x)$ is represented by $\phi_\#(e_m \otimes c \otimes c)$. Now

$$\begin{aligned}
\partial\phi_\#(e_m \otimes c \otimes c) &= \phi_\#(\partial(e_m \otimes c \otimes c)) \\
&= \phi_\#(\partial e_m \otimes c \otimes c) + (-1)^m \phi_\#(e_m \otimes \partial c \otimes c) \\
&\qquad\qquad + (-1)^{m+q}\phi_\#(e_m \otimes c \otimes \partial c) \\
&= \phi_\#(e_{m-1} \otimes c \otimes c) + (-1)^m \phi_\#(Te_{m-1} \otimes c \otimes c) \\
&\quad + 2(-1)^m \phi_\#(e_m \otimes b \otimes c) + 2(-1)^{m+q}\phi_\#(e_m \otimes c \otimes b) \\
&= \phi_\#(e_{m-1} \otimes c \otimes c) + (-1)^{m+q^2}\phi_\#(e_{m-1} \otimes c \otimes c) \\
&\quad + 2(-1)^m[\phi_\#(e_m \otimes b \otimes c) + (-1)^q \phi_\#(e_m \otimes c \otimes b)].
\end{aligned}$$

[1] In [4], Proposition 4.2, part (iv) should have $1 \leqq i \leqq n - 1$.

Hence if $m + q^2 \equiv m + q \equiv 0 \pmod 2$, then

$$\partial\phi_\#(e_m \otimes c \otimes c) = 2[\phi_\#(e_{m-1} \otimes c \otimes c) + (-1)^m(\phi_\#(e_m \otimes b \otimes c) + (-1)^q\phi_\#(e_m \otimes c \otimes b))],$$

so that $\beta_2 Q_m(x)$ is represented by

$$\phi_\#(e_{m-1} \otimes c \otimes c) + \phi_\#(e_m \otimes b \otimes c) + \phi_\#(Te_m \otimes b \otimes c) \pmod 2,$$

or $\beta_2 Q_m(x) = Q_{m-1}(x) + \psi_m(\beta_2 x, x)$, and since $\psi_m(\ ,\ ) = 0$ identically if $m < n$, we have proved (2).

Part (3) is proved similarly, by taking the chain representing $Q_m(x + y)$.

Clearly the homology operations Q_i and ψ_n are natural in the category of H_n-spaces. That is, if X and Y are H_n-spaces, $f: X \to Y$ such that the diagram

$$\begin{tikzcd} S^n \times X \times X \arrow[r, "\phi"] \arrow[d, "1 \times f \times f"'] & X \arrow[d, "f"] \\ S^n \times Y \times Y \arrow[r, "\phi'"] & Y \end{tikzcd}$$

is commutative, then $Q_i f = f \cdot Q_i$ and $f\psi_n = \psi_n(f \otimes f)$.

The importance of these operations for computing homology rings of loop spaces arises from the following theorem.

Let X be an H_n-space, E the space of paths over X, Ω the loop space of X, $\Omega = p^{-1}(e)$ where $p: E \to X$. Let σ be the homology suspension associated with the acyclic fibre space E. That is, σ is defined by the diagram

$$\begin{tikzcd} H_*(\Omega) \arrow[d, dashed, "\sigma"] & H_*(E, \Omega) \arrow[l, "\partial_*"'] \arrow[d, "p_*"] \\ H_*(X) \arrow[r, "j_*"] & H_*(X, e). \end{tikzcd}$$

The homomorphism ∂_* comes from the exact sequence of the pair (E, Ω) and is an isomorphism since E is contractible, while j_* is the inclusion and is an isomorphism for dimensions greater than two. Hence $\sigma = j_*^{-1} p_* \partial_*^{-1}$ is defined on positive-dimensional elements of $H_*(\Omega)$.

THEOREM 2. *Let X be an H_n-space, E the space of paths of X based at e, Ω the loop space. Then*

(1) $$(-1)^p\sigma\psi_{n+1}(x, y) = \psi_n(\sigma x, \sigma y),$$

(2) $$\sigma Q_i(z) = Q_{i-1}\,\sigma(z),$$

where $x \in H_p(\Omega; A)$, $y \in H_q(\Omega; A)$, $z \in H_r(\Omega; Z_2)$.

Proof. (1) Let $a \in C_{p+1}(E; A)$, $b \in C_{q+1}(E; A)$ be such that $\{\partial a\} = x$, $\{\partial b\} = y$ ($\{\cdots\}$ denotes homology class). Then $\{p_\#(a)\} = \sigma(x)$, $\{p_\#(b)\} = \sigma(y)$, $\{p_\# \phi_\#(g_n \otimes a \otimes b)\} = \psi_n(\sigma x, \sigma y)$, where g_n is the n-cycle of S^n, $g_n = e_+^n + (-1)^n e_-^n$, e_+^n = oriented upper hemisphere, e_-^n = oriented lower hemisphere. Let $h_{n+1} = (n+1)$-chain of E^{n+1} which is bounded by g_n. Then (if $\bar{\phi}_\#$ is the induced chain map of $\bar{\phi}$ composed with ∇)

$$\begin{aligned}\partial\bar{\phi}_\#(h_{n+1} \otimes \partial a \otimes b) &= \bar{\phi}_\#(\partial(h_{n+1} \otimes \partial a \otimes b))\\ &= \bar{\phi}_\#(g_n \otimes \partial a \otimes b + (-1)^{n+1}(-1)^p h_{n+1} \otimes \partial a \otimes \partial b)\\ &= \bar{\phi}_\#(g_n \otimes \partial a \otimes b) + (-1)^{n+p+1}\bar{\phi}_\#(h_{n+1} \otimes \partial a \otimes \partial b).\end{aligned}$$

Similarly,

$$\partial\bar{\phi}_\#(h_{n+1} \otimes \partial b \otimes a) = \bar{\phi}_\#(g_n \otimes \partial b \otimes a) + (-1)^{n+q+1}\bar{\phi}_\#(h_{n+1} \otimes \partial b \otimes \partial a).$$

Now

$$\bar{\phi}_\#(g_n \otimes \partial b \otimes a) = (-1)^n(-1)^{q(p+1)}\bar{\phi}_\#(g_n \otimes a \otimes \partial b).$$

Set

$$\begin{aligned}U = \bar{\phi}_\#(g_n \otimes a \otimes b) - (-1)^n\bar{\phi}_\#(h_{n+1} \otimes \partial a \otimes b)&\\ - (-1)^{(p+1)(q+1)}\bar{\phi}_\#(h_{n+1} \otimes \partial b \otimes a)&.\end{aligned}$$

Then

$$\begin{aligned}\partial U &= (-1)^n\bar{\phi}_\#(g_n \otimes \partial a \otimes b) + (-1)^{n+p+1}\bar{\phi}_\#(g_n \otimes a \otimes \partial b)\\ &\quad - (-1)^n\bar{\phi}_\#(g_n \otimes \partial a \otimes b) - (-1)^{p+1}\bar{\phi}_\#(h_{n+1} \otimes \partial a \otimes \partial b)\\ &\quad - (-1)^{n+p+1}\bar{\phi}_\#(g_n \otimes a \otimes \partial b) - (-1)^{n+pq+p}\bar{\phi}_\#(h_{n+1} \otimes \partial b \otimes \partial a)\\ &= (-1)^p(\bar{\phi}_\#(h_{n+1} \otimes \partial a \otimes \partial b) + (-1)^{n+1}(-1)^{pq}\bar{\phi}_\#(h_{n+1} \otimes \partial b \otimes \partial a))\\ &= (-1)^p\bar{\phi}(g_{n+1} \otimes \partial a \otimes \partial b) \in C(\Omega; A),\end{aligned}$$

and its homology class is $(-1)^p\psi_{n+1}(x, y)$. Finally it is clear from Theorem 1 (3) that $p_\#\bar{\phi}_\#(h_{n+1} \otimes u \otimes c)$ is degenerate, and thus $p_\# U = p_\# \bar{\phi}_\#(g_n \otimes a \otimes b)$ and $\{p_\# U\} = \psi_n(\sigma x, \sigma y)$. Thus (1) is proved.

In this section we work over Z_2, and signs are ignored.

(2) $Q_i(x) = \bar{\phi}_\flat(e_i \otimes u \otimes u)$ where u is cycle representing x. But

$$\bar{\phi}_\flat(e_i \otimes u \otimes u) = \bar{\phi}_\#(e_i^+ \otimes u \otimes u),$$

where e_i^+ is the chain represented by the upper hemisphere of S^i. Let $a \in C_n(E; Z_2)$ such that $\partial a = u$. Then $\{p_\# a\} = \sigma(x)$. Then

$$\{p_\# Q_{i-1}(a)\} = Q_{i-1}(\sigma(x)).$$

Now

$$\begin{aligned}\partial Q_{i-1}(a) &= \partial\bar{\phi}_\#(e_{i-1}^+ \otimes a \otimes a)\\ &= \bar{\phi}_\#(g_{i-2} \otimes a \otimes a) + \bar{\phi}_\#(e_{i-1}^+ \otimes \partial(a \otimes a))\\ &= \bar{\phi}_\#(e_{i-1}^+ \otimes u \otimes a) + \bar{\phi}_\#(e_{i-1}^+ \otimes a \otimes u)\end{aligned}$$

since $\bar{\phi}_\#(g_{i-2} \otimes a \otimes a) = 0$. But $\bar{\phi}_\#(e_{i-1}^+ \otimes a \otimes u) = \bar{\phi}_\#(e_{i-1}^- \otimes u \otimes a)$ and $e_{i-1}^+ + e_{i-1}^- = g_{i-1}$, which implies $\partial Q_{i-1}(a) = \bar{\phi}(g_{i-1} \otimes u \otimes a)$. But

$$\partial\bar{\phi}_\#(h_i \otimes u \otimes a) = \bar{\phi}_\#(g_{i-1} \otimes u \otimes a) + \bar{\phi}_\#(h_i \otimes u \otimes u).$$

So

$$\partial(Q_{i-1}(a) + \bar{\phi}_\#(h_i \otimes u \otimes a)) = \bar{\phi}_\#(h_i \otimes u \otimes u) = Q_i(u).$$

Since $p_\# \bar{\phi}_\#(h_i \otimes u \otimes a)$ is degenerate, we are done.

IV. $H_*(\Omega^n s^n X; Z_2)$, $H^*(\Omega^{n-1} s^n X; Z_2)$

Let sX denote the suspension of X; then there is a canonical map $\Sigma_1 : X \to \Omega sX$. Similarly there is a map $\Sigma_n : X \to \Omega^n s^n X$. These maps have the property that

$$\sigma \Sigma_{n*} = s^n,$$

where Σ_{n*} is the induced map in homology and s = homology suspension associated with the pair (cX, X), $s : H_q(X; A) \to H_{q+1}(sX; A)$. Thus Σ_{n*} is a monomorphism. (See [2], 22).

Now let us assume that $H_*(X; Z_2)$ is finite in each dimension, and that X is arcwise connected. Then it is well known that

$$H_*(\Omega s^n X; Z_2) = T(H_*(s^{n-1}X; Z_2)), \qquad n \geqq 1,$$

where $T(M)$ = tensor algebra of a graded module M over Z_2 (see, e.g., [2], 22 or [1]). One can write $T(M) = \otimes_i P(x_i)$ as Z_2-modules, where the $\{x_i\}$ form a basis of the graded Lie algebra generated by M in $T(M)$ (see for instance [3]). This is actually a special case of the Poincaré-Birkhoff-Witt Theorem. Then where $M = H_*(s^{n-1}X; Z_2)$, $n > 1$, each x_i is transgressive. In fact if $x_i = [a_1, [a_2, [a_3, \cdots], \cdots]]$, where $a_i \in H_*(s^{n-1}X; Z_2)$, and if $\sigma^n \alpha_i = a_i$, $\alpha_i \in H_*(X; Z_2) \subseteqq H_*(\Omega^n s^n X; Z_2)$, set

$$\xi_i = \psi_n(\alpha_1, \psi_n(\alpha_2, \psi_n(\alpha_3, \cdots), \cdots)).$$

Then $\sigma^n \xi_i = x_i$. We may now calculate $H_*(\Omega^n s^n X; Z_2)$ in the manner of Kudo and Araki. For the set $\{x_i^{2^r}\}$ forms a simple system of transgressive generators for $H_*(\Omega s^n X; Z_2)$.

THEOREM 3. *$H_*(\Omega^n s^n X; Z_2) = P(Q(H_*(X; Z_2)))$, $n \geqq 2$, where $P(M)$ = the graded polynomial ring generated over Z_2 by the module M, and $Q(H_*(X; Z_2))$ = submodule of $H_*(\Omega^n s^n X; Z_2)$ generated by all elements*

$$Q_1^{i_1} \cdots Q_{n-1}^{i_{n-1}}(\xi_j),$$

where ξ_j is defined above, Q_i are the operations of Kudo and Araki (see Section III), $(i_1, \cdots, i_{n-1})$ is any sequence of nonnegative integers (Q_i^0 = identity).

The proof proceeds by induction on n, and uses repeatedly the following comparison theorem.

THEOREM. *Let E^r, $'E^r$, $r \geqq 2$ be two spectral sequences over a field K,*

$$E^2 = E^2_{0,*} \otimes E^2_{*,0}, \qquad 'E^2 = 'E^2_{0,*} \otimes 'E^2_{*,0},$$

and let $\phi_r: E^r \to 'E^r$ *be a map of spectral sequences,* $r \geqq 2$. *Then*

(i) $\phi_\infty: E^\infty \to 'E^\infty$ *an isomorphism and*
(ii) $\phi_2 \mid E^2_{*,0}: E^2_{*,0} \to 'E^2_{*,0}$ *an isomorphism*

imply that

$$\phi_2 \mid E^2_{0,*}: E^2_{0,*} \to 'E^2_{0,*}$$

is an isomorphism also.

This is a special case of a more general theorem, and will not be proved here (see [4] and [7]).

Proof of Theorem 3. By Theorem 2 we have

$$\sigma\, Q_1^{i_1} \cdots Q_{n-1}^{i_{n-1}} (\xi_i) = (Q_1^{i_2} \cdots Q_{n-2}^{i_{n-1}} (\sigma\, \xi_i))^{2^{i_1}}.$$

By induction, these elements are a simple system of generators of

$$H_*(\Omega^{n-1} s^n X; Z_2);$$

hence σ is a monomorphism on $Q(H_*(X; Z_2))$, and in particular the elements of $Q(H_*(X;Z_2))$ are linearly independent. Now define a differential graded filtered algebra

$$A = P(Q(H_*(X;Z_2))) \otimes H_*(\Omega^{n-1} s^n X; Z_2),$$

filtering by degree in the second factor, and setting the boundary $d = \sigma^{-1}$ on the image of σ in the second factor, $d = 0$ in the first, and extending the definition by making d a derivation. Then $E^2(A) = A$, and we define a map ϕ of $E^r(A)$ into $'E^r =$ spectral sequence of the space of paths over $\Omega^{n-1} s^n X$ for $r = 2$, by setting $\phi =$ identity on $H_*(\Omega^{n-1} s^n X; Z_2)$, mapping $Q(H_*(X;Z_2))$ by inclusion into $H_*(\Omega^n s^n X; Z_2)$, and extending by multiplication.

Thus we have a map ϕ of graded filtered algebras over Z_2, and it is clear that ϕ is a map of spectral sequences. Both E^∞ and $'E^\infty$ are zero, and $\phi: E^2_{*,0} \to 'E^2_{*,0}$ is the identity isomorphism. Then by the comparison theorem, $\phi: E^2_{0,*} \to 'E^2_{0,*}$ is an isomorphism, and the theorem is proved.

Since the generators of $H_*(\Omega^{n-1} s^n X; Z_2)$ are images under σ, they are primitive in the Hopf-algebra structure, that is, $\Delta_*(x) = x \otimes 1 + 1 \otimes x$ where $\Delta: \Omega^{n-1} s^n X \to (\Omega^{n-1} s^n X) \times (\Omega^{n-1} s^n X)$ is the diagonal map, $\Delta(\eta) = (\eta, \eta)$, $\eta \in \Omega^{n-1} s^n X$, and where x is a generator of $H_*(\Omega^{n-1} s^n X; Z_2)$. For if $\sigma y = x$, $y \in H_*(\Omega^n s^n X; Z_2)$, and if we represent the singular cycle y as the image of a cycle on a polyhedron under a continuous map, then x is the image of suspension of the cycle on the suspension of the polyhedron, and since all cycles on a suspension are primitive, x is primitive. Then since $H_*(\Omega^{n-1} s^n X; Z_2)$ is the tensor product of polynomial rings with primitive generators, the dual algebra, $H^*(\Omega^{n-1} s^n X; Z_2)$ is simply the tensor product of the dual algebra of a polynomial ring. If $A = P(x)$ over Z_2, $\bar{y} \in \mathrm{Hom}(A, Z_2) = A^*$, with $\bar{y}(y) = 1$, $y \in A$, then $A^* = E(\bar{x}, \bar{x}^2, \bar{x}^4, \cdots) =$ the graded exterior algebra generated by the dual elements to the $(2^n)^{\text{th}}$ powers of x (see [5]).

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