ON COVERING DIMENSION AND INVERSE LIMITS OF COMPACT SPACES

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In 1937 H. Freudenthal proved that every metrizable compact space X is homeomorphic with the inverse limit of an inverse sequence of compact polyhedra P_i, whose dimension $\dim P_i \leqq \dim X$ ([3], Satz 1, p. 229).[1] In this paper we ourselves propose to generalize Freudenthal's theorem to the case of Hausdorff compact spaces. Throughout the paper dimension is taken in the sense of finite open coverings and is denoted by dim.

It is well known that Hausdorff compact spaces can be characterized as inverse limits of inverse systems (over general directed sets) of compact polyhedra (see [2], Theorem 10.1, p. 284). This fact, together with Freudenthal's theorem, leads naturally to the conjecture that every compact Hausdorff space X is homeomorphic with the limit of an inverse system of compact polyhedra P_α, subjected to the additional requirement $\dim P_\alpha \leqq \dim X$. However, this conjecture is shown false in Section 5 of this paper, where we produce examples of 1-dimensional Hausdorff compact spaces which are not expressible as limits of polyhedra P_α with $\dim P_\alpha \leqq 1$.

Nevertheless, in Section 3 we show that every Hausdorff compact space X is an inverse limit of metrizable compacta X_α with $\dim X_\alpha \leqq \dim X$ (Theorem 1). Combining this result with the theorem of Freudenthal we conclude that every Hausdorff compact space X is a double iterated inverse limit of polyhedra $P_{\alpha i}$, satisfying $\dim P_{\alpha i} \leqq \dim X$.

Section 4 is devoted to another generalization of Freudenthal's theorem. This time we prove that every nonmetrizable Hausdorff compact space X can be obtained as the inverse limit of a well-ordered system of Hausdorff compact spaces X_α, where $\dim X_\alpha \leqq \dim X$, and in addition the weight[2] $w(X_\alpha)$ of every X_α is strictly smaller than the weight $w(X)$ of X (Theorem 3).

The proofs of Theorems 1 and 3 depend on establishing the existence of a factorization of mappings $f: X \rightarrow Y$ through a compact space Q, satisfying $\dim Q \leqq \dim X$, $w(Q) \leqq w(Y)$. The first results of this kind are proved in Section 2 (Lemmas 3 and 4); the question is resumed in Section 3. From one of our factorization theorems follows a recent result of E. Sklyarenko on the compactification of normal spaces (Corollary 3).

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[1] Actually, Freudenthal proved a stronger statement, giving additional information concerning the nature of the bonding maps that appear in the sequence. In particular, all the maps can be assumed to be onto.

[2] The definition of weight is given in Section 1.

1. Preliminaries

1. All spaces in this paper are topological Hausdorff spaces. By a polyhedron we understand a triangulable compactum. Whenever f and g are two mappings into a metric space with metric d, then we denote Sup $(d(f(x), g(x)))$ by $d(f, g)$. $k(S)$ denotes the cardinal number of the set S.

By a covering we always mean a finite open covering $u = \{U_i\}$. To every covering u belongs the corresponding nerve $N(u)$; dimension of a covering u is the dimension of the nerve $N(u)$. We denote by $|N(u)|$ the geometric realization of $N(u)$. Coverings are ordered by the relation $u < v$, which means that v refines u. A space X is said to have (covering) dimension $\dim X \leqq n$ if the set of coverings u of dimension $\leqq n$ is cofinal in the set of all coverings (with respect to $<$).[3] If $Y \subset X$ is a closed subset, then $\dim Y \leqq \dim X$.

Whenever $u < v$, there is at least one natural projection

$$p_{uv}: |N(v)| \to |N(u)|,$$

induced by a simplicial mapping $p_{uv}: N(v) \to N(u)$ (see [2], Definition 2.8, p. 234). To every mapping $\varphi: X \to |N(u)|$, belongs a system of continuous real-valued functions φ_i, where $\varphi_i(x)$ is the barycentric coordinate of the point $\varphi(x)$ corresponding to the vertex $U_i \in N(u)$. A mapping φ is said to be canonical with respect to the covering u if the set $\Phi_i = \{x \mid x \in X, \varphi_i(x) \neq 0\}$ is contained in U_i. Every covering of a normal space admits canonical mappings (see [2], Theorem 118, p. 286).[4]

2. The weight of a space X is the least cardinal which is the cardinal number of a basis of open sets for the topology of X; we denote the weight of X by $w(X)$. If $w(X)$ is finite, then X is a finite set of points; $w(X) \leqq \aleph_0$ means that X satisfies the second axiom of countability. If $w(x)$ is infinite, then it is an aleph, $w(X) = \aleph_{\tau(X)}$.

If β is any ordinal, let I^β denote the Cartesian product $\prod I_\alpha$ of copies $I_\alpha = I$ of the unit segment [0, 1], where α ranges through the set of all ordinals $\alpha < \beta$. For infinite β, $w(I^\beta)$ is the cardinal $k(\beta)$ belonging to the ordinal β, i.e., $k(\{\alpha \mid \alpha < \beta\})$. Let $I^{w(X)}$ denote I^β, with $\beta = \omega_{\tau(X)}$, where $\omega_{\tau(X)}$ is the initial ordinal number belonging to $\aleph_{\tau(X)} = w(X)$. In other words, $I^{w(X)}$ is a product of segments I_α, where α ranges through a set of cardinality $w(X)$. A well-known theorem of A. Tychonoff asserts that every completely

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[3] We recall that, for Hausdorff compact spaces X and finite (covering) dimension, $\dim X$ coincides with the cohomological dimension (with integer coefficients) (see e.g. [1], Theorem 5.1, p. 31).

[4] Observe that the nerves of coverings of a space X together with the corresponding projections do not form an inverse system, because the projections are not unique. Notice also that the canonical mappings are not unique.

regular space X can be homeomorphically imbedded in $I^{w(X)}$ ([10], Proposition 2, p. 550).

3. A compact (Hausdorff) space X^* is said to be the Čech compactification of a space X provided X is a dense subset of X^* and every map $f: X \rightarrow I$ admits an extension $f^*: X^* \rightarrow I$. Every completely regular space X admits a unique Čech compactification. Every map f of a completely regular space X into a compact space Y admits an extension $f^*: X^* \rightarrow Y$ (see 8, Chapter X of [2]). If X is normal, then $\dim X = \dim X^*$ (see e.g. [1], Corollary 6.3, p. 35 and [5], Proposition 5, p. 84).

4. Let $(A, <)$ be a directed set, and $\{X_\alpha, \pi_{\alpha\alpha'}\}$, $\alpha < \alpha'$, $\alpha, \alpha' \in A$, an inverse system of spaces ($\pi_{\alpha\alpha}$ is the identity). We denote by $\lim X_\alpha$ the associated inverse limit, and by π^α the natural projections of $\lim X_\alpha$ into X_α.[5] A basis for the topology of $\lim X_\alpha$ is given by sets of the form $(\pi^\alpha)^{-1}(U_\alpha)$, where $\alpha \in A$ and U_α ranges through a basis for X_α. If all X_α are Hausdorff compact spaces, then so is $\lim X_\alpha$.

If k_1 and k_2 are infinite cardinals and $k(A) \leqq k_1$, while $w(X_\alpha) \leqq k_2$ for all $\alpha \in A$, then clearly

$$w(\lim X_\alpha) \leqq \operatorname{Max}(k_1, k_2). \tag{1}$$

If all X_α are compact, then every covering u of $\lim X_\alpha$ can be refined by a covering of the form $\{(\pi^\alpha)^{-1}(U_{\alpha i})\}$, where $\alpha \in A$ is fixed and $\{U_{\alpha i}\}$ is a covering of X_α. Consequently, if all X_α are Hausdorff compact spaces of dimension $\dim X_\alpha \leqq n$, then

$$\dim(\lim X_\alpha) \leqq n. \tag{2}$$

If the directed set A is the set of positive integers, then we speak of an inverse sequence $\{X_i, \pi_{ij}\}$, $i = 1, 2, \cdots$; its limit is metrizable and compact.

5. An ordinal γ is said to be of the first kind if it has an immediate predecessor $\gamma - 1$. The remaining ordinals $\gamma \neq 0$ are said to be of the second kind or limit ordinals.

Let γ be any ordinal of the second kind. Then the set $\{\beta \mid \beta < \gamma\}$ of all ordinals strictly smaller than γ is a well-ordered set. We associate to every $\beta < \gamma$ the cube I^β (see **1**, 2). Let $\pi_{\beta\beta'}: I^{\beta'} \rightarrow I^\beta$, $\beta < \beta'$, be the mapping which does not change the first β coordinates t_α, $\alpha < \beta$, of a point $t = \{t_\alpha\} \in I^{\beta'}$, while it sends the remaining coordinates into 0. If $\beta < \beta' < \beta''$, then clearly $\pi_{\beta\beta''} = \pi_{\beta\beta'}\pi_{\beta'\beta''}$. Hence, we have an inverse system $\{I^\beta, \pi_{\beta\beta'}\}$, $\beta < \gamma$. The limit of this system is readily seen to be I^γ; the corresponding projections $\pi^\beta: I^\gamma \rightarrow I^\beta$, are the maps $\pi_{\beta\gamma}$. A particular case is the case of the Hilbert cube I^{ω_0}, where $\gamma = \omega_0$. I^{ω_0} is the limit of finite-dimensional cubes I^i, $i = 1, 2, \cdots$.

[5] These notions are discussed in detail in Chapter VIII of [2].

2. Lemmas on factorization of maps

1. LEMMA 1. *Let X be a (Hausdorff) compact space, P a (compact) polyhedron with a given metric d, $r > 0$ a real number, and $f:X \to P$ a mapping. Then there exists a (compact) polyhedron Q with*

$$\dim Q \leqq \dim X; \tag{1}$$

furthermore, there exist a map $g:X \to Q$, which is onto, and a map $p:Q \to P$, such that

$$d(f, p\,g) \leqq r. \tag{2}$$

$p\,g$ denotes as usual the composite mapping.

Proof. Let K be a triangulation of P of mesh not greater than r. Let a_i be the vertices of K, and let St a_i be the open star of K around the vertex a_i. $\{\text{St } a_i\}$ is an open covering for P, and so is $u = \{f^{-1}(\text{St } a_i)\}$ for X. Let v be a refinement of u of dimension not greater than $\dim X$. Consider the nerve $N(v)$, and let $g':X \to |N(v)|$ be a canonical mapping belonging to v (see **1,** 1). Let $p:|N(v)| \to |K|$ be a simplicial mapping sending each vertex $V_j \in v$ of $N(v)$ into a vertex a_i, having the property that $V_j \subset f^{-1}(\text{St } a_i)$. It is readily seen that the (open) simplex $\sigma(g'(x))$ of $N(v)$, which carries $g'(x)$, is mapped by p into a face of the (open) simplex $\sigma(f(x))$, which carries $f(x)$ in K.

In order to obtain a mapping g for which $g(X)$ will be a subcomplex of $N(v)$, we first consider an open simplex σ of $N(v)$, of the highest dimension, having the property that σ is not entirely covered by $g'(X)$. We choose a point in σ, not belonging to $g'(X)$ and compose g' with a mapping, which is the identity outside σ, while in σ it is the projection from the selected point into the boundary of σ. Repeating this procedure, we arrive at a mapping $g:X \to |N(v)|$, for which $g(X)$ is a subcomplex of $N(v)$ and thus $Q = g(X)$ is a polyhedron. Clearly, the (open) simplex $\sigma(g(x))$ carrying $g(x)$ is a face of $\sigma(g'(x))$ and therefore is mapped by p again into a face of $\sigma(f(x))$. Hence, both $f(x)$ and $p\,g(x)$ lie in the closure of $\sigma(f(x))$, and the distance $d(f(x), p\,g(x))$ is bounded by the mesh of K. This proves (2).

LEMMA 2. *Let X be a (Hausdorff) compact space, P_i, $i = 1, \cdots, n$, a finite collection of (compact) polyhedra with given metrics d_i, $r_i > 0$ real numbers, and $f_i:X \to P_i$ mappings, $i = 1, \cdots, n$. Then there exists a (compact) polyhedron Q with*

$$\dim Q \leqq \dim X; \tag{3}$$

furthermore, there exist a map $g:X \to Q$, which is onto, and mappings $p_i:Q \to P_i$, such that

$$d_i(f, p_i\,g) \leqq r_i, \qquad i = 1, \cdots, n. \tag{4}$$

Proof. Let P be the Cartesian product $P = P_1 \times \cdots \times P_n$, and let

$f: X \to P$ be the mapping $f = f_1 \times \cdots \times f_n$. Let d be the metric in P given by $d(x, y) = d_1(x_1, y_1) + \cdots + d_n(x_n, y_n)$, where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$; notice that $d_i(x_i, y_i) \leqq d(x, y)$, $i = 1, \cdots, n$. Now apply Lemma 1 with $r = \min(r_1, \cdots, r_n)$. One obtains a polyhedron Q satisfying (3) and maps $g: X \to Q$ and $p: Q \to P$ satisfying $g(X) = Q$ and (2). However, p splits into maps $p_i: Q \to P_i$, satisfying

$$d_i(f_i, p_i g) \leqq d(f, p g) \leqq r \leqq r_i, \qquad i = 1, \cdots, n.$$

2. LEMMA 3. *Let X be a (Hausdorff) compact space, I^{ω_0} the Hilbert cube, and $f: X \to I^{\omega_0}$ a mapping. Then there exists an inverse sequence $\{Q_i, q_{ij}\}$ of (compact) polyhedra Q_i satisfying*

$$\dim Q_i \leqq \dim X; \tag{5}$$

furthermore, there exist a mapping $g: X \to Q = \lim Q_i$, which is onto, and a mapping $p: Q \to I^{\omega_0}$, such that

$$f = p g. \tag{6}$$

Proof.

2.1. We know that $I^{\omega_0} = \lim \{I^i, \pi_{ij}\}$ (see **1**, 5). Choose a metric d on I^{ω_0} and a sequence of real numbers $r_i > 0$ satisfying

$$\lim r_i = 0, \quad i \to \infty, \tag{7}$$

and such that every subset $M_j \subset I^j$ of diameter $\operatorname{diam}(M_j) \leqq 2r_j$ is mapped by π_{ij}, $i < j$, into a subset of I^i of diameter not greater than $2^{i-j}r_i$; we write this condition in symbols as follows:

$$\operatorname{diam}(M_j) \leqq 2r_j \Rightarrow \operatorname{diam}(\pi_{ij}(M_j)) \leqq 2^{i-j}r_i, \qquad i < j. \tag{8}$$

Now we shall construct, by induction, a sequence of real numbers $s_i > 0$, and a sequence of polyhedra Q_i with metrics d_i and dimensions $\dim Q_i \leqq \dim X$. Furthermore, we shall construct sequences of maps $g_i: X \to Q_i$ and $q_{ij}: Q_j \to Q_i$, $p_i: Q_i \to I^i$, $i = 1, 2, \cdots$, $i < j$ (see Figure 1) in such a manner that g_i are mappings onto and

$$d_i(g_i, q_{i,i+1} g_{i+1}) \leqq \tfrac{1}{2} s_i, \tag{9}$$

$$d(\pi^i f, p_i g_i) \leqq \tfrac{1}{2} r_i, \tag{10}$$

and in addition, for every set $N_i \subset Q_i$,

$$\operatorname{diam}(N_i) \leqq s_i \Rightarrow \operatorname{diam}(p_i(N_i)) \leqq \tfrac{1}{2} r_i, \tag{11}$$

$$\operatorname{diam}(N_j) \leqq s_j \Rightarrow \operatorname{diam}(q_{ij}(N_j)) \leqq 2^{i-j}s_i, \qquad i < j. \tag{12}$$

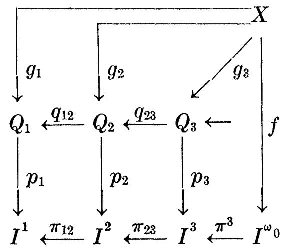

Figure 1.

We start the inductive construction by applying Lemma 1 to X, I^1, $\frac{1}{2} r_1$, and $\pi^1 f$; we obtain Q_1, g_1, and p_1 in accordance with (10). Next we determine s_1 in accordance with (11), using uniform continuity of p_1. Assume now that we have already determined Q_i, s_i, g_i, p_i, and $q_{i'i}$, for $i < k$, $k > 1$, $i' < i$. Then apply Lemma 2 (with $n = 2$) to X, I^k, Q_{k-1}, $\frac{1}{2} r_k$, $\frac{1}{2} s_{k-1}$, $\pi^k f$, and g_{k-1}. One obtains Q_k, g_k, p_k, and $q_{k-1,k}$; q_{ik}, $i < k$, is defined as the composite $q_{i,i+1} \cdots q_{k-1,k}$ (q_{kk} denotes the identity). Finally, s_k is determined in accordance with (11) and (12).

2.2. First we prove, by induction on $j - i$, that

$$(13) \qquad d_i(g_i, q_{ij} g_j) \leqq s_i, \qquad i \leqq j.$$

By assumption of induction $d_{i+1}(g_{i+1}, q_{i+1,j} g_j) \leqq s_{i+1}$, and thus (by (12)) $d_i(q_{i,i+1} g_{i+1}, q_{ij} g_j) \leqq \frac{1}{2} s_i$. This relation and (9) yield (13); moreover, (9) guarantees that (13) is true for $j - i = 1$. Applying (12) to (13) we obtain

$$(14) \qquad d_i(q_{ij} g_j, q_{ik} g_k) \leqq 2^{i-j} s_i, \qquad i < j \leqq k.$$

Consider now the mappings $g^i : X \to Q_i$, defined by

$$(15) \qquad g^i = \lim_{j \to \infty} q_{ij} g_j ;$$

relation (14) guarantees the existence of g^i. Notice that, for $j \to \infty$, (13) goes over into

$$(16) \qquad d_i(g_i, g^i) \leqq s_i .$$

The polyhedra Q_i and maps q_{ij} form an inverse sequence having a metrizable compact Q as its limit; let $q^i : Q \to Q_i$ denote the corresponding projections. It follows from (15) that

$$(17) \qquad g^i = q_{ij} g^j,$$

so that g^i induce a mapping $g : X \to Q$, defined by

$$(18) \qquad g^i = q^i g.$$

In order to prove that $g(X) = Q$, it suffices to show that $g(X)$ is dense in Q, because X is compact. Let $y \in Q$, and let U be an open set of Q containing y; we can assume that $U = (q^i)^{-1}(U_i)$, where U_i is an ε-neighborhood of Q_i around the point $q^i(y)$. Choose j so large that $2^{i-j}s_i < \varepsilon$, and consider the point $q^j(y) \in Q_j$. Since g_j is a mapping onto, there is an $x \in X$ such that $g_j(x) = q^j(y)$. By (16), $d_j(g_j(x), g^j(x)) \leqq s_j$, and therefore (see (12)) $d_i(q_{ij}\, g_j(x),\ q_{ij}\, g^j(x)) \leqq 2^{i-j}s_j < \varepsilon$. Since $q_{ij}\, g_j(x) = q^i(y)$, while $q_{ij}\, g^j(x) = q^i g(x)$, we conclude that $q^i g(x) \in U_i$ and $g(x) \in U$.

2.3. Observe that (16) and (11) imply $d(p_i\, g_i,\ p_i\, g^i) \leqq \frac{1}{2}\, r_i$. This relation combined with (10) yields

$$d(\pi^i f,\ p_i\, g^i) \leqq r_i . \tag{19}$$

Applying (8) to (19) we obtain

$$d(\pi^{i-1} f,\ \pi_{i-1,i}\, p_i\, g^i) \leqq \tfrac{1}{2}\, r_{i-1} . \tag{20}$$

Combining (19) and (20) (replacing i in (20) by $i + 1$) we obtain

$$d(p_i\, g^i,\ \pi_{i,i+1}\, p_{i+1}\, g^{i+1}) \leqq \tfrac{3}{2}\, r_i . \tag{21}$$

Now we can prove (by induction on $j - i$) that

$$d(p_i\, g^i,\ \pi_{ij}\, p_j\, g^j) \leqq 2r_i , \qquad i \leqq j. \tag{22}$$

By assumption of induction $d(p_{i+1}\, g^{i+1},\ \pi_{i+1,j}\, p_j\, g^j) \leqq 2r_{i+1}$, and thus (by (8)) $d(\pi_{i,i+1}\, p_{i+1}\, g^{i+1},\ \pi_{ij}\, p_j\, g^j) \leqq \frac{1}{2}\, r_i$. Combining this relation with (21) one obtains (22); moreover, (21) guarantees that (22) is true for $j - i = 1$. Applying (8) to (22) we obtain

$$d(\pi_{ij}\, p_j\, g^j,\ \pi_{ik}\, p_k\, g^k) \leqq 2^{i-j} r_i , \qquad i < j \leqq k. \tag{23}$$

This relation and (18) guarantee the existence of mappings $p^i : Q \to P_i$, defined by

$$p^i = \lim_{j\to\infty} \pi_{ij}\, p_j\, q^j. \tag{24}$$

Clearly, $p^i = \pi_{ij}\, p^j$, so that p^i induce a mapping $p : Q \to P$, defined by

$$p^i = \pi^i p. \tag{25}$$

Notice that, for $j \to \infty$, (22) goes over into

$$d(p_i\, q^i,\ p^i) \leqq 2r_i . \tag{26}$$

2.4. In order to show that p and g verify (6), choose a fixed $x \in X$ and a fixed $\varepsilon > 0$. Since $f(x) = \lim\, (\pi^i f)(x)$, $(pg)(x) = \lim\, (p^i g)(x)$, and $\lim r_i = 0$ (see (25) and (7)), there is an i such that each of the numbers $d(f(x),\ (\pi^i f)(x))$, $d((pg)(x),\ (p^i g)(x))$, and $3r_i$ is not greater than $\varepsilon/3$. Considering the points $f(x)$, $(\pi^i f)(x)$, $(p_i\, g^i)(x) = (p_i\, q^i g)(x)$, $(p^i g)(x)$, and $(pg)(x)$, and taking into account (19) and (26), we conclude that $d(f(x),\ (pg)(x)) \leqq \varepsilon$. This proves (6).

3. An easy consequence of Lemma 3 is

LEMMA 4. *Let X be a (Hausdorff) compact space, let P_i, $i = 1, \cdots, n$, be a finite collection of metrizable compact spaces, and $f_i : X \to P_i$, $i = 1, \cdots, n$, a collection of mappings. Then there exist a metrizable compact space Q and and mappings $g : X \to Q$, $p_i : Q \to P_i$, $i = 1, \cdots, n$, such that g is onto and*

$$\dim Q \leqq \dim X, \tag{27}$$

$$f_i = p_i g, \qquad i = 1, \cdots, n. \tag{28}$$

Proof. If $n = 1$, the assertion is an immediate consequence of Lemma 3. Indeed, consider P_1 as being homeomorphically imbedded in the Hilbert cube I^{ω_0}, and apply Lemma 3. It follows from (5) that Q satisfies (27) (see **1**, 4).

The case $n > 1$ reduces to the case $n = 1$ by considering the product space $P = P_1 \times \cdots \times P_n$ and the mapping $f = f_1 \times \cdots \times f_n : X \to P$ and applying the lemma (case $n = 1$) to this situation. The mapping $p : Q \to P$ splits into maps $p_i : Q \to P_i$, and (6) implies (28).

4. The theorem of Freudenthal (see footnote 1) is actually contained in Lemma 3. Indeed, let X be a metrizable compact space, and let $f : X \to I^{\omega_0}$ be a homeomorphic imbedding. Then f can be factored through Q, which is the limit of polyhedra Q_i of dimension $\dim Q_i \leqq \dim X$. However, g is a homeomorphism between X and Q, because $g(X) = Q$ and $p\, g = f$.

3. Expansion into inverse systems of metrizable compacta. Factorization theorems

1. In this section we ourselves propose to prove these two theorems:

THEOREM 1. *Every (Hausdorff) compact space X is homeomorphic with the inverse limit of an inverse system of metrizable compacta $\{Q_b, p_{bb'}\}$ with* $\dim Q_b \leqq \dim X$; *b ranges through a directed* set *B of cardinality $k(B) \leqq w(X)$.*[6]

THEOREM 2. *Let X and P be two (Hausdorff) compact spaces and $f : X \to P$ a mapping. Then there exist a (Hausdorff) compact space Q and mappings $g : X \to Q$, $p : Q \to P$ such that g is onto and*

$$\dim Q \leqq \dim X, \tag{1}$$

$$w(Q) \leqq w(P),^{6} \tag{2}$$

$$f = p\, g. \tag{3}$$

If P is metrizable, i.e., $w(P) \leqq \aleph_0$, then the statement of Theorem 2 reduces to case $n = 1$ of Lemma 4.

[6] $w(X)$ denotes the weight of X (see **1**, 2). Notice that if $\dim X$ is finite and $X = \lim Q_b$, then one can always choose a cofinal subsystem ranging over $B' \subset B$ in such a way that $\dim Q_{b'} = \dim X$, $b' \in B'$.

The proofs for both theorems are based on this

LEMMA 5. *Let X be a (Hausdorff) compact space, ω_τ an initial ordinal number, I^{ω_τ} the corresponding cube (see* **1**, 2), *and $f:X \to I^{\omega_\tau}$ a mapping. Then there exists an inverse system $\{Q_b, p_{bb'}\}$, $b \in B$, where Q_b are metrizable compacta with* $\dim Q_b \leqq \dim X$ *and $k(B) = \aleph_\tau$. Furthermore, there exist a mapping $g:X \to Q =$* lim Q_b, *which is onto, and a mapping $p:Q \to I^{\omega_\tau}$, such that $f = p\, g$.*

2. *Proof of Lemma* 5. Let A be the set of all ordinals α strictly smaller than ω_τ; in this proof we disregard the order of A and consider A merely as a set. I^{ω_τ} is the Cartesian product $\prod I_\alpha$, $\alpha \in A$, of copies I_α of the segment $I = [0, 1]$. Let $f_\alpha:X \to I_\alpha$ be the composite of f and of the projection $I^{\omega_\tau} \to I_\alpha$. Let $B = (B, <)$ be the set of all nonempty finite subsets of A, ordered by inclusion $\subset$, and let $B_i \subset B$ consist of all subsets of A having precisely $i + 1$ (different) elements. We can identify A with B_0 in the obvious way. $(B, <)$ is a directed set containing A as the set of initial elements of B ($b_0 \in B$ is initial if it has no predecessors in B other than b_0 itself). Clearly, $B = \bigcup B_i$, $i = 0, 1, \cdots$. Every element $b \in B$ has only finitely many predecessors. The cardinal $k(B) = k(A) = \aleph_\tau$.

Now we shall define, by induction on i, for every $b \in B$, a metrizable compact space Q_b such that whenever $b \in B_i$ and $i > 0$, then $\dim Q_b \leqq \dim X$. Furthermore, we shall define mappings $g_b:X \to Q_b$, which are onto, and mappings $p_{bb'}:Q_{b'} \to Q_b$, $b < b'$, in such a way that $\{Q_b, p_{bb'}\}$ will be an inverse system and that

$$g_b = p_{bb'}\, g_{b'}, \qquad b < b'. \tag{4}$$

We start by setting $Q_\alpha = I_\alpha$, $\alpha \in A = B_0$ and $g_\alpha = f_\alpha$. Assume that Q_b, g_b, $p_{bb'}$ have already been defined (in accordance with our requirements) for all $b, b' \in B_i$, $i < k$, $k > 0$. Take any $b \in B_k$ and consider all its immediate predecessors $b(1), \cdots, b(n)$; there are finitely many of these, and they all belong to B_{k-1}. Apply Lemma 4 to X, all $Q_{b(j)}$, and all $g_{b(j)}:X \to Q_{b(j)}$. One obtains a metrizable compact space Q_b with $\dim Q_b \leqq \dim X$ and a mapping $g_b:X \to Q_b$, which is onto; moreover, one obtains maps

$$p_{b(j),b}:Q_b \to Q_{b(j)}$$

satisfying

$$g_{b(j)} = p_{b(j),b}\, g_b. \tag{5}$$

If $b'' \in B_i$ and $i < k - 1$, we choose a $b(j) \in B_{k-1}$ such that $b'' < b(j)$ and define $p_{b''b}$ by composing $p_{b(j),b}$ with $p_{b'',b(j)}$; this last mapping is by assumption of induction already defined. $p_{b''b}$ is independent of the choice of $b(j)$, because g_b is a mapping onto, and we have (4) and (5).

Let Q be the limit of the inverse system $\{Q_b, p_{bb'}\}$ obtained in this way, and let $p^b:Q \to Q_b$ be the corresponding projections. By (4), the maps

$g_b:X \to Q_b$ induce a mapping $g:X \to Q$, defined by

$$g_b = \overset{b}{p} g, \qquad b \in B. \tag{6}$$

Since all g_b are mappings onto, so is g. Finally, if $b = \alpha \in B_0$, then (6) goes over into

$$f_\alpha = \overset{\alpha}{p} g, \tag{7}$$

proving that $\overset{\alpha}{p}$, $\alpha \in A$, define a mapping $p:Q \to I^{\omega_\tau} = \prod I_\alpha$, satisfying $f = p\,g$. B_0 can now be removed from B without affecting the limit Q.

Remark. The directed set B, which appears in Lemma 5, has a special structure. It is the set of all finite subsets of A having at least two elements; A is any set of cardinality $\aleph_\tau$.

3. *Proof of Theorem* 1. We can assume that the weight of X is infinite and thus $w(X) = \aleph_{\tau(X)}$. Let $f:X \to I^{\omega_{\tau(X)}}$ be a homeomorphic imbedding of X (see **1,** 2). Apply Lemma 5 and observe that $g:X \to Q = \lim Q_b$ is a homeomorphism, because $f = p\,g$ is a homeomorphism and $g(X) = Q$. The above remark also applies to Theorem 1, with $\tau = \tau(X)$.

If we combine Theorem 1 with the theorem of Freudenthal (**2,** 4), then we obtain

COROLLARY 1. *Every (Hausdorff) compact space X is homeomorphic with a double iterated inverse limit* $\lim_b (\lim_i P_{bi})$ *of (compact) polyhedra P_{bi} satisfying* $\dim P_{bi} \leqq \dim X$; *i ranges through positive integers.*

4. *Proof of Theorem* 2. By the theorem of Tychonoff (see **1,** 2) we can consider P as being a subset of $I^{w(P)}$, and therefore $f:X \to I^{w(P)}$. Assuming that $w(P)$ is infinite, we apply Lemma 5 and obtain a factorization of f through $Q = \lim Q_b$, $b \in B$. (1) follows from $\dim Q_b \leqq \dim X$. On the other hand, $w(Q_b) \leqq \aleph_0$ for all $b \in B$, and $k(B) = w(P) \geqq \aleph_0$, so that $w(Q) \leqq w(P)$ (see **1,** 4).

Using properties of the Čech compactification, we can derive from Theorem 2 a factorization theorem for normal spaces:

COROLLARY 2. *Let X be a (Hausdorff) normal space, P a (Hausdorff) compact space, and $f:X \to P$ a mapping. Then there exist a (Hausdorff) compact space Q and mappings $g:X \to Q$, $p:Q \to P$ such that $g(X)$ is dense in Q and* (1), (2), *and* (3) *hold.*

Proof. Let X^* be the Čech compactification of X, and let $f^*:X^* \to P$ be an extension of f. Applying Theorem 2 to this situation, one obtains Q and maps $g^*:X^* \to Q$, $p:Q \to P$ such that $f^* = p\,g^*$. Let $g = g^* \mid X$. Since g^* is onto and X is dense in X^*, it follows that $g(X)$ is dense in Q and that (3) holds. Moreover, we have (2) and $\dim Q \leqq \dim X^* = \dim X$ (see **1,** 3).

From Corollary 2 follows immediately

COROLLARY 3. *Every (Hausdorff) normal space X admits a compactification X' such that* $\dim X' \leqq \dim X$ *and* $w(X') \leqq w(X)$.

Proof. Let $P = I^{w(X)}$, and let f be a homeomorphic imbedding of X into $I^{w(X)}$ (see **1**, 2). Apply Corollary 2 to obtain a compact space $X' = Q$ and a factorization $f = p\,g$. Since f is a homeomorphic imbedding, it follows that g is a homeomorphic imbedding of X into X'. Moreover, $g(X)$ is dense in X', and $\dim X' \leqq \dim X$, $w(X') \leqq w(I^{w(X)}) = w(X)$.

This result has been recently obtained by E. Sklyarenko [9].

Another immediate consequence of Theorem 2 (needed in the sequel) is

COROLLARY 4. *Let X and P_i, $i = 1, \cdots, n$, be a finite collection of (Hausdorff) compact spaces, and let $f_i: X \to P_i$ be maps; we assume that $w(P_i)$ is infinite at least for one i. Then there exist a (Hausdorff) compact space Q and mappings $g: X \to Q$, $p_i: Q \to P_i$, $i = 1, \cdots, n$, such that g is onto and* $\dim Q \leqq \dim X$, $w(Q) \leqq \operatorname{Max}(w(P_1), \cdots, w(P_n))$, $f_i = p_i\, g$, $i = 1, \cdots, n$.

To prove this statement it suffices to consider $P = P_1 \times \cdots \times P_n$ and $f = f_1 \times \cdots \times f_n$ and apply Theorem 2.

4. Expansion of compact spaces into well-ordered inverse systems

1. In this section we prove

THEOREM 3. *Every nonmetrizable (Hausdorff) compact space X is homeomorphic with the inverse limit of an inverse system $\{X_\beta, p_{\beta\beta'}\}$, where β ranges through all the ordinals $\beta < \omega_{\tau(X)}$,*[7] *while X_β are (Hausdorff) compact spaces satisfying*

$$\dim X_\beta \leqq \dim X, \tag{1}$$

$$w(X_\beta) < w(X). \tag{2}$$

Moreover,

$$w(X_\beta) \leqq k(\beta),^{8} \qquad \omega_0 \leqq \beta < \omega_{\tau(X)}. \tag{3}$$

If β is of the second kind,[9] *then*

$$X_\beta = \lim \{X_\alpha, p_{\alpha\alpha'}\}, \qquad \alpha < \beta, \tag{4}$$

$p_{\alpha\beta}: X_\beta \to X_\alpha$ being the corresponding projections.

Proof. Consider the cube I^{ω_τ}, where $\tau = \tau(X)$. I^{ω_τ} is the limit of the inverse system $\{I^\beta, \pi_{\beta\beta'}\}$, $1 \leqq \beta < \omega_\tau$.[9] Let $f: X \to I^{\omega_\tau}$ be a homeomorphic

[7] See **1**, 2.

[8] $k(\beta)$ denotes the cardinal of the set $\{\alpha \mid \alpha < \beta\}$.

[9] See **1**, 5.

imbedding denoted also by f^{ω_τ} (see **1,** 2) and let $f^\beta : X \to I^\beta$ be the composite mapping $\pi^\beta f$, where $\pi^\beta : I^{\omega_\tau} \to I^\beta$ is the corresponding projection.[9] We shall construct, by transfinite induction, for every $\beta \leqq \omega_\tau$ a compact space X_β, a mapping $g_\beta : X \to X_\beta$, which is onto for $\beta > 1$, and a mapping $q_\beta : X_\beta \to I^\beta$; furthermore, for $\beta < \beta' \leqq \omega_\tau$, we shall construct maps $p_{\beta\beta'} : X_{\beta'} \to X_\beta$ in such a way that

(5) $$p_{\beta\beta''} = p_{\beta\beta'} \, p_{\beta'\beta''}, \qquad \beta < \beta' < \beta'',$$

(6) $$g_\beta = p_{\beta\beta'} \, g_{\beta'}, \qquad \beta < \beta',$$

(7) $$f^\beta = q_\beta \, g_\beta .$$

Furthermore, if $\beta > 1$ is of the first kind,[9] we require (1) and

(8) $$w(X_\beta) \leqq \operatorname{Max}\,(w(X_{\beta-1}), k(\beta)),$$

while for β of the second kind we require (4), the projections $X_\beta \to X_\alpha$ being $p_{\alpha\beta}$.

We start the construction by setting $X_1 = I^1$, $g_1 = f^1$ and taking the identity for q_1. Now assume that X_α, g_α, $p_{\alpha'\alpha}$, q_α have been already defined for $\alpha < \beta$, in accordance with all our requirements. If β is of the second kind, define X_β by (4), and define $p_{\alpha\beta} : X_\beta \to X_\alpha$ as the corresponding projection $(\lim X_\alpha) \to X_\alpha$; clearly, $p_{\alpha\beta} = p_{\alpha\alpha'} \, p_{\alpha'\beta}$, as required by (5). g_α, $\alpha < \beta$, induce a mapping $g_\beta : X \to (\lim X_\alpha)$ (see (6)), defined by

(9) $$g_\alpha = p_{\alpha\beta} \, g_\beta .$$

The fact that all g_α are onto and (6) guarantee that g_β is also. Now observe that $I^\beta = \lim \{I^\alpha, \pi_{\alpha\alpha'}\}$, $\alpha < \beta$, the projections being $\pi_{\alpha\beta}$ (see **1,** 5). By definition, $f^{\alpha'} = \pi^{\alpha'} f$, and thus $\pi_{\alpha\alpha'} f^{\alpha'} = \pi^\alpha f = f^\alpha$. Therefore, (6) and (7) imply $q_\alpha \, p_{\alpha\alpha'} \, g_{\alpha'} = f^\alpha = \pi_{\alpha\alpha'} f^{\alpha'} = \pi_{\alpha\alpha'} \, q_{\alpha'} \, g_{\alpha'}$. Since $g_{\alpha'}$ is onto, we obtain $q_\alpha \, p_{\alpha\alpha'} = \pi_{\alpha\alpha'} \, q_{\alpha'}$. This shows that mappings $q_\alpha : X_\alpha \to I^\alpha$ induce a mapping $q_\beta : (\lim X_\alpha) \to (\lim I^\alpha)$, defined by

(10) $$\pi_{\alpha\beta} \, q_\beta = q_\alpha \, p_{\alpha\beta} .$$

To show that q_β satisfies (7) it suffices to show that $\pi_{\alpha\beta} f^\beta = \pi_{\alpha\beta} \, q_\beta \, g_\beta$, for all $\alpha < \beta$. However, $\pi_{\alpha\beta} f^\beta = f^\alpha$ and (7) (for $\alpha < \beta$), (9), and (10) imply $f^\alpha = q_\alpha \, g_\alpha = q_\alpha \, p_{\alpha\beta} \, g_\beta = \pi_{\alpha\beta} \, q_\beta \, g_\beta$.

Assume now that β is of the first kind, i.e., that $\beta - 1$ exists. Then we can apply Corollary 4 to compact spaces X, $X_{\beta-1}$, and I^β and to mappings $g_{\beta-1} : X \to X_{\beta-1}$, $f^\beta : X \to I^\beta$; we obtain a compact space X_β satisfying (1) and (8). We obtain also maps g_β, $p_{\beta-1,\beta}$, and q_β in accordance with (6) and (7). We define $p_{\beta'\beta}$, for $\beta' < \beta - 1$, by $p_{\beta'\beta} = p_{\beta',\beta-1} \, p_{\beta-1,\beta}$. This completes the construction.

Notice that $f = f^{\omega_\tau} = q_{\omega_\tau} \, g_{\omega_\tau}$ is by assumption a homeomorphic imbedding, while $g_{\omega_\tau} : X \to X_{\omega_\tau}$ is a mapping onto. Therefore, g_{ω_τ} establishes a homeomorphism between X and $X_{\omega_\tau} = \lim \{X_\beta, p_{\beta\beta'}\}$, $\beta < \omega_\tau$. That (1) holds for all β is now proved by transfinite induction. By assumption of our construction, (1) is true for all $\beta > 1$ of the first kind. On the other hand, if β is of the second kind, then we have (4), and thus the assumption $\dim X_\alpha \leqq \dim X$, for all $\alpha < \beta$, implies (1) for β (see **1,** 4). To complete the proof of Theorem 3, it remains only to prove (3) and (2). We prove (3) by transfinite induction on β. Since $w(I^i) = \aleph_0$, $i = 1, 2, \cdots$, it follows from (8) that $w(X_i) \leqq \aleph_0$, and thus $w(X_{\omega_0}) \leqq \aleph_0 = k(\omega_0)$, because $X_{\omega_0} = \lim X_i$. Assume now that (3) is true for $\omega_0 \leqq \alpha < \beta$. If β is of the first kind, then (3) follows from (8), because $w(X_{\beta-1}) \leqq k(\beta - 1) \leqq k(\beta)$. If β is of the second kind, then we have (4), and we know that $w(X_\alpha) \leqq k(\alpha) \leqq k(\beta)$. Applying (1) of **1,** 4 we obtain (3) in this case too. (2) is an immediate consequence of (3). Indeed, ω_τ is an initial ordinal, and thus $\beta < \omega_\tau$ implies $k(\beta) < \aleph_\tau = w(X)$.

5. Inverse systems of polyhedra

1. Let Fr V denote the frontier of V; if V is open then $\text{Fr } V = \bar{V} - V$. The inductive dimension of a space X, denoted as ind X, is defined by induction as follows. $\text{ind } X = -1$ if the space is vacuous. $\text{ind } X \leqq n$ if for every $x \in X$ and open set $U \subset X$, $x \in U$, there exists an open set V, $x \in V \subset U$, such that $\text{ind Fr } V \leqq n - 1$. It is readily seen that $\dim X = 0$ implies $\text{ind } X = 0$. For separable metric space ind X and dim X coincide.

LEMMA 6. *Let $\{X_\alpha, p_{\alpha\alpha'}\}$, $\alpha \in A$, be an inverse system of metrizable compacta X_α, having the property that each X_α can be homeomorphically imbedded in a polyhedron P_α of dimension* $\dim P_\alpha \leqq 1$. *Then* $X = \lim X_\alpha$ *satisfies*

$$\text{ind } X \leqq 1. \tag{1}$$

The proof depends on the following proposition.

Let P be a polyhedron of dimension n, let C be a closed subset of P and U an open subset of C. Then $\dim (\text{Fr } U) \leqq n - 1$. This statement is easily derived from the fact that the boundary of an open set in the Euclidean n-space has dimension not greater than $n - 1$ (see [4], Theorem IV 3, p. 44).

Proof of Lemma 6. Let $x \in X = \lim X_\alpha$, and let U be an open set of X, $x \in U$. Choose an open set V, $x \in V \subset U$ of the form $V = (p^\alpha)^{-1}(V_\alpha)$, where V_α is an open set of X_α containing $x_\alpha = p^\alpha(x)$. Let $V_{\alpha'} = (p_{\alpha\alpha'})^{-1}(V_\alpha)$, and let $F_{\alpha'} = X_{\alpha'} \backslash V_{\alpha'}$. Clearly, $p_{\alpha'\alpha''}$ maps $V_{\alpha''}$, $\bar{V}_{\alpha''}$, and $F_{\alpha''}$ into $V_{\alpha'}$, $\bar{V}_{\alpha'}$, and $F_{\alpha'}$, respectively, $\alpha' < \alpha''$, while $p^{\alpha'}$ maps V, $\bar{V}$, and $F = X \backslash V$ into $V_{\alpha'}$, $\bar{V}_{\alpha'}$, and $F_{\alpha'}$, respectively. Since $\bar{V}_{\alpha'} \cap F_{\alpha'} = \text{Fr } V_{\alpha'}$ and $\bar{V} \cap F = \text{Fr } V$, we conclude that $\{\text{Fr } V_{\alpha'}, p_{\alpha'\alpha''}\}$, $\alpha < \alpha' \in A$, is an inverse system whose limit $\lim \text{Fr } V_{\alpha'}$ contains Fr V. By assumption of the lemma and by the above proposition we know that $\dim \text{Fr } V_{\alpha'} \leqq 0$, and thus (by **1,** 4)

$$\dim (\text{Fr } V) \leqq \dim (\lim \text{Fr } V_{\alpha'}) \leqq 0.$$

This implies that $\text{ind } (\text{Fr } V) = 0$ and proves that $\text{ind } X \leqq 1$.

2. In 1949 A. Lunc [7] and then O. V. Lokucievskiĭ [6] established the existence of Hausdorff compact spaces X having $\dim X = 1$ and $\operatorname{ind} X = 2$. In a recent paper P. Vopěnka [11] has extended this result by constructing Hausdorff compact spaces X with $\dim X = m$, $\operatorname{ind} X = n$, for arbitrary integers satisfying $0 < m < n$. These results, together with Lemma 6, prove

THEOREM 4. *There exist 1-dimensional (Hausdorff) compact spaces X, which cannot be obtained as inverse limits of inverse systems $\{P_\alpha, p_{\alpha\alpha'}\}$, where all P_α are (compact) polyhedra of dimension* $\dim P_\alpha \leqq 1$. *In particular, this is the case whenever* $\dim X = 1$, *but* $\operatorname{ind} X > 1$.[10]

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[10] This result has been obtained by the author during the winter of 1957–58 (see Notices Amer. Math. Soc., vol. 5 (1958), p. 785). It has been obtained also by B. Pasynkov [8].