ON THE NORM OF A GROUP

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The object of this note is to simplify and improve on the results of Wos (cf. [3]) about the norm of a group. We prove the following theorem.

THEOREM. *The norm of a group is in the second center of the group; the centralizer of the norm includes the commutator subgroup of the group.*

Proof. It should be recalled that the norm $N(G)$ of a group G (cf. [1]) is the set of elements a such that for each subgroup H of G, $aH = Ha$. Thus the norm comprises precisely those elements a such that aga^{-1} is, for every g in G, a power of g. It follows that if $[a, g]$ designates the commutator $aga^{-1}g^{-1}$, then $[[a, g], g] = 1$ for all g in G. In particular, $[[a, ga], ga] = 1$. But

$$[[a, ga], ga] = [[a, g], ga] = [[a, g], g]g[[a, g], a]g^{-1}$$

(cf. [4], p. 80), and hence $[[a, g], a] = 1$, which implies that a commutes with its conjugates. Thus

(A) *Every element of the norm is in an Abelian normal subgroup of* G; *and* $[a, g^m] = [a, g]^m$ *for every integer* m.

On the basis of the above, we next show that

$$(1) \qquad [[h, a], g^{-1}] = [[g, a], h].$$

For $[a, g] = g^r$, a power of g; $[a, h] = h^s$, a power of h; and $[a, hg] = (hg)^t$, a power of hg. Furthermore $aga^{-1} = g^{r+1}$, $aha^{-1} = h^{s+1}$, $ahga^{-1} = (hg)^{t+1}$, whence it follows that $h^{s+1}g^{r+1} = (hg)^{t+1}$. But h^s and g^r commute because they are in the Abelian normal subgroup spanned by a and its conjugates. It follows that $hg^rh^sg = (hg)^{t+1}$ and

$$g^{r-1}h^sg = (hg)^t = [a, hg] = [a, h]h[a, g]h^{-1} = h^shg^rh^{-1}.$$

Then $h^{-s}g^{-1}h^sg = g^{-r}hg^rh^{-1}$, and since $[a, h]^{-1} = [h, a]$,

$$(1) \qquad [[h, a], g^{-1}] = [[g, a], h],$$

as was to be shown.

Now $[[g, a], h]$ is a power of h since a, and consequently $[g, a]$, belongs to the norm; and the above equation (1) shows that $[[g, a], h]$ is a power of g as well. Hence

(B) $[[g, a], h]$ *is in the center of the group* K *generated by* g *and* h.

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We note next that $[a, g]$ commutes with any conjugate of g; for

$$[[a, g], hgh^{-1}] = [[a, g], h][[a, g], g][[a, g], h^{-1}] = 1.$$

From this and (A) we conclude, since hg is a conjugate of gh, that

$$\begin{aligned}[a, [g, h]] = [a, gh(g^{-1}h^{-1})] &= [a, gh][a, g^{-1}h^{-1}] \\ &= [a, g][g, [a, h]][a, h][a, g^{-1}][g^{-1}, [a, h^{-1}]][a, h^{-1}] \\ &= [g, [a, h]][g^{-1}, [a, h^{-1}]],\end{aligned}$$

and hence by (1)

$$(2) \qquad [a, [g, h]] = [g, [a, h]]^2.$$

From (2) we see that

$$[a, [[g, h], h]] = [[g, h], [a, h]]^2 = [[a, h], [g, h]]^{-2} = [g, [[a, h], h]^2]^{-2} = 1$$

since $[[a, h], h] = 1$. Since a is an arbitrary element of the norm, it follows that $[[g, h], h]$ is in T, the centralizer of the norm, which is a normal subgroup of G. Similarly, $[[h, g], g]$ is in T, and hence $[[g, h], g]$ is in T. Thus modulo $T \cap K$, $[g, h]$ is in the center of K. Hence

(C) *The elements of the third member of the descending central series of* K *commute with the elements of the norm.*

Now we may assume that the norm has only elements of finite order; for if the norm contains an element of infinite order, then it equals the center (cf. [1] and [2]). In order to show that the norm belongs to the second center, we need only show that its elements of prime power order belong to the second center. Accordingly we assume that a has prime power order and shall arrive at a contradiction if we assume that $[[g, a], h]$ is not 1. We do this by picking out generators g and h for K so that the product of the orders of $[a, g]$ and $[a, h]$ is minimal, and then exhibiting generators g and f so that the product of the orders of $[a, g]$ and $[a, f]$ is still lower.

First we note that $[a, g]$ and $[g, [a, h]]$ are both powers of g of p-power order since a and $[a, h]$ are both in an Abelian normal p-subgroup contained in the norm; and hence one of the two elements, $[a, g]$ and $[g, [a, h]]$, is a power of the other. By our assumption, $[a, g]$ does not commute with h, whereas $[g, [a, h]]$ must commute with h by (1); accordingly, we can conclude that for some $t \geqq 1$

$$(3) \qquad [a, g]^{p^t} = [g, [a, h]]^r,$$

where r is prime to p. Similarly,

$$(4) \qquad [a, h]^{p^k} = [g, [a, h]]^s,$$

where s is prime to p, and for definiteness $t \geqq k \geqq 1$. Then let $f = g^m h$, where $mr \equiv -s \pmod p$ if $k = t$, and where $m = p$ if $t > k$. By [4], p. 81,

$$(g^m h)^{p^t} = g^{mp^t} h^{p^t} [h, g^m]^{p^t(p^t-1)/2} z, \tag{5}$$

where z is in the third member of the descending central series of K and hence commutes with a by (C). Then

$$[a, f]^{p^t} = [a, f^{p^t}] = [a, g^{mp^t} h^{p^t} [h, g^m]^{p^t(p^t-1)/2} z].$$

But by (3), (4), (B), and (C),

$$[a, f]^{p^t} = [a, g]^{mp^t} [a, h]^{p^t} [a, [h, g]]^{p^t(p^t-1)m/2}.$$

Then by (3), (4), (2), and the fact that $[a, [h, g]] = [a, [g, h]]^{-1}$,

$$[a, f]^{p^t} = [g, [a, h]]^{rm+sp^t-k-p^t(p^t-1)m}.$$

Thus $[a, f]^{p^t}$ is a power of a positive p^{th} power of $[g, [a, h]]$, and hence $[a, f]$ has lower order than $[a, h]$, as was to be shown. We conclude that $[[g, a], h] = 1$, and hence a is in the second center of G as the theorem asserts.

The second statement of the theorem now follows from (2).

Bibliography

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