THE RADIUS OF UNIVALENCE OF BESSEL FUNCTIONS I[1]

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1. Introduction

In this paper we begin a study of the radius of univalence of Bessel functions. It is necessary to normalize the function, and a natural form is

$$(1.1)\qquad \tilde{J}_\nu(z) = z^{1-\nu}J_\nu(z) = a_1^{(\nu)}z - a_3^{(\nu)}z^3 + a_5^{(\nu)}z^5 - \cdots,$$

where the coefficients $a_{2m+1}^{(\nu)}$ are defined by the recurrence relation

$$(1.2)\qquad a_1^{(\nu)} = 2^{-\nu}/\Gamma(1+\nu), \qquad a_{2m+1}^{(\nu)} = a_{2m-1}^{(\nu)}/4m(\nu+m), \qquad m = 1, 2, \cdots.$$

It is well known that $\tilde{J}_\nu(z)$ is an entire function for any ν. With respect to the normalization factor $z^{1-\nu}$, we note that it is unique in the sense that $1 - \nu$ is the only exponent for which $z^{1-\nu}J_\nu(z)$ is schlicht in some neighborhood of the origin when $\nu > -1$.

The index ν is assumed to be real.

We present here a complete solution for $\nu > -1$. In §5 we state some results for $\nu < -1$ which appear plausible in the light of our computational experiments; we expect to handle these in a later paper.

2. Some general properties of Bessel functions

We require various standard results from the theory of Bessel functions and one from the theory of conformal representation. These will be quoted with references, but without proof. We have quoted Watson [1], but the results will be found in many places, in particular in Erdélyi, Magnus, Oberhettinger, and Tricomi [2].

LEMMA 1. *For $\nu > -1$ the functions $J_\nu(z)$ and $\tilde{J}_\nu(z)$ have infinitely many zeros, and all are real.* Cf. Watson [1], pp. 478, 483.

As usual we shall denote the positive zeros of $\tilde{J}_\nu(z)$, in order of magnitude, as $j_{\nu,1} < j_{\nu,2} < j_{\nu,3} < \cdots$. We note that, in addition, 0 and $-j_{\nu,m}$ $(m = 1, 2, \cdots)$ are zeros of $\tilde{J}_\nu(z)$.

LEMMA 2. *For fixed m, $j_{\nu,m}$ is an increasing function of ν.* Cf. Watson [1], p. 508.

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LEMMA 3. *$\sum_{m=1}^{\infty} j_{\nu,m}^{-2}$ is convergent for any $\nu > -1$.*

Proof. For fixed ν, it follows from the asymptotic formulae that the large zeros are spaced at an interval of approximately π. Convergence follows by comparison with $\sum_{m=1}^{\infty} m^{-2}$. Cf. also Watson [1], pp. 495–497.

The canonical product of $\tilde{J}_\nu(z)$ is

$$\tilde{J}_\nu(z) = (z/2^\nu\Gamma(1+\nu))\prod_{m=1}^{\infty}(1 - z^2 j_{\nu,m}^{-2}). \tag{2.1}$$

Cf. Watson [1], pp. 497–499.

LEMMA 4. *Let C be a simple closed contour and D its interior. Suppose that $f(z)$ is regular in D and continuous in $D \cup C$. Suppose that as z describes C in the positive direction, $w = f(z)$ describes a simple closed contour Γ once. Then Γ is described positively, and $w = f(z)$ gives a one-to-one and conformal representation of D on Δ, the interior of Γ.* Cf. Littlewood [4], p. 121.

3. Determination of the radius of univalence

THEOREM 1. *For $0 \leqq \theta \leqq \frac{1}{2}\pi$, $r < j_{\nu,1}$, and $\nu > -1$, the function*

$$h(\theta) = |\tilde{J}_\nu(re^{i\theta})|$$

increases.

Proof. We have

$$\begin{aligned} h^2(\theta) &= \tilde{J}_\nu(re^{i\theta})\tilde{J}_\nu(re^{-i\theta}) \\ &= (2^{-\nu}/\Gamma(1+\nu))re^{i\theta}\prod_{m=1}^{\infty}(1 - r^2e^{2i\theta}j_{\nu,m}^{-2})\,(2^{-\nu}/\Gamma(1+\nu))\,re^{-i\theta} \\ &\qquad\cdot\prod_{m=1}^{\infty}(1 - r^2e^{-2i\theta}j_{\nu,m}^{-2}) \\ &= (2^{-\nu}/\Gamma(1+\nu))^2r^2\prod_{m=1}^{\infty}(1 - r^2j_{\nu,m}^{-2}\cos 2\theta - ir^2j_{\nu,m}^{-2}\sin 2\theta) \\ &\qquad\cdot(1 - r^2j_{\nu,m}^{-2}\cos 2\theta + ir^2j_{\nu,m}^{-2}\sin 2\theta) \\ &= (2^{-\nu}/\Gamma(1+\nu))^2r^2\prod_{m=1}^{\infty}(1 + r^4j_{\nu,m}^{-4} - 2r^2j_{\nu,m}^{-2}\cos 2\theta). \end{aligned}$$

As θ increases from 0 to $\frac{1}{2}\pi$, $\cos 2\theta$ decreases from 1 to -1. Hence each factor in the last product increases, and so therefore does the product itself, provided each factor is positive—and this is certainly the case when $r < j_{\nu,1}$.

Consider $\tilde{J}_\nu(x)$ for $x \geqq 0$. Since

$$\tilde{J}_\nu(x) = (x/2^\nu\Gamma(1+\nu)) + O(x^3),$$

the function starts by increasing. Since $\tilde{J}_\nu(x)$, $\nu > -1$, has positive zeros, there is a least positive number $\rho_\nu < j_{\nu,1}$ for which $\tilde{J}_\nu(x)$ is maximum. Clearly the radius of univalence cannot exceed ρ_ν because values of $\tilde{J}_\nu(x)$, $x < \rho_\nu$, are repeated for $x > \rho_\nu$. We shall show that ρ_ν is the radius of univalence of $\tilde{J}_\nu(z)$.

By differentiating (2.1) logarithmically we can specify ρ_ν more quantitatively as the least positive zero of the function

$$g_\nu(x) \equiv \frac{d}{dx}[\log_e \tilde{J}_\nu(x)] = \frac{1}{x} - 2x \sum_{m=1}^{\infty} \frac{j_{\nu,m}^{-2}}{1 - x^2 j_{\nu,m}^{-2}}. \tag{3.1}$$

Hence

$$\tfrac{1}{2}\rho_\nu^{-2} = \textstyle\sum_{m=1}^{\infty} (j_{\nu,m}^2 - \rho_\nu^2)^{-1}. \tag{3.2}$$

THEOREM 2. *If* $0 < \theta < \frac{1}{2}\pi$, *then, for* $\nu > -1$,

$$0 < \arg \tilde{J}_\nu(\rho_\nu e^{i\theta}) < \theta.$$

Proof. From (2.1) we obtain

$$P_\nu(\theta) \equiv \rho_\nu^{-1} e^{-i\theta} \tilde{J}_\nu(\rho_\nu e^{i\theta}) = (2^{-\nu}/\Gamma(1+\nu)) \textstyle\prod_{m=1}^{\infty} (1 - \rho_\nu^2 e^{2i\theta} j_{\nu,m}^{-2}).$$

We have to show that

$$-\theta < \arg P_\nu < 0, \quad (0 < \theta < \tfrac{1}{2}\pi, \quad \nu > -1).$$

We set

$$\vartheta_{\nu,m} = \rho_\nu^2 j_{\nu,m}^{-2}, \qquad m = 1, 2, \cdots,$$

and

$$p_{\nu,m} = 1 - \vartheta_{\nu,m} e^{2i\theta}.$$

Then

$$-\arg p_{\nu,m} = \arctan\big(\vartheta_{\nu,m} \sin 2\theta/(1 - \vartheta_{\nu,m} \cos 2\theta)\big).$$

Now for $0 < \theta < \frac{1}{2}\pi$ we have

$$0 < \sin 2\theta < 2\theta \quad \text{and} \quad 1 > \cos 2\theta > -1.$$

Since $\vartheta_{\nu,m} > 0$, we thus obtain

$$\vartheta_{\nu,m} \sin 2\theta/(1 - \vartheta_{\nu,m} \cos 2\theta) < 2\theta\vartheta_{\nu,m}/(1 - \vartheta_{\nu,m}).$$

Since

$$\vartheta_{\nu,m} < 1, \qquad m = 1, 2, \cdots,$$

and $\arctan x < x$ $(x > 0)$, it follows that

$$0 < -\arg p_{\nu,m} < 2\theta\vartheta_{\nu,m}/(1 - \vartheta_{\nu,m}).$$

Summing with respect to m and using (3.2) we have

$$0 < -\arg P_\nu < \theta \quad (0 < \theta < \tfrac{1}{2}\pi, \quad \nu > -1).$$

This proves Theorem 2.

THEOREM 3. *The radius of univalence of* $\tilde{J}_\nu(z)$, $\nu > -1$, *is* ρ_ν.

Proof. We consider $\tilde{J}_\nu(z)$ for any fixed $\nu > -1$. Let C denote the curve consisting of three arcs, namely

the segment $C_1 : 0 \leqq x \leqq \rho_\nu$ of the real axis,
the arc $C_2 : 0 < \theta < \frac{1}{2}\pi$ of the circle $|z| = \rho_\nu$, and
the segment $C_3 : \rho_\nu \geqq y > 0$ of the imaginary axis.

In virtue of what has already been said, the reflection properties of $\tilde{J}_\nu(z)$, and Lemma 4, it is only necessary to prove the map Γ of C by $\tilde{J}_\nu(z)$ has no double points.

Let Γ_i be the map of C_i, $i = 1, 2, 3$. On C_1, the function $\tilde{J}_\nu$ is real and increases steadily with x, since ρ_ν is the first positive maximum of $\tilde{J}_\nu(x)$. Hence Γ_1 is simple. On C_2, the absolute value $|\tilde{J}_\nu|$ increases steadily with θ (cf. Theorem 1). Hence Γ_2 is simple. From the power series in (1.1) it follows that on C_3, $\tilde{J}_\nu$ is purely imaginary, and its imaginary part decreases steadily with decreasing y. Hence Γ_3 is simple. By Theorem 2, $\tilde{J}_\nu$ is genuinely complex on C_2. Hence Γ_1 or Γ_3 cannot have points in common with Γ_2. Since $\tilde{J}_\nu$ is real on C_1 and purely imaginary on C_3, the arcs Γ_1 and Γ_3 cannot have common points. This completes the proof.

4. The radius of univalence ρ_ν considered as a function of ν

We now consider ρ_ν as a function of ν for real values of $\nu > -1$.

THEOREM 4. *For $\nu > -1$, the radius of univalence ρ_ν increases steadily with ν.*

Proof. From (3.1) we obtain

$$g_\mu(x) = g_\nu(x) + 2x\sum_{m=1}^{\infty} \{(j_{\nu,m}^2 - x^2)^{-1} - (j_{\mu,m}^2 - x^2)^{-1}\}. \tag{4.1}$$

From (3.1) and (3.2) it follows that

$$g_\nu(x) > 0 \quad \text{for} \quad 0 \leqq x < \rho_\nu \qquad \text{and} \quad g_\nu(\rho_\nu) = 0.$$

Take $\mu > \nu$. Then, by Lemma 2, $j_{\mu,m} > j_{\nu,m}$, $m = 1, 2, \cdots$, and for $x \leqq \rho_\nu$ the terms of the series in (4.1) are positive. Hence

$$g_\mu(x) > 0 \qquad (0 \leqq x \leqq \rho_\nu),$$

and therefore $\rho_\mu > \rho_\nu$. This completes the proof.

THEOREM 5. *Suppose that ν is real, and $\nu > -1$. Then*

$$\text{(a)} \quad \lim_{\nu\to\infty} \rho_\nu = \infty, \qquad \text{(b)} \quad \lim_{\nu\to-1} \rho_\nu = 0. \tag{4.2}$$

Proof. We prove (4.2a). From (1.1) we obtain

$$\tilde{J}'_\nu(x) = \sum_{m=0}^{\infty} (-1)^m b_{2m}^{(\nu)} x^{2m} \equiv \sum_{m=0}^{\infty} (-1)^m \beta_{2m}^{(\nu)}(x), \tag{4.3}$$

where

$$b_0^{(\nu)} = 2^{-\nu}/\Gamma(1+\nu), \qquad b_{2m}^{(\nu)} = \frac{2m+1}{2m-1}\frac{1}{4m(\nu+m)} b_{2m-2}^{(\nu)}. \tag{4.4}$$

Denoting the sum of the first two terms in (4.3) by $\alpha_\nu(x)$ and the remainder after these terms by $\omega_\nu(x)$, we have

$$\tilde{J}'_\nu(x) = \alpha_\nu(x) + \omega_\nu(x).$$

Let

$$x_0^{(\nu)} = 2\sqrt{(\nu+1)/3}.$$

Then

$$\alpha_\nu(x_0^{(\nu)}) = 0,$$

as follows from (4.3) and (4.4). We prove that for $0 < x \leqq x_0^{(\nu)}$ the absolute values of the terms of $\omega_\nu(x)$ form a monotone decreasing sequence whose limit is zero. From (4.3) and (4.4) we have

$$(4.5) \qquad q_m^{(\nu)}(x) \equiv \frac{\beta_{2m}^{(\nu)}(x)}{\beta_{2m-2}^{(\nu)}(x)} = \frac{2m+1}{2m-1}\frac{x^2}{4m(\nu+m)}.$$

Hence

$$q_m^{(\nu)}(x) \leqq \frac{2m+1}{2m-1}\frac{\nu+1}{3m(\nu+m)} \qquad (0 \leqq x \leqq x_0^{(\nu)}),$$

and thus

$$q_1^{(\nu)}(x) \leqq 1, \qquad q_m^{(\nu)}(x) < 1 \qquad (0 \leqq x \leqq x_0^{(\nu)};\, m = 2, 3, \cdots).$$

Further

$$q_m^{(\nu)} = O(m^{-2}) \qquad (m \to \infty).$$

Hence

$$\omega_\nu(x) > 0 \qquad (0 \leqq x \leqq x_0^{(\nu)}),$$

and therefore

$$\tilde{J}'_\nu(x) > 0 \qquad (0 \leqq x \leqq x_0^{(\nu)}).$$

This means that $\rho_\nu > x_0^{(\nu)}$. Since $x_0^{(\nu)}$ tends to infinity as ν tends to infinity, (4.2a) is proved. We prove (4.2b). When $\nu \sim -1$, it is clear that $\tilde{J}_\nu(z)$ vanishes at points near $x = \pm x_0^{(\nu)}$. Since $x_0^{(\nu)}$ tends to zero as ν tends to -1, those points tend to zero, too. This proves (4.2b).

Similar arguments lead to an upper bound for ρ_ν which shows that ρ_ν $(\nu > -1)$ is of order $\nu^{1/2}$ exactly. We prove that

$$(4.6) \qquad \rho_\nu < x_1^{(\nu)} = \sqrt{12(\nu+2)/5}.$$

For this purpose we consider (4.3), written in the form

$$(4.7) \qquad \tilde{J}'_\nu(x) = a_\nu(x) - b_\nu(x) - \textstyle\sum_{m=3}^{\infty} c_{\nu,m}(x),$$

where

$$a_\nu(x) = 2^{-\nu}/\Gamma(1+\nu) - \beta_2^{(\nu)}(x) + \beta_4^{(\nu)}(x) - \tfrac{2}{3}\beta_6^{(\nu)}(x),$$

$$b_\nu(x) = \tfrac{1}{3}\beta_6^{(\nu)}(x) - \beta_8^{(\nu)}(x), \qquad c_{\nu,m}(x) = \beta_{4m-2}^{(\nu)}(x) - \beta_{4m}^{(\nu)}(x).$$

The above value of $x_1^{(\nu)}$ is chosen as it is the value of x for which

$$2^{-\nu}/\Gamma(1+\nu) - \beta_2^{(\nu)}(x) + \beta_4^{(\nu)}(x)$$

is minimum; this minimum is $2^{-\nu}(\nu - 8)/10\Gamma(\nu+2)$. Using the values

$$b_2^{(\nu)} = 3\cdot 2^{-\nu}/4\Gamma(\nu+2),$$

$$b_4^{(\nu)} = 5\cdot 2^{-\nu}/32\Gamma(\nu+3),$$

$$b_6^{(\nu)} = 7\cdot 2^{-\nu}/384\Gamma(\nu+4),$$

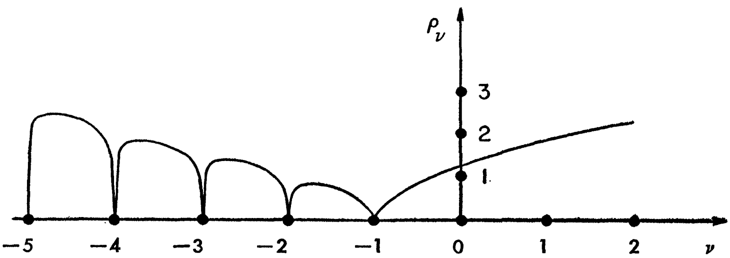

Figure 1

we find that

$$(4.8)\quad \begin{aligned} &2^\nu\Gamma(1+\nu)a_\nu(x_1^{(\nu)}) \\ &= -(17\nu^2+293\nu+768)/250(\nu+1)(\nu+3) < 0 \quad (\nu > -1) \end{aligned}$$

Furthermore,

$$q_m^{(\nu)}(x_1^{(\nu)}) = 3(2m+1)(\nu+2)/5m(2m-1)(\nu+m),$$

as follows from (4.5) and (4.6). Hence

$$q_m^{(\nu)}(x_1^{(\nu)}) < 1/m \qquad (m \geqq 4).$$

and therefore

$$(4.9)\qquad b_\nu(x_1^{(\nu)}) > 0, \qquad c_{\nu,m}(x_1^{(\nu)}) > 0, \qquad m = 3, 4, \cdots .$$

From (4.7)–(4.9) we obtain

$$\tilde{J}'(x_1^{(\nu)}) < 0.$$

From this, (4.6) follows.

5. Numerical results

The results which have been established rigorously above were suggested by the results of a series of calculations made on the Datatron 205 at the California Institute of Technology. Three basic subroutines were prepared; the first produced the coefficients of $\tilde{J}_\nu(z)$ when ν was assigned; the second computed the real and imaginary parts of $\tilde{J}_\nu(\rho e^{i\theta})$ when ρ and θ were assigned; the third computed the extrema of $\tilde{J}_\nu(z)$ on the real and imaginary axes.

With a little experience, reasonable estimates of ρ_ν could be obtained rapidly. Curve plotting equipment would have been convenient. Figure 1 shows ρ_ν in the interval $-5 \leqq \nu \leqq 2$.

Nothing further need be said about the range $\nu > -1$, except to note that the critical point is on the real axis of the z-plane.

In the intervals $-1 > \nu > -2$ and $-3 > \nu > -4$ the critical point appears to be on the imaginary axis. In the intervals $-2 > \nu > -3$ and $-4 > \nu > -5$ the critical point is genuinely complex.

Added in proof November 21, 1959. R. K. Brown [3] has discussed the radius of univalence of $J_\nu(z)$ and $[J_\nu(z)]^{1/\nu}$ for certain complex values of ν (with $\Re\nu > 0$) using the methods of Z. Nehari and M. S. Robertson.

REFERENCES

1. G. N. WATSON, *A treatise on the theory of Bessel functions*, New York, Macmillan, 1944.
2. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. TRICOMI (Bateman Manuscript Project), *Higher transcendental functions, Vol. II*, New York, McGraw-Hill, 1953.
3. R. K. BROWN, *Univalence of Bessel functions*, Notices, Amer. Math. Soc., Abstract 559-118, vol. 6 (1959), p. 512.
4. J. E. LITTLEWOOD, *Lectures on the theory of functions*, Oxford University Press, 1944.

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