LIMITING THEOREMS FOR AGE-DEPENDENT BRANCHING PROCESSES[1]

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1. To motivate the theorems that will be stated and proved here, consider particles which are assumed to have a life span with cumulative probability distribution function $G(t)$. At the end of its life a particle is assumed to split into n particles with probability q_n, where each particle has the same properties as the original. It is assumed $q_n \geqq 0$ and $n \geqq 0$. The generating function associated with $\{q_n\}$ is

$$(1.1) \qquad h(s) = \sum_{j=0}^{\infty} q_j s^j, \qquad h(1) = 1.$$

Given a particle at $t = 0$, let the probability that there are n particles at time $t \geqq 0$ be $p_n(t) \geqq 0$. The generating function is

$$(1.2) \qquad F(s, t) = \sum_0^{\infty} p_j(t) s^j, \qquad F(1, t) = 1.$$

Then the above description suggests that $F(s, t)$ satisfies

$$(1.3) \qquad F(s,t) = \int_{0-}^{t} h(F(s, t - y))\, dG(y) + s[1 - G(t)]\,.$$

This problem with $h(s) = s^2$ and with mild restrictions on $G(t)$ has been studied by Bellman and Harris [1]. References to the literature will be found in [1].

In the special case where $G(t)$ is a step function with one discontinuity, the process becomes the Galton-Watson branching process. For this case the author has shown [2] that a best possible condition on $h(s)$ for the desired limiting theorems to hold is just a little more stringent than the existence of the first moment

$$(1.4) \qquad \mu = h'(1) = \sum_1^{\infty} j q_j < \infty\,.$$

It will be shown here that, with $\mu > 1$, essentially the same condition on $h(s)$ as given in [2] is sufficient to yield the basic limit theorem in the age-dependent case subject to restrictions on $G(t)$.

If, following [1], the random variable representing the number of particles at time t, starting with one particle at $t = 0$, is denoted by $Z(t)$, then for $t \geqq 0$ and $|s| \leqq 1$

$$(1.5) \qquad \begin{aligned} F(s, t) &= E[s^{Z(t)}], \\ E[Z(t)] &= m(t) = \partial F(1, t)/\partial s. \end{aligned}$$

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It will be shown that there is a limiting distribution of $Z(t)/m(t)$ as $t \to \infty$. In terms of F, what will be shown is that for Rl $s \geqq 0$ there exists

$$\lim_{t\to\infty} F(e^{-s/m(t)}, t) = \phi(s), \tag{1.6}$$

so that ϕ is the Laplace Stieltjes transform of the limiting cumulative distribution function.

If a is chosen so that

$$\mu \int_0^\infty e^{-ay}\, dG(y) = 1, \tag{1.7}$$

where μ is defined in (1.4), then it will be shown that

$$\phi(s) = \int_0^\infty h(\phi(se^{-ay}))\, dG(y), \qquad \phi(0) = 1, \qquad \phi'(0) = -1. \tag{1.8}$$

For given G and h, the main object of this paper is to prove (1) that (1.3) has a solution $F(s, t)$, (2) that (1.8) has a solution ϕ, and (3) that (1.6) is valid, that is, that as $t \to \infty$, $Z(t)/m(t)$ has a limiting cumulative distribution with Laplace Stieltjes transform $\phi(s)$. These results will now be stated and proved.

2. It will be assumed that $G(0+) = 0$ and that there exists a continuous $g(t) \geqq 0$ on $t \geqq 0$ such that

$$G(t) = \int_0^t g(y)\, dy, \qquad G(\infty) = 1. \tag{2.1}$$

It will be assumed that

$$\int_0^\infty e^{-2at} g^2(t)\, dt < \infty, \tag{2.2}$$

where a is given by (1.7). The requirement (2.2) can be weakened considerably as is indicated in [1, §3].

The equation (1.3) now becomes

$$F(s,t) = \int_0^t h(F(s,t-y))g(y)\, dy + s[1 - G(t)]. \tag{2.3}$$

THEOREM 2.1. *With* (1.4) *and* (2.1) *valid, the equation* (2.3) *has a solution* $F(s, t)$, *continuous in* (s, t) *for* $|s| \leqq 1$, $0 \leqq t < \infty$. $|F(s, t)| \leqq 1$, *and, for each* t, $F(s, t)$ *is analytic for* $|s| < 1$, *and*

$$\partial^j F(0, t)/\partial s^j \geqq 0, \qquad j = 0, 1, 2, \cdots. \tag{2.4}$$

The solution $F(s, t)$ *is unique, and* $F(1, t) = 1$. *Hence for each* t, $F(s, t)$ *is a generating function.*

Proof. Successive approximations will be used. Let $F_0(s, t) = 0$, and let

$$F_{n+1}(s,t) = \int_0^t h(F_n(s,t-y))g(y)\, dy + s[1 - G(t)]. \tag{2.5}$$

By (1.1), for complex s, $|s| \leqq 1$, $|h(s)| \leqq 1$, and by (1.4)

$$|h'(s)| \leqq \mu. \tag{2.6}$$

Hence $F_1(s, t) = h(0)G(t) + s[1 - G(t)]$, and therefore for $|s| \leqq 1$

$$|F_1(s, t)| \leqq G(t) + |s|[1 - G(t)] \leqq 1. \tag{2.7}$$

From (2.5) follows so long as $|F_n(s, t)| \leqq 1$,

$$|F_{n+1}(s, t)| \leqq h(1)G(t) + |s|[1 - G(t)] \leqq 1.$$

Hence by induction, $|F_n(s, t)| \leqq 1$.

Since $F_0 = 0$, (2.7) shows that

$$|F_1(s, t) - F_0(s, t)| \leqq 1. \tag{2.8}$$

From (2.5) follows

$$\begin{aligned} &|F_{n+1}(s, t) - F_n(s, t)| \\ &\qquad \leqq \int_0^t |h(F_n(s, t - y)) - h(F_{n-1}(s, t - y))| g(y)\, dy. \end{aligned} \tag{2.9}$$

If $t_1 > 0$ is chosen, and if K_1 is chosen so that $|g(t)| \leqq K_1$ for $0 \leqq t \leqq t_1$, then (2.9) gives for $t \leqq t_1$

$$|F_{n+1}(s, t) - F_n(s, t)| \leqq \mu K_1 \int_0^t |F_n(s, t - y) - F_{n-1}(s, t - y)|\, dy.$$

If we use (2.8), it follows by induction that

$$|F_{n+1}(s, t) - F_n(s, t)| \leqq (\mu^n K_1^n/n!)t^n. \tag{2.10}$$

Thus for $|s| \leqq 1$, $0 \leqq t \leqq t_1$, $F_n(s, t)$ converges uniformly to a limit $F(s, t)$ which must be continuous since $F_n(s, t)$ is. By letting $n \to \infty$ in (2.5) it follows that $F(s, t)$ satisfies (2.3).

Since, as is readily verified by induction, $|F_n(s, t| < 1$ for $|s| < 1$ and is analytic in s for fixed t, it follows by the uniform convergence that $F(s, t)$ is analytic in s for $|s| < 1$ for each t. Moreover, as is readily verified by induction,

$$\partial^j F_n(0, t)/\partial s^j \geqq 0, \qquad j \geqq 0.$$

Using the Cauchy formula yields

$$\frac{\partial^j F_n}{\partial s^j}(0, t) = \frac{j!}{2\pi} \int_0^{2\pi} F_n(e^{i\theta}, t) e^{-in\theta}\, d\theta .$$

Hence the uniform convergence of F_n to F shows that

$$\partial^j F(0, t)/\partial s^j \geqq 0.$$

If for any s_1, $|s_1| \leqq 1$, (2.3) has two continuous solutions $F(s_1, t)$ and $\tilde{F}(s_1, t)$, where $|F(s_1, t)| \leqq 1$ and $|\tilde{F}(s_1, t)| \leqq 1$, then

(2.11) $|F(s_1, t) - \tilde{F}(s_1, t)| \leqq \mu \int_0^t |F(s_1, t-y) - \tilde{F}(s_1, t-y)| g(y)\, dy,$

That $F - \tilde{F} = 0$ on $[0, \infty)$ will follow from Lemma 2.1 given below with $K = 0$. Hence uniqueness is demonstrated.

Finally, since for $s = 1$ a solution of (2.3) is obviously $F = 1$, it follows from uniqueness that $F(1, t) = 1$.

LEMMA 2.1. *Let $A(t)$ be real and continuous on $[0, \infty)$, let $B(t) \geqq 0$, and let*

$$\int_0^\infty B(t)\, dt < \infty .$$

Let $K \geqq 0$ be a constant. Let

$$A(t) \leqq \int_0^t A(t-y)B(y)\, dy + K.$$

Then there is a constant k such that

(2.12) $$A(t) \leqq 2Ke^{kt}.$$

Proof. Choose $k \geqq 0$ so that

$$\int_0^\infty e^{-ky}B(y)\, dy < \frac{1}{2}.$$

Let $R(t) = A(t)e^{-kt}$. Then

$$R(t) \leqq \int_0^t R(t-y)e^{-ky}B(y)\, dy + K.$$

Let $t_1 > 0$, and let max $R(t)$ on $[0, t_1]$ be denoted by M. Then clearly

$$M \leqq M \int_0^{t_1} e^{-ky}B(y)\, dy + K \leqq \frac{1}{2} M + K.$$

Hence $M \leqq 2K$ on $[0, t_1]$ for any choice of t_1. Hence $R(t) \leqq 2K$ on $[0, \infty)$, which proves the lemma.

THEOREM 2.2. *With* (1.4), (2.1), *and* (2.2) *valid, there exists a continuous function $m(t)$, $0 \leqq t < \infty$, such that in $|s| \leqq 1$*

(2.13) $$\partial F(1, t)/\partial s = m(t)$$

and

(2.14) $$\lim_{\sigma \to 0+} \frac{1 - F(e^{-\sigma}, t)}{\sigma} = m(t)$$

uniformly on any finite interval $0 \leqq t \leqq t_1$. Moreover $m(t)$ satisfies

(2.15) $$m(t) = \mu \int_0^t m(t-y)g(y)\, dy + 1 - G(t),$$

and if a is defined by (1.7), *then*

$$\lim_{t\to\infty} e^{-at}m(t) = c, \tag{2.16}$$

where the constant

$$c = (\mu - 1) \Big/ \left[a\mu^2 \int_0^\infty y g(y) e^{-ay}\, dy \right], \tag{2.17}$$

Proof. Formula (2.13) is an immediate consequence of (2.14) since $F(1, t) = 1$ and hence will be proved once (2.14) is proved. By (2.4)

$$F(s, t) = \textstyle\sum_0^\infty p_n(t) s^n, \tag{2.18}$$

where $p_n(t) \geqq 0$. (By the Cauchy formula for the coefficients, the continuity of $F(s, t)$ implies that the $p_n(t)$ are continuous on $[0, \infty)$, but this will not be required here.) Since $F(1, t) = 1$,

$$\textstyle\sum_0^\infty p_n(t) = 1. \tag{2.19}$$

From (2.3) follows for $\sigma > 0$

$$\frac{1 - F(e^{-\sigma}, t)}{\sigma} = \int_0^t \frac{1 - h(F(e^{-\sigma}, t - y))}{\sigma} g(y)\, dy + \frac{1 - e^{-\sigma}}{\sigma}(1 - G(t)). \tag{2.20}$$

Since

$$\frac{1 - h(F(e^{-\sigma}, t))}{1 - F(e^{-\sigma}, t)} \leqq \mu,$$

it follows from (2.20) that

$$\frac{1 - F(e^{-\sigma}, t)}{\sigma} \leqq \mu \int_0^t \frac{1 - F(e^{-\sigma}, t - y)}{\sigma} g(y)\, dy + (1 - G(t)).$$

Hence by Lemma 2.1 there exists $k > 0$ such that

$$\frac{1 - F(e^{-\sigma}, t)}{\sigma} \leqq 2e^{kt}, \qquad \sigma > 0. \tag{2.21}$$

From (2.18)

$$\frac{1 - F(e^{-\sigma}, t)}{\sigma} = \sum_0^\infty n p_n(t) \frac{1 - e^{-n\sigma}}{n\sigma}. \tag{2.22}$$

From (2.21) and (2.22)

$$\textstyle\sum_0^\infty n p_n(t) \leqq 2e^{kt}, \qquad t \geqq 0. \tag{2.23}$$

Let

$$m(t) = \textstyle\sum_0^\infty n p_n(t). \tag{2.24}$$

Then from (2.22)–(2.24) follows, since $1 - e^{-n\sigma} \leqq n\sigma$,

$$\frac{1 - F(e^{-\sigma}, t)}{\sigma} \leqq m(t), \tag{2.25}$$

and

$$\lim_{\sigma \to 0+} \frac{1 - F(e^{-\sigma}, t)}{\sigma} = m(t). \tag{2.26}$$

From (2.20)

$$\frac{1 - F(e^{-\sigma}, t)}{\sigma} = \int_0^t \frac{1 - h(F(e^{-\sigma}, y))}{\sigma} g(t - y)\, dy + \frac{1 - e^{-\sigma}}{\sigma}(1 - G(t)). \tag{2.27}$$

Since

$$\frac{1 - h(F(e^{-\sigma}, y))}{\sigma} = \frac{1 - h(F)}{1 - F} \frac{1 - F}{\sigma} \leqq \mu m(y) \leqq 2\mu e^{ky},$$

it follows from (2.27) that since g is continuous over any interval $[0, t_1]$,

$$\frac{1 - F(e^{-\sigma}, t)}{\sigma}$$

is continuous in t over $[0, t_1]$ and that this continuity is uniform in σ, $0 < \sigma \leqq \sigma_1$, as well as in t, $0 \leqq t \leqq t_1$. This uniform continuity implies that the convergence in (2.26) must be uniform in t over any finite interval of t, and that $m(t)$ must be continuous. Writing

$$\frac{1 - h}{\sigma} = \frac{1 - h}{1 - F} \frac{1 - F}{\sigma}$$

in (2.20) and letting $\sigma \to 0+$ yields (2.15) as a result of the uniform convergence of (2.26).

It remains now to use (2.15) to obtain (2.16) and (2.17). This well known renewal equation is treated by the Laplace transform. Here it is already known that $m(t)$ satisfies (2.15), and hence it is not necessary to establish the existence of a solution. Moreover Lemma 2.1 shows the uniqueness of $m(t)$ as a solution of (2.15).

[*Addendum to* §2, *October 30, 1958.* The referee has remarked that it would be desirable to show that $m(t)$ is an increasing function since the probabilistic origin of the problem indicates this to be the case. To show this analytically, one takes the derivative of (2.15) formally and finds that m', if it exists, is a solution of

$$\xi(t) = \int_0^t \xi(y) g(t - y)\, dy + (\mu - 1) g(t). \tag{$*$}$$

(Here use is made of $m(0) = 1$.) But since $g \geqq 0$, the usual Volterra procedure shows that $(*)$ has a solution $\xi \geqq 0$ which is unique. Setting $y = t - x$ in $(*)$ and integrating with respect to t gives, if one defines

$$\tilde{m}(t) = 1 + \int_0^t \xi(y)\, dy,$$

$$\begin{aligned}\tilde{m}(t) &= 1 + \mu \int_0^t dy \int_0^y \xi(y - x) g(x)\, dx + (\mu - 1) G(t) \\ &= 1 + \mu \int_0^t g(x) \left(\int_x^t \xi(y - x)\, dy \right) dx + (\mu - 1) G(t) \\ &= 1 + \mu \int_0^t g(x) [\tilde{m}(t - x) - 1]\, dx + (\mu - 1) G(t),\end{aligned}$$

or

$$\tilde{m}(t) = \mu \int_0^t \tilde{m}(t - y) g(y)\, dy + 1 - G(t).$$

Hence, $\tilde{m}(t) = m(t)$, and thus $m'(t) = \xi(t) \geqq 0$.]

With a defined by (1.7), let

$$f(t) = e^{-at} m(t). \tag{2.28}$$

Then by (2.15)

$$f(t) = \mu \int_0^t f(t - y) e^{-ay} g(y)\, dy + e^{-at} [1 - G(t)]. \tag{2.29}$$

From (2.23), (2.24), and (2.28) it follows that the Laplace transform

$$\Phi(w) = \int_0^\infty f(t) e^{-tw}\, dt$$

exists. Let $w = u + iv$, and let

$$\Gamma(w) = \int_0^\infty g(t) e^{-tw}\, dt.$$

Then (2.29) yields

$$\Phi(w) = \mu \Gamma(a + w) \Phi(w) + \frac{1 - \Gamma(a + w)}{a + w}.$$

Since $\Gamma(w)$ is analytic for $u > 0$, the above shows that

$$\Phi(w) = \frac{1 - \Gamma(a + w)}{[1 - \mu \Gamma(a + w)](a + w)}$$

is analytic for $u > 0$. Since $g(t) \geqq 0$, it follows easily that

$$1 - \mu \Gamma(a + iv) \neq 0, \qquad v \neq 0,$$

and by (1.7)

$$\mu \mid \Gamma(a + u + iv) \mid < 1, \qquad u > 0.$$

Hence $\Phi(w)$ is analytic for $u \geqq 0$ except for a pole at $w = 0$. The residue of this pole is

$$\frac{1-\Gamma(a)}{-\mu a\Gamma'(a)} = c,$$

where c is given by (2.17). Hence

$$\Phi(w) - c/w$$

is analytic for $u \geqq 0$. By the Riemann-Lebesgue theorem

$$\lim_{v\to\infty} \frac{1-\Gamma(a+iv)}{1-\mu\Gamma(a+iv)} = 1 .$$

Clearly

$$\Phi(w) - \frac{c}{w} - \frac{1-c}{a+w} = \frac{(\mu-1)\Gamma(a+w)}{(a+w)[1-\mu\Gamma(a+w)]} - \frac{ac}{w(a+w)} .$$

The right side is analytic on $u = 0$, and for large v, $\Gamma(a+iv)$ is $L^2(-\infty, \infty)$ by (2.2). Hence by the Schwarz inequality the right side is absolutely integrable on $u = 0$. Thus its Fourier transform

$$f(t) - c - (1-c)e^{-at} \to 0$$

as $t \to \infty$. Thus (2.16) and (2.17) are proved.

3. Let $h(s)$ be defined by (1.1) for $|s| \leqq 1$. Let $\beta(t)$ be a continuous, positive, monotone decreasing function such that

$$\int_1^\infty \frac{\beta(t)}{t}\,dt \leqq \infty , \tag{3.1}$$

and let

$$\textstyle\sum_{j\geqq n}^{\infty} jq_j \leqq \beta(n). \tag{3.2}$$

(Examples of $\beta(t)$ are $Ct^{-\delta}$, $C[\log(1+t)]^{-1-\delta}$, etc., where C and $\delta > 0$ are constants. Clearly (3.1) and (3.2) are considerably weaker than the requirement of the existence of the second moment which would imply (3.2) with $\beta(n) = C/n$.) It will also be convenient to assume that β is of class C'' and that

$$3\beta'(t) + t\beta''(t) \leqq 0 \tag{3.3}$$

for large t.

In [2, Lemma 2.1] it is shown that (3.1) and (3.2) imply the existence of a continuous nondecreasing function $\alpha(\sigma)$, $0 \leqq \sigma < \infty$, $\alpha(0) = 0$, such that

$$\int_0^1 \frac{\alpha(\sigma)}{\sigma}\,d\sigma < \infty , \tag{3.4}$$

$$\left| \mu - \frac{1-h(e^{-\sigma})}{\sigma} \right| \leqq \alpha(\sigma) , \qquad 0 \leqq \sigma < \infty . \tag{3.5}$$

The condition (3.3) implies that α is C'' and that for small σ

$$\alpha''(\sigma) \leqq 0. \tag{3.6}$$

In the course of proving (1.6) it will be convenient to consider

$$B(s, t) = F(\exp(-se^{-at}/c), t). \tag{3.7}$$

(Since $m(t)e^{-at} \to c$, it will be easy to relate the above to (1.6).) From (2.14) follows

$$\lim_{\sigma \to 0+} \frac{1 - B(\sigma, t)}{\sigma} = \lim_{\sigma \to 0+} \frac{1 - F(\exp(-\sigma e^{-at}/c), t)}{\sigma} = \frac{m(t)e^{-at}}{c} = \frac{f(t)}{c},$$

where $f(t)$ is defined in (2.28). Moreover this convergence is uniform over any finite interval of t. It will be convenient to set

$$\pi(t) = f(t)/c, \tag{3.8}$$

so that

$$\lim_{\sigma \to 0+} \frac{1 - B(\sigma, t)}{\sigma} = \pi(t). \tag{3.9}$$

As already remarked, the convergence in (3.9) is uniform over any finite interval of t. An important step of the proof of (1.6) is the following theorem.

THEOREM 3.1. *The convergence in* (3.9) *is uniform in t for* $0 \leqq t < \infty$; *that is, for given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left| \pi(t) - \frac{1 - B(\sigma, t)}{\sigma} \right| < \varepsilon$$

for $0 < \sigma < \delta$ *independent of* t.

Proof of Theorem 3.1. From (2.3) and (3.7) follows for Rl $s \geqq 0$

$$B(s, t) = \int_0^t h(B(se^{-ay}, t - y))g(y)\, dy + \exp(-se^{-at}/c)\,[1 - G(t)], \tag{3.10}$$

and from (3.10)

$$\frac{1 - B(s, t)}{s} = \int_0^t \frac{1 - h(B(se^{-ay}, t - y)}{s} g(y)\, dy + \frac{1 - \exp(-se^{-at}/c)}{s}\,[1 - G(t)]. \tag{3.11}$$

Let

$$Q(\sigma, t) = \pi(t) - \frac{1 - B(\sigma, t)}{\sigma}.$$

Then $Q(\sigma, t)$ is continuous in (σ, t) for $\sigma \geqq 0$, $t \geqq 0$, and by (3.9)

$$Q(0, t) = 0. \tag{3.12}$$

By (2.29)

$$\pi(t) = \mu \int_0^t \pi(t - y)e^{-ay}g(y)\,dy + \frac{e^{-at}}{c}[1 - G(t)]. \tag{3.13}$$

Hence by (3.11) and (3.13)

$$\begin{aligned} Q(\sigma, t) &= \mu \int_0^t Q(\sigma e^{-ay}, t - y)e^{-ay}g(y)\,dy \\ &+ \int_0^t \left[\mu \frac{1 - B(\sigma e^{-ay}, t - y)}{\sigma e^{-ay}} - \frac{1 - h(B(\sigma e^{-ay}, t - y))}{\sigma e^{-ay}}\right] e^{-ay}g(y)\,dy \\ &+ \left[\frac{e^{-at}}{c} - \frac{1 - \exp(-\sigma e^{-at}/c)}{\sigma}\right][1 - G(t)]. \end{aligned} \tag{3.14}$$

From (2.18) and (3.7)

$$B(s, t) = \textstyle\sum_0^\infty p_n(t) \exp(-nse^{-at}/c). \tag{3.15}$$

From (3.15) and $p_n(t) \geqq 0$, (3.9) implies

$$\pi(t) = (1/c) \textstyle\sum_0^\infty np_n(t)e^{-at} \tag{3.16}$$

and

$$Q(\sigma, t) = \sum_0^\infty \frac{ne^{-at}}{c} p_n(t) \left[1 - \frac{1 - \exp(-n\sigma e^{-at}/c)}{n\sigma e^{-at}/c}\right]. \tag{3.17}$$

The formula (3.17) shows that $Q(\sigma, t)$ is an increasing function of σ since $(1 - e^{-x})/x$ is decreasing in x. Let

$$\max_{0 \leqq t \leqq t_0} Q(\sigma, t) = M(\sigma, t_0). \tag{3.18}$$

Then M is continuous since Q is. Since Q is increasing in σ, $M(\sigma, t_0)$ is an increasing function of σ.

From (3.5) it is readily verified, since $1 - x \leqq \log(1/x)$, that

$$0 < \mu - \frac{1 - h(x)}{1 - x} \leqq \alpha\left(\log \frac{1}{x}\right),$$

and hence

$$\left|\mu - \frac{1 - h(x)}{1 - x}\right| \leqq \alpha[2(1 - x)], \qquad \tfrac{1}{2} \leqq x < 1. \tag{3.19}$$

Note that since $1 - x < e^{-x} < 1 - x + x^2/2$,

$$0 < \frac{e^{-at}}{c} - \frac{1 - \exp(-\sigma e^{-at}/c)}{\sigma} \leqq \frac{\sigma}{2c^2}. \tag{3.20}$$

Let τ be defined by

$$\mu \int_0^\tau e^{-ay}g(y)\,dy = \frac{1}{2}.$$

Choose $t_0 > \tau$, and choose t so that $Q(\sigma, t) = M(\sigma, t_0)$ in (3.17). If we denote $M(\sigma, t_0)$ by $M_0(\sigma)$, (3.14) yields, on dividing the range of integration in the first integral into $(0, \tau)$ and (τ, t) (or simply taking $(0, t)$ if $t < \tau$),

$$M_0(\sigma) \leqq \tfrac{1}{2}M_0(\sigma) + \tfrac{1}{2}M_0(\sigma e^{-a\tau}) + \sigma/2c^2$$
$$+ \int_0^t \left[\mu - \frac{1 - h(B(\sigma e^{-ay}, t - y))}{1 - B}\right] \frac{1 - B(\sigma e^{-ay}, t - y)}{\sigma e^{-ay}} e^{-ay} g(y)\, dy.$$

Since by (3.17) $Q \geqq 0$, $1 - B(\sigma, t) \leqq \pi(t)\sigma$. Hence

$$\frac{1 - B(\sigma e^{-ay}, t)}{\sigma e^{-ay}} \leqq \pi(t) < C_0,$$

where C_0 is a constant. Hence by using (3.19) and taking σ small,

$$M_0(\sigma) \leqq M_0(\sigma e^{-a\tau}) + \frac{\sigma}{c^2} + 4C_0 \int_0^t \alpha(2C_0 \sigma e^{-ay}) e^{-ay} g(y)\, dy,$$

or

$$M_0(\sigma) \leqq M_0(\sigma e^{-a\tau}) + \sigma/c^2 + 4C_0\, \alpha(2C_0\, \sigma).$$

The above with σ replaced by $\sigma e^{-a\tau}$, $\sigma e^{-2a\tau}$, etc. yields

$$M_0(\sigma e^{-a\tau}) \leqq M_0(\sigma e^{-2a\tau}) + \sigma e^{-a\tau}/c^2 + 4C_0\, \alpha(2C_0\, \sigma e^{-a\tau}),$$
$$M_0(\sigma e^{-2a\tau}) \leqq M_0(\sigma e^{-3a\tau}) + \sigma e^{-2a\tau}/c^2 + 4C_0\, \alpha(2C_0\, \sigma e^{-2a\tau}),$$

etc. Since $M_0(0) = 0$, adding gives

$$M_0(\sigma) \leqq \frac{\sigma}{c^2} \frac{1}{1 - e^{-a\tau}} + 4C_0 \sum_{j=0}^{\infty} \alpha(2C_0 \sigma e^{-ja\tau}).$$

Denote $\sup_{0 \leqq t < \infty} Q(\sigma, t)$ by $M(\sigma)$. Then since the right side above does not depend on t_0,

(3.21) $$M(\sigma) \leqq \frac{\sigma}{c^2} \frac{1}{1 - e^{-a\tau}} + 4\, C_0 \sum_{j=0}^{\infty} \alpha(2C_0 \sigma e^{-ja\tau}).$$

(Since $\partial B/\partial\sigma < 0$, it is easily seen that $\partial Q/\partial\sigma < 1/\sigma^2$ for $\sigma > 0$, and hence $M(\sigma)$ is continuous for $\sigma > 0$, but this is not required here.) Noting that for $j - 1 \leqq u \leqq j$

$$\alpha(2C_0\, \sigma e^{-ja\tau}) \leqq \alpha(2C_0\, \sigma e^{-ua\tau}),$$

we obtain

$$\sum_{j=1}^{\infty} \alpha(2C_0 \sigma e^{-ja\tau}) \leqq \int_0^{\infty} \alpha(2C_0 \sigma e^{-ua\tau})\, du \leqq \frac{1}{a\tau} \int_0^{2C_0\sigma} \alpha(y) \frac{dy}{y}.$$

Hence (3.21) shows that $M(0+) = 0$. This proves Theorem 3.1.

4. In proving (1.6) it will first be shown that

(4.1) $$\lim_{t \to \infty} B(s, t) = \phi(s).$$

In order to prove (4.1), it will be convenient first to define $\phi(s)$ independent of any limiting process. Letting $t \to \infty$ in (3.10) suggests that if (4.1) is true, then

$$\phi(s) = \int_0^\infty h(\phi(se^{-ay}))g(y)\,dy. \tag{4.2}$$

Moreover $B(0, t) = 1$ suggests that $\phi(0) = 1$, and since $\pi(t) = f(t)/c \to 1$ as $t \to \infty$, (3.9) suggests

$$\phi'(0) = -1. \tag{4.3}$$

THEOREM 4.1. *With $h(s)$ subject to (3.1), (3.2), and (3.3), the equation*

$$\phi(\sigma) = \int_0^\infty h(\phi(\sigma e^{-ay}))g(y)\,dy \tag{4.4}$$

has a continuous solution for $0 \leqq \sigma < \infty$ with $|\phi(\sigma)| \leqq 1$, $\phi(0) = 1$, and $\phi'(0) = -1$. Moreover this solution is unique.

Proof. Let $\phi_0(\sigma) = e^{-\sigma}$, and let

$$\phi_{n+1}(\sigma) = \int_0^\infty h(\phi_n(\sigma e^{-ay}))g(y)\,dy \tag{4.5}$$

for $n \geqq 0$. By induction, $0 \leqq \phi_n(\sigma) \leqq 1$. Clearly

$$\phi_1(\sigma) = \int_0^\infty h\,(\exp\,(-\sigma e^{-ay}))g(y)\,dy,$$

and

$$\frac{1 - \phi_1(\sigma)}{\sigma} = \int_0^\infty \frac{1 - h\,(\exp\,(-\sigma e^{-ay}))}{\sigma e^{-ay}}\, e^{-ay} g(y)\,dy.$$

By using (3.5)

$$\left|\frac{1 - \phi_1(\sigma)}{\sigma} - 1\right| \leqq \int_0^\infty \alpha(\sigma e^{-ay})e^{-ay}g(y)\,dy.$$

Hence since $\alpha(\sigma)$ is nondecreasing, using (1.7) shows

$$|(1 - \phi_1(\sigma))/\sigma - 1| \leqq \alpha(\sigma). \tag{4.6}$$

It is readily verified that

$$|(1 - \phi_0(\sigma))/\sigma - 1| \leqq \tfrac{1}{2}\sigma.$$

Hence

$$|\phi_1(\sigma) - \phi_0(\sigma)| \leqq \sigma[\alpha(\sigma) + \tfrac{1}{2}\sigma].$$

If we denote $\alpha(\sigma) + \frac{1}{2}\sigma$ by $\alpha_1(\sigma)$, it is clear that α_1 has the required properties of α, namely (3.4) and (3.6). Hence

$$|\phi_1(\sigma) - \phi_0(\sigma)| \leqq \sigma\alpha_1(\sigma). \tag{4.7}$$

By (4.5)

$$\phi_2(\sigma) - \phi_1(\sigma) = \int_0^\infty [h(\phi_1(\sigma e^{-ay})) - h(\phi_0(\sigma e^{-ay}))]g(y)\,dy.$$

Since $h' \leqq \mu$, this and (4.7) give

$$(4.8) \qquad |\phi_2(\sigma) - \phi_1(\sigma)| \leqq \mu \int_0^\infty \sigma e^{-ay} \alpha_1(\sigma e^{-ay}) g(y)\, dy.$$

With τ defined as below (3.20), this gives

$$|\phi_2(\sigma) - \phi_1(\sigma)| \leqq (\sigma/2)\alpha_1(\sigma) + (\sigma/2)\alpha_1(\sigma e^{-a\tau}).$$

Since $\alpha_1'' \leqq 0$,

$$|\phi_2(\sigma) - \phi_1(\sigma)| \leqq \sigma\alpha_1(\sigma(1 + e^{-a\tau})/2).$$

An easy induction now gives

$$(4.9) \qquad |\phi_{n+1}(\sigma) - \phi_n(\sigma)| \leqq \sigma\alpha_1[\sigma((1 + e^{-a\tau})/2)^n].$$

Hence denoting $(1 + e^{-a\tau})/2$ by $e^{-\delta}$, we have

$$(4.10) \qquad \begin{aligned} \sum_{j=n+1}^\infty |\phi_{j+1}(\sigma) - \phi_j(\sigma)| &\leqq \sigma \sum_{j=n+1}^\infty \alpha_1[\sigma e^{-\delta j}] \\ &\leqq \sigma \int_n^\infty \alpha_1(\sigma e^{-\delta t})\, dt = \frac{\sigma}{\delta} \int_0^{\sigma e^{-\delta n}} \alpha_1(u) \frac{du}{u} \end{aligned}$$

if $u = \sigma e^{-\delta t}$. Hence over any finite interval of σ, $0 \leqq \sigma \leqq \sigma_1$, $\phi_n(\sigma)$ converges uniformly. Since each $\phi_n(\sigma)$ is continuous by induction, it follows that the limit $\phi(\sigma)$ is also continuous. By (4.5) it is clear that $\phi(\sigma)$ satisfies (4.4). By induction it is clear that $\phi_n(0) = 1$, and hence $\phi(0) = 1$.

From (4.10) with $n = 0$

$$\left| \frac{1 - \phi(\sigma)}{\sigma} - \frac{1 - \phi_0(\sigma)}{\sigma} \right| \leqq \frac{1}{\delta} \int_0^\sigma \alpha_1(u) \frac{du}{u} + \alpha_1(\sigma).$$

Letting $\sigma \to 0+$ and recalling that $\phi_0(\sigma) = e^{-\sigma}$ gives $\phi'(0) = -1$.

Now assume that (4.4) has two solutions ϕ and ϕ^* satisfying the conditions of the theorem. Then

$$\phi(\sigma) - \phi^*(\sigma) = \int_0^\infty [h(\phi(\sigma e^{-ay})) - h(\phi^*(\sigma e^{-ay}))] g(y)\, dy.$$

Hence

$$(4.11) \qquad \left| \frac{\phi(\sigma) - \phi^*(\sigma)}{\sigma} \right| \leqq \mu \int_0^\infty \left| \frac{\phi(\sigma e^{-ay}) - \phi^*(\sigma e^{-ay})}{\sigma e^{-ay}} \right| e^{-ay} g(y)\, dy.$$

Let

$$\max_{0 \leqq u \leqq \sigma} \left| \frac{\phi(u) - \phi^*(u)}{u} \right| = N(\sigma).$$

Then since $\phi(0) = \phi^*(0) = 1$ and since $\phi'(0) = \phi^{*\prime}(0) = -1$, $N(0+) = 0$. Also from (4.11) simple considerations including the introduction of τ lead to

$$N(\sigma) \leqq \tfrac{1}{2}N(\sigma) + \tfrac{1}{2}N(\sigma e^{-a\tau}).$$

Hence

$$N(\sigma) \leqq N(\sigma e^{-a\tau}).$$

Since $N(0+) = 0$, iterations of the above lead to $N(\sigma) \equiv 0$, which proves the uniqueness of $\phi(\sigma)$ and completes the proof of Theorem 4.1.

It will now be shown that

$$\lim_{t\to\infty} B(\sigma, t) = \phi(\sigma), \qquad \sigma \geqq 0.$$

This is a consequence of the following theorem since $B(0, t) = \phi(0) = 1$.

THEOREM 4.2. *For* $0 < \sigma < \infty$

$$\lim_{t\to\infty}(B(\sigma, t) - \phi(\sigma))/\sigma = 0. \tag{4.12}$$

Proof. From the definition of Q

$$(B(\sigma, t) - \phi(\sigma))/\sigma = Q(\sigma, t) + [(1 - \phi(\sigma))/\sigma - 1] + [1 - \pi(t)]. \tag{4.13}$$

Let

$$\rho(\sigma, t) = | B(\sigma, t) - \phi(\sigma) |/\sigma.$$

Let

$$R(\sigma) = \limsup_{t\to\infty} \rho(\sigma, t).$$

Theorem 3.1, $\phi'(0) = -1$, and $\pi(\infty) = 1$ used in (4.13) imply

$$R(0+) = 0. \tag{4.14}$$

Since $0 < \partial Q/\partial\sigma < 1/\sigma^2$, it follows readily that $R(\sigma)$ is continuous for $\sigma > 0$ and hence by (4.14) for $\sigma \geqq 0$.

Using (3.10) and (4.4) yields

$$(B(\sigma, t) - \phi(\sigma))/\sigma = J_1 + J_2, \tag{4.15}$$

where

$$J_1 = \int_0^t [h(B(\sigma e^{-ay}, t - y)) - h(\phi(\sigma e^{-ay}))]g(y)\frac{dy}{\sigma},$$

$$J_2 = -\int_t^\infty h(\phi(\sigma e^{-ay}))g(y)\frac{dy}{\sigma} + \frac{\exp(-\sigma e^{-at}/c)}{\sigma}[1 - G(t)].$$

Clearly

$$J_1 = \int_0^t \frac{h(B(\sigma e^{-ay}, t - y)) - h(\phi(\sigma e^{-ay}))}{B(\sigma e^{-ay}, t - y) - \phi(\sigma e^{-ay})} \cdot \frac{B(\sigma e^{-ay}, t - y) - \phi(\sigma e^{-ay})}{\sigma e^{-ay}} e^{-ay}g(y)\,dy.$$

Hence

$$| J_1 | \leqq \mu \int_0^t \rho(\sigma e^{-ay}, t - y)e^{-ay}g(y)\,dy.$$

Also

$$J_2 = \int_t^\infty \frac{1 - h(\phi(\sigma e^{-ay}))}{\sigma e^{-ay}} e^{-ay}g(y)\,dy - \frac{1 - \exp(-\sigma e^{-at}/c)}{\sigma e^{-at}}[1 - G(t)]e^{-at}.$$

Hence

$$| J_2 | \leqq \mu \int_t^\infty \frac{1 - \phi(\sigma e^{-ay})}{\sigma e^{-ay}} e^{-ay} g(y)\, dy + e^{-at} \int_t^\infty g(y)\, dy/c.$$

Thus (4.15) gives

$$\rho(\sigma, t) \leqq \mu \int_0^t \rho(\sigma e^{-ay}, t - y) e^{-ay} g(y)\, dy \tag{4.16}$$
$$+ C_1 \int_t^\infty e^{-ay} g(y)\, dy + e^{-at} \int_t^\infty g(y)\, dy/c$$

for some constant C_1 since $(1 - \phi(\sigma))/\sigma$ is uniformly bounded.

By recalling $R(\sigma)$ above (4.14) and its continuity for $0 \leqq \sigma < \infty$, it follows that if $R(\sigma) \not\equiv 0$, then there exists some $\delta > 0$ such that for some $\sigma > 0$, $R(\sigma) = \delta$. Let σ_1 be the least σ such that $R(\sigma_1) = \delta$. Hence $R(\sigma) < \delta$ for $0 \leqq \sigma < \sigma_1$. In particular if τ is defined as below (3.20) there exists $\eta < \delta$ such that

$$R(\sigma) \leqq \eta, \qquad 0 \leqq \sigma \leqq \sigma_1 e^{-a\tau}. \tag{4.17}$$

From Theorem 3.1 and $\phi'(0) = -1$, it follows readily that there exists K_0 such that

$$\rho(\sigma, t) \leqq K_0 .$$

From (4.16) for large t

$$\rho(\sigma_1, t) \leqq \mu \int_0^\tau \rho(\sigma_1 e^{-ay}, t - y) e^{-ay} g(y)\, dy + \mu \int_\tau^{t/2} \rho(\sigma_1 e^{-ay}, t - y) e^{-ay} g(y)\, dy$$
$$+ \mu K_0 \int_{t/2}^t e^{-ay} g(y)\, dy + C_1 \int_t^\infty e^{-ay} g(y)\, dy + e^{-at} \int_t^\infty g(y)\, dy/c.$$

If we let $t \to \infty$, this gives by (4.17)

$$\delta \leqq \tfrac{1}{2}\delta + \tfrac{1}{2}\eta,$$

or $\delta \leqq \eta$, which is impossible. Hence $R(\sigma) \equiv 0$, and the theorem is proved.

5. It remains now to prove (1.6) and to extend some of the results from real σ to complex s. Both are rather simple to do at this point.

It was proved in Theorem 2.1 that $F(s, t)$ is a generating function so that

$$F(s, t) = \textstyle\sum_0^\infty p_n(t) s^n, \qquad | s | \leqq 1,$$

and $F(1, t) = 1$ for $0 \leqq t < \infty$. Hence

$$B_1(s, t) = F(e^{-s/m(t)}, t) = \textstyle\sum_0^\infty p_n(t) e^{-ns/m(t)}, \tag{5.1}$$

and

$$B(s, t) = F[\exp(-se^{-at}/c, t] = \textstyle\sum_0^\infty p_n(t) \exp(-nse^{-at}/c). \tag{5.2}$$

Thus $B_1(s, t)$ is the Laplace Stieltjes transform of

$$H_1(u, t) = \sum_{n \leq um(t)} p_n(t) \tag{5.3}$$

and $B(s, t)$ of

$$H(u, t) = \sum_{n \leq uc \exp(at)} p_n(t); \tag{5.4}$$

that is, for Rl $s \geq 0$,

$$B(s, t) = \int_{0-}^{\infty} e^{-su}\, dH(u, t),$$

where t is held fixed in H, and

$$B_1(s, t) = \int_{0-}^{\infty} e^{-su}\, dH_1(u, t). \tag{5.5}$$

Since $B(\sigma, t) \to \phi(\sigma)$ as $t \to \infty$, the fact that $H(u, t)$ is a cumulative distribution function for each t implies that $\phi(\sigma)$ is the Laplace Stieltjes transform of a cumulative distribution function $\Psi(u)$, and that at points of continuity of Ψ

$$\lim_{t\to\infty} H(u, t) = \Psi(u). \tag{5.6}$$

Hence $B(s, t) \to \phi(s)$ for Rl $s \geq 0$ as $t \to \infty$. Also in (1.8) both sides are defined for Rl $s \geq 0$ and are analytic for Rl $s > 0$. Since both sides are equal for real s, it follows they are equal for all s, Rl $s \geq 0$.

Since $m(t)e^{-at} \to c$, (5.3), (5.4), and (5.6) imply that at all points of continuity of Ψ

$$\lim_{t\to\infty} H_1(u, t) = \Psi(u).$$

In (5.5) this implies $B_1(s, t) \to \phi(s)$ as $t \to \infty$, Rl $s \geq 0$, which proves (1.6).

Addendum, November 3, 1958. The referee suggested weakening the requirement that $g(t)$ be continuous. By adding another stage to the proof, it can be shown that the results of this paper are valid for $G(t)$ absolutely continuous and $G(0+) = 0$. This will now be done.

6. It will be assumed here that $G'(t) = g(t) \geq 0$ almost everywhere. Hence, there exists a sequence of continuous functions $\{g_j(t)\}$ such that

$$g_j(t) \geq 0, \qquad \int_0^{\infty} g_j(t)\, dt = 1,$$

and

$$\lim_{j\to\infty} \int_0^{\infty} |g(t) - g_j(t)|\, dt = 0. \tag{6.1}$$

Let

$$G_j(t) = \int_0^t g_j(y)\, dy.$$

By Theorem 2.1 there exist the generating functions $F_j(s, t)$, and

$$F_j(s, t) = \int_0^t h(F_j(s, t - y))g_j(y)\, dy + s[1 - G_j(t)]; \tag{6.2}$$

that is, to each G_j can be associated an F_j satisfying the conclusion of Theorem 2.1.

The argument used in Lemma 2.1 shows that if k is chosen so that

$$\mu \int_0^\infty e^{-kt} g(t)\, dt < \tfrac{1}{4},$$

then by virtue of (6.1) it follows that if j is large, then the inequality

$$A(t) \leqq \mu \int_0^t A(t-y) g_j(y)\, dy + K \tag{6.3}$$

implies that

$$A(t) \leqq 2Ke^{kt}. \tag{6.4}$$

By using (6.2) for j and i, where i is also a positive integer,

$$\begin{aligned} F_j(s,t) - F_i(s,t) &= \int_0^t [h(F_j(s, t-y)) - h(F_i(s, t-y))] g_j(y)\, dy \\ &+ \int_0^t h(F_i(s, t-y))[g_j(y) - g_i(y)]\, dy + s[G_i(t) - G_j(t)]. \end{aligned} \tag{6.5}$$

Hence, since $|h'| \leqq \mu$ and $|h| \leqq 1$,

$$\begin{aligned} |F_j(s,t) - F_i(s,t)| \leqq \mu \int_0^t |F_j(s, t-y) - F_i(s, t-y)| g_j(y)\, dy \\ + (1 + |s|) \int_0^t |g_j(y) - g_i(y)|\, dy. \end{aligned}$$

Since the last term can be made as small as desired by taking i and j large enough, (6.3) implies that, for any t_0, $F_j(s, t)$ converges uniformly for $0 \leqq t \leqq t_0$, $|s| \leqq 1$ as $j \to \infty$. If

$$F(s,t) = \lim_{j\to\infty} F_j(s,t),$$

then the uniform convergence guarantees that $F(s, t)$ is continuous for $|s| \leqq 1$, $0 \leqq t \leqq \infty$, and that $F(s, t)$ is a generating function for each t. Moreover, writing (6.2) as

$$\begin{aligned} F_j(s,t) &= \int_0^t h(F_j(s, t-y)) g(y)\, dy \\ &+ \int_0^t h(F_j(s, t-y))[g_j(y) - g(y)]\, dy + s[1 - G_j(t)], \end{aligned}$$

it follows on letting $j \to \infty$, that $F(s, t)$ is a solution of (2.3).

Corresponding to each $g_j(t)$ there is, by Theorem 2.2, an $m_j(t)$ satisfying

$$m_j(t) = \mu \int_0^t m_j(t-y) g_j(y)\, dy + 1 - G_j(t).$$

Proceeding with the aid of (6.3) much as before, it follows that $m_j(t)$ converges uniformly for $0 \leqq t \leqq t_0$, and hence

$$m(t) = \lim_{j\to\infty} m_j(t)$$

is a continuous nondecreasing function for $0 \leqq t \leqq \infty$ and satisfies (2.15). Also for large j, (6.4) implies that

$$m_j(t) \leqq 2e^{kt}. \tag{6.6}$$

From (6.5)

$$\frac{1 - F_i(e^{-\sigma}, t)}{\sigma} - \frac{1 - F_j(e^{-\sigma}, t)}{\sigma}$$

$$= \int_0^t [h(F_j(e^{-\sigma}, t - y)) - h(F_i(e^{-\sigma}, t - y))]g_j(y)\, dy/\sigma$$

$$+ \int_0^t \frac{1 - h(F_i(e^{-\sigma}, t - y))}{1 - F_i} [g_i(y) - g_j(y)] \frac{1 - F_i}{\sigma}\, dy$$

$$+ \frac{1 - e^{-\sigma}}{\sigma} \int_0^t [g_j(y) - g_i(y)]\, dy.$$

If we recall that by (2.25) $1 - F_i(e^{-\sigma}, t) \leqq \sigma m_i(t)$ and let

$$|(1 - F_i)/\sigma - (1 - F_j)/\sigma| = W_{ij}(\sigma, t),$$

the above yields

$$W_{ij}(\sigma, t) \leqq \mu \int_0^t W_{ij}(\sigma, t - y) g_j(y)\, dy$$

$$+ \mu \int_0^t |g_i(y) - g_j(y)|\, m_i(t - y)\, dy + \int_0^t |g_i(y) - g_j(y)|\, dy.$$

By (6.6) for $0 \leqq t \leqq t_0$,

$$W_{ij}(\sigma, t) \leqq \mu \int_0^t W_{ij}(\sigma, t - y) g_j(y)\, dy$$

$$+ (1 + 2\mu e^{kt_0}) \int_0^\infty |g_i(y) - g_j(y)|\, dy.$$

Since the last term can be made as small as desired by taking i and j large enough, (6.3) implies that $W_{ij}(\sigma, t)$ converges uniformly to zero for $0 < \sigma < \infty$ and $0 \leqq t \leqq t_0$. Thus

$$(1 - F_i(e^{-\sigma}, t))/\sigma - (1 - F(e^{-\sigma}, t))/\sigma$$

converges uniformly to zero as $i \to \infty$ for $0 < \sigma < \infty$ and $0 \leqq t \leqq t_0$, and hence as $i \to \infty$

$$m_i(t) - (1 - F_i(e^{-\sigma}, t))/\sigma \tag{6.7}$$

converges uniformly for $0 < \sigma < \infty$, $0 \leqq t \leqq t_0$, to

$$m(t) - (1 - F(e^{-\sigma}, t))/\sigma. \tag{6.8}$$

Since (6.7) is continuous and converges uniformly to zero as $\sigma \to 0+$, the uniform convergence of (6.7) to (6.8) implies that (6.8) is continuous on $0 < \sigma < \infty$, $0 \leqq t \leqq t_0$ and that

$$\lim_{\sigma \to 0+} (1 - F(e^{-\sigma}, t))/\sigma = m(t),$$

with the convergence uniform for $0 \leqq t \leqq t_0$, for any t_0.

From (2.28) onward, the argument now applies to the case $G(t)$ absolutely continuous by virtue of the above.

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