MAXIMALLY UNCLEFT RINGS AND ALGEBRAS

BY

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I. Definitions and elementary properties

1. We shall follow the definitions in [7] for semiprimary, primary, and completely primary rings. Our rings are semiprimary with nilpotent radical, and our algebras are of finite dimension over a base field F. On these rings and algebras we shall place an additional restriction which for commutative completely primary rings makes them exactly coefficient rings, not fields (see [4] for definition of coefficient ring).

In a ring A, with radical $N \subset A$ (proper inclusion), there may exist a subring A^* such that A^* is semisimple and maps onto A/N in the natural homomorphism of A onto A/N. A is called *cleft* if A^* exists, otherwise, *uncleft*. The existence of and relation between such subalgebras when A is an algebra and A/N is separable is known as the Wedderburn-Malcev Theorem [3]. When A is a complete equal characteristic local ring, the existence of A^* is due to I. S. Cohen [2]. (See also [4], [5], [9].) We shall say that the ring A is *maximally uncleft* (briefly, m.u.) if A/J is uncleft for every ideal J, $J \subset N$, $N \neq \{0\}$. The corresponding statement defining an m.u. algebra is of course relative to the base field.

Our main purpose here is to prove that (a) the study of m.u. rings and algebras reduces to a study of m.u. completely primary rings and algebras, (b) the presence of the m.u. property in algebras with central radicals is determined by the 2-dimensional cohomology groups of pure inseparable field extensions, and (c) the product of two division algebras is m.u. if and only if the product of their centers is m.u. Before proceeding, we note some examples of m.u. rings and algebras:

(1) $C/(p^r)$, where C is the ring of integers, p a prime, and $r > 1$.

(2) $F(\alpha)$, where α is algebraic over the field F with minimum function $(x^q - c)^n$, $q = p^e$, $n > 1$, $e > 0$, p the characteristic of F, $x^q - c$ irreducible over F.

(3) Any coefficient ring of an unequal characteristic complete local ring, hence of a commutative unequal characteristic completely primary ring.

(4) B_n, a primary matric ring with B an m.u. completely primary ring. The second example is m.u. as an algebra over F, not as a ring.

2. We now give some convenient equivalent definitions and properties of an m.u. ring or algebra. We always assume $N \neq \{0\}$, $N \subset A$.

Let T denote a residue system (system of representatives) for the classes of A/N. Let (T) denote the subring of A generated by T. If A is an algebra,

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we shall understand that T is a linear set over the base field, so (T) is a subalgebra.

(I) *A is m.u. if and only if $(T) = A$ for every choice of T.*

Proof. Let $N_1 = (T) \cap N$. If $a \in A$, then $a = t + n$, where $t \in T$ and $n \in N$. If $n_1 \in N_1$, then $an_1 = tn_1 + nn_1$ and $n_1 a = n_1 t + n_1 n$. But nn_1 and $n_1 n$ are in N^2, while tn_1 and $n_1 t$ are in N_1. Hence $I = (N_1, N^2)$, the subring generated by N_1 and N^2, is an ideal of A such that A/I is cleft. If A is m.u., we may infer that $I = N$, so $N_1 \equiv N \pmod{N^2}$. Thus N_1 contains a residue system for N/N^2. But every element of N is a sum of power products of such a residue system, hence $N_1 = N$ and $(T) = A$. On the other hand, let $(T) = A$ for every T. If A/J were cleft for some ideal $J \subsetneq N$, then there would exist a system T such that $(T) \cap N \subset J$, a contradiction, Q.E.D.

(II) *A is m.u. if and only if A contains no proper subring B such that $(B, N)/N = A/N$, where (B, N) is the subring generated by B and N.*

Proof. If B exists in A, then B contains a residue system T for A and $(T) \subset B \subsetneq A$. Hence A is not m.u. On the other hand, if A is not m.u., then there exists a residue system T in A such that $(T) \subsetneq A$. Hence $(T) = B$ is a proper subring of A, Q.E.D.

(III) *A is m.u. if and only if A/J is uncleft (in fact, m.u.) for every ideal J such that $N^2 \subset J \subsetneq N$. Thus, A is m.u. if and only if A/N^2 is m.u.*

Proof. Suppose that A is m.u., and let $A^* = A/J$ for any J such that $N^2 \subset J \subsetneq N$. If A^*/J^* is cleft for some J^* properly contained in the radical of A^*, then to J^* there corresponds an ideal J' of A properly contained in N and such that $J'/J = J^*$. But then A/J' is cleft, contrary to the assumption on A. Suppose, on the other hand, that A^* is uncleft for every J such that $N^2 \subset J \subsetneq N$. If there is an ideal J' of A properly contained in N and such that A/J' is cleft, let $I = (J', N^2)$ be the ideal of A generated by J' and N^2 together in A, and let $K = I/N^2$. Since $J' \not\equiv N \pmod{N^2}$ we have $I \not\equiv N \pmod{N^2}$. But then $(A/N^2)/K$ is cleft, contrary to assumption, Q.E.D.

3. Since a semiprimary ring with nilpotent radical is an SBI ring, we can investigate the superficial structure of m.u. rings by application of idempotent decompositions.

THEOREM 1. *A ring is m.u. if and only if it is the sum of primary rings at east one of which is m.u. while the others are simple if not m.u.*

Proof. The ring A is the sum of a finite number of primary rings plus a module in the radical: $A = \sum \oplus P_i + N_0$. If $N_0 \neq \{0\}$, then there clearly exists a system T, selected piecewise from the P_i's, such that $(T) \cap N_0 = \{0\}$. If some P_i is neither simple nor m.u., then a system T clearly exists with the property $(T) \not\supset P_i$. On the other hand, if $P_1 \oplus P_2$ is a sum of m.u. primary rings, let T be a system of representatives and T_1 the subset of T representing P_1. Let e_1 be the identity of P_1 and $t_1 \in T_1$ the representative of the class $e_1 + N$. Then $t_1 = e_1 + n_1 + n_2 = f_1 + n_2$, $f_1 = e_1 + n_1 \in P_1$, $n_2 \in P_2 \cap N$. Now $f_1 n_2 = n_2 f_1 = 0$, so $t_1^k = (f_1 + n_2)^k = f_1^k$ for suitably high k. But $f_1^k = (e_1 + n_1)^k = e_1 + n_1'$, that is, (T) contains $e_1' = e_1 + n_1$, $n_1 \in P_1$. Now if $x = x_1 + m_2 \in T$ represents a class of $(P_1, N)/N$, with $x_1 \in P_1$ and $m_2 \in P_2 \cap N$, then $e_1' x = x_1 + n_1 x_1 \in P_1$ and represents the same class. Hence (T) contains a system T_1 such that $(T_1) = P_1$. The same applies to P_2. Hence $(T) = A$. The extension of this method is clear. This completes the proof.

Every primary ring has the structure of a matric ring B_n, where B is a completely primary ring. Concerning B_n we prove the following theorem.

THEOREM 2. *A primary ring B_n is m.u. if and only if the completely primary ring B is m.u.*

Proof. Every ideal of B_n is of the form J_n, where J is an ideal of B. In particular the radical of B_n is N_n, where N is the radical of B. If $J \subset N$, then $J_n \subset N_n$, and $B_n/J_n \cong (B/J)_n$. Hence B_n/J_n is cleft if B/J is cleft. Thus if B is not m.u., then B_n is not m.u. On the other hand, if B_n is not m.u., there is an ideal $J_n \subset N_n$ such that the primary ring B_n/J_n is cleft. But by Lemma 5 in [13] this is equivalent to cleavage of B/J; hence B is not m.u., Q.E.D.

4. As is well known, a commutative ring with identity is a direct sum of completely primary rings. Somewhat more generally one has the following lemma (wherein A^A = center of A).

LEMMA 1. *If A is a ring with identity and $N \subset A^A$, then $A = \sum \oplus C_i$, where C_i is simple if its radical is $\{0\}$, and C_i is completely primary with C_i/N_i a field if its radical is not $\{0\}$.*

Proof. In terms of the idempotent decomposition $A = \sum \oplus P_i + \sum P_{ij}$, where P_i is primary and $P_{ij} = e_i A e_j$ is in N, we see that $e_i A e_j = e_i e_i A e_j e_j = e_i e_j P_{ij} = 0$. Therefore $A = \sum \oplus P_i$. Now consider any $P_i = P$ and call its radical N. Assume $N \neq \{0\}$. Let $n \neq 0$, $n \in N$, and let $K_n = (J_n, N)$, where $J_n = \{x \in A \mid xn = 0\}$. K_n is an ideal; hence $K_n = N$ or $K_n = A$. If $K_n = N$, then $xy - yx \in N$ for all x, y because $(xy)n = x(ny) = n(yx) = (yx)n$, and hence A/N is commutative. If $K_n = A$, then the identity e of A is of form $e = j_n + n$, $j_n \in J_n$ and $n \in N$. This implies that $en = n = n^2 = \cdots = 0$, contrary to the assumption $n \neq 0$. Hence $K_n = N$. (Actually we used only the existence of a cyclic module $AnA \neq \{0\}$ in the center of A mod N^2.) This completes the proof.

Since a completely primary ring is (under our restrictions) complete local if it is commutative, we know that it cleaves as a ring (although not necessarily as an algebra over a given field) if and only if its characteristic equals the characteristic of its residue class field. If commutativity is replaced by centrality of the radical, we encounter uncleft rings.

LEMMA 2. *If A is a primary ring, $N \subset A^A$, and A has the same characteristic as A/N, then A is cleft if and only if A is commutative.*

Proof. If A is cleft, that is, $A = A^* + N$, $A^* \cong A/N$, then the commutativity of A^* and the centrality of N imply that A is commutative. Conversely, if A is commutative, it is an equal characteristic complete local ring, and hence it cleaves, Q.E.D.

COROLLARY. *If A is a noncommutative primary algebra over F and $N \subset A^A$, then A does not cleave as a ring or algebra.*

Proof. Since A is a primary algebra, it is equal characteristic; hence it cannot be cleft, Q.E.D.

When dealing with m.u. algebras it is convenient to have the following generalization of Theorem 1 of [10].

THEOREM 3. *Let A be a primary algebra over a field F. If $N \subset A^A$, then A/N is a field, and there exists a unique separable field K which is contained in A and is an extension of F such that either A is an algebra over K and A/N is pure inseparable over K, or $A = K + N$.*

Proof. The fact that A/N is a field over F is a consequence of Lemma 1, interpreted for algebras. There exists a unique maximal separable subfield K^*, $F \subset K^* \subset A/N$, over F. Let A_0 be the subalgebra of A generated by the elements of the classes $\{\alpha + N\}$ which define K^*. Then A_0 is separable modulo N, hence cleft: $A_0 = K + N$, where $K \cong K^*$ over F. By Lemma 2, A_0 is commutative. Now let $\alpha \in K$, $a \notin N$, and $n = \alpha a - a\alpha$. Since a^{-1} exists, $\alpha = a\alpha a^{-1} + n'$ for $n' \in N$. If the characteristic of F is zero, then $K^* = A/N$ and $A = K + N$. If the characteristic of F is $p \neq 0$, then $\alpha^{p^f} = a\alpha^{p^f}a^{-1}$ for suitably large f, that is, $a\alpha^{p^f} = \alpha^{p^f}a$ for *all* $a \in A$. But K is separable over F, hence $K^{(p^f)} = K$ for all f, which implies that $a\alpha = \alpha a$ for all $\alpha \in K$ and $a \in A$. Thus A is an algebra over K and A/N is pure inseparable over K. If $K \cong \hat{K} \subset A$, $\hat{K}$ a subfield over F, let $k \leftrightarrow \hat{k} = \sigma(k)$ denote the isomorphism. Since K^* is unique, we have $\sigma(k) = \hat{k} = k + n(k)$. If $f(x)$ is the minimum function of k over F, then $f(k) = f(\hat{k}) = 0$. Hence if $n(k) \neq 0$ for some $k \in K$, then $0 = f(\hat{k}) = f(k + n(k)) = f(k) + n(k)\,g(k) = n(k)\,g(k)$. Now $g(k) \neq 0$ (because $f(x)$ is separable) and is a polynomial in elements of K; hence $g(k)$ has an inverse. Thus $n(k) = 0$, a contradiction if $\hat{K} \neq K$, Q.E.D.

5. As noted above, if A is a commutative completely primary ring, then it is a local ring which in our case is complete. Therefore, from the known embedding theory we may deduce the following elementary facts concerning the structure of m.u. commutative primary rings.

THEOREM 4. *Let A be a commutative primary ring. Then A is m.u. if and only if A is of unequal characteristic and a coefficient ring, hence if and only if it is a principal ideal extension of $C/(p^r)$ with same index of radical and $r > 1$.*

Proof. A complete local ring must be of unequal characteristic in order that it be uncleft, and a coefficient ring which is not a field is of unequal characteristic modulo any ideal contained properly in the radical (because N is a principal ideal generated by the prime p, [9]). Hence, coefficient rings are m.u. Furthermore, every unequal characteristic local ring A contains a coefficient ring. Thus in order that A be m.u., A must equal its coefficient ring. As for the structure of the coefficient rings, we have given the one implied by [12], p. 50 and p. 54, Q.E.D.

The proof that Example 2 is the most general m.u. polynomial algebra may be found in [14].

II. Singular m.u. algebras

1. Since an algebra A is m.u. if and only if A/N^2 is m.u., we consider the structure of A/N^2 in this part. The algebra A/N^2 may be regarded as a singular extension [6] of R by the R-R module N/N^2 with $R \cong A/N$ and 2-cocycle $g(x, y)$. Here, $g(x, y) = A_x A_y - A_{xy} \in Z^2(R, N/N^2)$, $x \leftrightarrow A_x$ is a linear correspondence over F (hence g is normalized) between R and a residue system $T = \{A_x\}$, and $xn = A_x n, nx = nA_x$ for $n \in N/N^2$, $x \in R$. Furthermore, since by Lemmas 1 and 2 a completely primary algebra is the basic type of m.u. algebra, we shall consider singular m.u. completely primary algebras, so $N^2 = \{0\}$, and R is a division algebra. We shall write $A = \mathrm{Ext}(R, N, g)$ for these singular extensions. N is an R-R module of equal left and right dimension over R. We call this dimension the *defect* of A.

We seek in particular the relations between the defect and $g(x, y)$ when A is m.u., especially for the case $N \subset A^A$, and apply these relations to the construction of m.u. algebras.

2. By $g(R, R)$ we mean the subalgebra of A generated by $g(x, y)$ as x and y vary through R. Then $N_g = Rg(R, R)R$ $(= Rg(R, R) = g(R, R)R$ because $\delta g = 0)$ is an ideal of A contained in the radical. (The following theorem is also true when A is a ring, with $x \leftrightarrow A_x$ a linear correspondence [8].)

THEOREM 5. *If $A = \mathrm{Ext}(R, N, g)$, then A is m.u. if and only if $N_{g'} = N$ for every g' cohomologous to g.*

Proof. If A is m.u., consider the residue system $T = \{A_x\}$. Then $(T) = A$. If $n \in N$, then n is a polynomial in the elements A_x. But $A_x A_y = A_{xy} + g(x, y)$, and A_x is linear in x. Hence $n \in Rg(R, R)R$. On the other hand, suppose $N_{g'} = N$ for all g' cohomologous to g. Suppose also that some residue system $T'' = \{A''_x\}$ is such that $(T'') \cap N \subsetneq N$, that is, A is not m.u. But $g''(x, y) = A''_x A''_y - A''_{xy} \in (T'') \cap N$ which implies that $Rg''(R, R)R \subset (T'') \cap N$. This is a contradiction, Q.E.D.

3. If $n_1, \cdots, n_d$ is a left basis of N over R, then for every $x, y \in R$ we have uniquely

$$g(x, y) = \sum_{i=1}^{d} p_i(x, y)n_i, \qquad p_i(x, y) \in R.$$

Each p_i is a single valued bilinear function on R to R. We call p_i a *projection* of g, and the set $p_1, \cdots, p_d$ the projections of g with respect to $n_1, \cdots, n_d$. We derive some preliminary properties of these projections.

LEMMA 3. *If $N = N_g$, and $e_1, \cdots, e_s$ is a linear basis of R over F, then there exists a finite set of pairs (a_i, b_i) of these basis elements such that $g_i = g(a_i, b_i)$, $i = 1, \cdots, d$, is a left basis of N over R.*

Proof. Since g is a cocycle, $\delta g = 0$. Hence, in particular,

$$g(e_b, e_c)e_d = e_b\, g(e_c, e_d) - g(e_b e_c, e_d) + g(e_b, e_c e_d), \qquad \text{Q.E.D.}$$

LEMMA 4. *If $N = N_g$, let $g_1, \cdots, g_d$ be the basis described in Lemma 3 and $n_1, \cdots, n_d$ any other basis. Let $g(x, y) = \sum \pi_i(x, y)g_i = \sum p_i(x, y)n_i$. Then $p_j(x, y) = \sum_i \pi_i(x, y)a_{ij}$ for a nonsingular matrix (a_{ij}) with $a_{ij} \in R$. Furthermore $\pi_i(a_j, b_j) = \delta_{ij}$.*

Proof. $g_i = \sum_j a_{ij}\, n_j$ for a nonsingular (a_{ij}), Q.E.D.

LEMMA 5. *If $N = N_g$ and π_i, p_i are as in Lemma 4, then*

$$(\delta p_j)(x, y, z) = \sum_i (\delta\pi_i)(x, y, z)a_{ij} + \sum_i \pi_i(x, y)(za_{ij} - a_{ij}\, z)$$

for all $x, y, z \in R$ and $j = 1, \cdots, d$.

Proof. Let $f_i(x, y) = \pi_i(x, y)a_{ij}$. Then

$$\begin{aligned}(\delta f_i)(x, y, z) &= xf_i(y, z) - f_i(xy, z) + f_i(x, yz) - f_i(x, y)z \\ &= x\pi_i(y, z)a_{ij} - \pi_i(xy, z)a_{ij} + \pi_i(x, yz)a_{ij} - \pi_i(x, y)a_{ij}\, z \\ &= (\delta\pi_i)(x, y, z) + \pi_i(x, y)(za_{ij} - a_{ij}\, z).\end{aligned}$$

Apply this to $\sum_i \pi_i(x, y)a_{ij}$ termwise. This completes the proof.

4. All projections are in $C^2(R, R)$. We now ask for the conditions equivalent to all projections being in $Z^2(R, R)$.

LEMMA 6. *Let $N = N_g$. If every projection of g is a cocycle, then the matrix (a_{ij}) has coefficients in R^R.*

Proof. By Lemma 5, $\sum_i \pi_i(x, y)(za_{ij} - a_{ij}\, z) = 0$ for all x, y, z, j. Take $x = a_k$, $y = b_k$ to get $za_{kj} - a_{kj}\, z = 0$ for all z, k, j, Q.E.D.

LEMMA 7. *The projections π_i, $i = 1, \cdots, d$, are cocycles if and only if $g_1, \cdots, g_d$ are in A^A.*

Proof. $\delta g = 0$ implies $\sum (\delta\pi_i)(x, y, z)g_i + \sum \pi_i(x, y)(zg_i - g_i\, z) = 0$ for all x, y, z. Let $x = a_j$, $y = b_j$ to get

$$\sum_i (\delta\pi_i)(a_j, b_j, z)g_i + (zg_j - g_j\, z) = 0$$

for all z and j. Now, if $\delta\pi_i = 0$ for $i = 1, \cdots, d$, then $zg_j - g_j z = 0$ for all j and z, that is, $A_z g_j = g_j A_z$. Since $N^2 = \{0\}$, we have $g_j \in A^A$. On the other hand, if $zg_j - g_j z = 0$ for all z and j, then $\sum (\delta\pi_i)(x, y, z)g_i = 0$; hence $(\delta\pi_i)(x, y, z) = 0$ for all x, y, z, i, Q.E.D.

THEOREM 6. *Let $N = N_g$. Then every projection of g is a cocycle if and only if $N \subset A^A$.*

Proof. Suppose all projections of g are cocycles. Then in particular $\{g_1, \cdots, g_d\} \subset A^A$ and $\{a_{ij}\} \subset R^R$. Now $(b_{ij}) = (a_{ij})^{-1}$ has coefficients in R^R, and

$$0 = zg_i - g_i z = z(\textstyle\sum_j a_{ij} n_j) - (\textstyle\sum_j a_{ij} n_j)z = \textstyle\sum_j a_{ij}(zn_j - n_j z)$$

for all $z \in R$. Hence $zn_k - n_k z = 0$ for all z, k. Since every $n \neq 0$ can occur in some basis of N over R, we conclude that N is in A^A. On the other hand, let N be in A^A. Since $\delta g = 0$, we have

$$\sum (\delta p_i)(x, y, z)n_i + \sum p_i(x, y)(zn_i - n_i z) = 0$$

for all x, y, z in R. Hence $\sum (\delta p_i)(x, y, z)n_i = 0$, or $(\delta p_i)(x, y, z) = 0$, $i = 1, \cdots, d$, Q.E.D.

COROLLARY. *If $N = N_g$, then every projection of g is a cocycle if and only if N is in A^A and R is a field.*

Proof. That R is a field is a consequence of N being in A^A, by Theorem 3, Q.E.D.

5. Let us refer to an algebra as *almost commutative* if $N \subset A^A$ and A/N is a field. In the rest of Part II we shall consider almost commutative algebras only.

LEMMA 8. *If $A = \mathrm{Ext}(R, N, g)$ is almost commutative, then every projection of g is cobounding if and only if g is cobounding.*

Proof. Suppose g is a coboundary, that is, there exists a 1-cocycle $f \in Z^2(R, N)$ such that $g(x, y) = (\delta f)(x, y) = xf(y) - f(xy) + f(x)y$. If $n_1, \cdots, n_d$ is a left basis of N, then $f(x) = \sum f_i(x)n_i$, hence $g(x, y) = \sum (\delta f_i)(x, y)n_i = \sum p_i(x, y)n_i$. Thus $p_i = \delta f_i$, $i = 1, \cdots, d$. On the other hand, if $p_i = \delta f_i$, $i = 1, \cdots, d$ for $f_i \in Z^2(R, R)$, then, defining $f(x) = \sum_i f_i(x)n_i$, we have $g(x, y) = \sum_i (\delta f_i)(x, y)n_i = (\delta f)(x, y)$, Q.E.D.

LEMMA 9. *Let $A = \mathrm{Ext}(R, N, g)$ be almost commutative. If p is any coboundary of $C^2(R, R)$, then for any u in N the function defined by $p(x, y)u$ is a coboundary of $C^2(R, N)$.*

Proof. Since $p(x, y) = (\delta f)(x, y)$ for some $f \in C^1(R, R)$, we have

$$p(x, y)u = x[f(y)u] - [f(xy)u] + [f(x)u]y = (\delta g)(x, y),$$

where we define $g(x)$ by $g(x) = f(x)u$, Q.E.D.

6. The elements of $C^2(R, R)$ constitute a vector space over R, as do therefore the elements of $Z^2(R, R)$, $B^2(R, R)$, and $H^2(R, R) = Z^2(R, R)/B^2(R, R)$. When we talk of linear independence of projections in what follows, we mean modulo $B^2(R, R)$. As usual $A = \text{Ext}(R, N, g)$.

LEMMA 10. *If $A = \text{Ext}(R, N, g)$ is an m.u. almost commutative algebra, then the projections with respect to a basis of N are linearly independent non-cobounding cocycles.*

Proof. If A is m.u., then g must be non-cobounding (else A would be cleft). Again letting $n_1, \cdots, n_d$ be a basis of N and $g(x, y) = \sum p_i(x, y)n_i$, suppose p_1 is a coboundary. Then $g'(x, y) = g(x, y) - p_1(x, y)n_1$ is a non-coboundary. A being m.u., we know that $N = Rg'(R, R)$. But then the dimension of N is less than d, a contradiction; hence p_i is non-cobounding. Further, suppose the p_i's are linearly dependent, say $p_1(x, y) = \sum_{k=2}^{d} r_k\, p_k(x, y) + (\delta f)(x, y)$ for some $\{r_k\} \subset R$ and all x, y. Then $g''(x, y) = g(x, y) - (\delta f)(x, y)n_1 = \sum_{k=2}^{d} p_k(x, y)(r_k\, n_1 + n_k)$ is a non-coboundary, which implies N is of dimension less than d, a contradiction, Q.E.D.

LEMMA 11. *If $p_1, \cdots, p_s$ are given linearly independent non-cobounding cocycles of $Z^2(R, R)$, then there exists an m.u. almost commutative algebra $A =$ $\text{Ext}(R, N, g)$ of defect s and with $p_1, \cdots, p_s$ the projections of g with respect to a basis of N.*

Proof. Choose any trivially two-sided R-R module of dimension s and call it N. Let $n_1, \cdots, n_s$ be a basis of N and define $g(x, y)$ by $g(x, y) = \sum_{i=1}^{s} p_i(x, y)n_i$. Also define $N^2 = \{0\}$. Then by the proof of Lemma 8, g is a non-cobounding cocycle. Now if $N_g = Rg(R, R) \neq N$, then we have a proper subspace N_g of N with basis say $m_1, \cdots, m_{s'}$, $s' < s$, such that $g(x, y) = \sum_{i=1}^{s'} q_i(x, y)m_i$ for all x, y. This implies the existence of a matrix (d_{ij}) with coefficients in R such that $m_i = \sum_{j=1}^{s} d_{ij}\, n_j$, $i = 1, \cdots, s'$. Then $g(x, y) = \sum_{i,j} q_i(x, y)d_{ij}\, n_j = \sum p_j(x, y)n_j$ yields $p_j(x, y) = \sum_{i=1}^{s'} q_i(x, y)d_{ij}$, $j = 1, \cdots, s$. Now the q_i's are cocycles of $C^2(R, R)$; hence we have that $p_1, \cdots, p_s$ are linearly dependent, contrary to assumption. Hence $N = N_g$. Suppose $g'(x, y) = g(x, y) + (\delta f)(x, y)$. If we write $g'(x, y) = \sum p_i'(x, y)n_i$ and $f(x) = \sum f_i(x)n_i$, we get $g'(x, y) - g(x, y) = \sum [p_i'(x, y) - p_i(x, y)]n_i = (\delta f)(x, y) = \sum (\delta f_i)(x, y)n_i$. Hence $p_i'(x, y) - p_i(x, y) = (\delta f_i)(x, y)$. If $\sum_i r_i\, p_i'(x, y) \in B^2(R, R)$, then $\sum r_i\, p_i(x, y) \in B^2(R, R)$, a contradiction. Thus the projections $p_1', \cdots, p_s'$ are linearly independent. Consequently, $N_{g'} = N$, and, by Theorem 5, A is m.u., Q.E.D.

THEOREM 7. *Let $A = \text{Ext}(R, N, g)$ be almost commutative. Then A is m.u. if and only if every set $p_1, p_2, \cdots, p_s$ of cochains of $C^2(R, R)$ generated by $g(x, y) = \sum_{i=1}^{s} p_i(x, y)n_i$ for a basis $n_1, \cdots, n_s$ of N over R is a linearly independent (over R) set of non-cobounding cocycles of $Z^2(R, R)$.*

Proof. If A is m.u. then the claim on the p_i's follows from Lemma 10. On the other hand, if the p_i's are as claimed, construct an m.u. extension

$A' = \text{Ext}'(R, N, g)$ using Lemma 11 with the N of our given algebra as the two-sided module. But there is a 1-1 correspondence between classes of isomorphic singular extensions and 2-dimensional cohomology classes [6]. Thus $A' \cong A$, and A is therefore m.u., Q.E.D.

7. An example of construction of an m.u. singular extension as mentioned in Lemma 11 follows:

Let F be the field $P(\alpha_1 \alpha_2)$ of characteristic 2 obtained by adjoining to the prime field of integers P_2 the algebraically independent indeterminates α_1 and α_2. Let R be the field $F(a_1, a_2) = F(a_1) \times F(a_2)$ obtained by adjoining a_1 and a_2 to F, where the minimum polynomials over F of a_1 and a_2 are $x^2 + \alpha_1$ and $x^2 + \alpha_2$, respectively. The multiplication table for R is of course

1	a_1	a_2	$a_1 a_2$
	α_1	$a_1 a_2$	$\alpha_1 a_2$
		α_2	$\alpha_2 a_1$
			$\alpha_1 \alpha_2$.

Let us define a bilinear map p of R into R by

$$p(a_2, a_1) = 1, \qquad p(a_1 a_2, a_1) = a_1,$$
$$p(a_2, a_1 a_2) = a_2, \qquad p(a_1 a_2, a_1 a_2) = a_1 a_2,$$

$p(x, y) = 0$ for all other pairs (x, y) taken from the basis of R over F. Then p is a cocycle. If p were cobounding, then there would exist an f in $C^1(R, R)$ such that $p(x, y) = xf(y) - f(xy) + f(x)y$ for all x, y in R. But $p(a_1, a_2) = a_1 f(a_2) - f(a_1 a_2) + a_2 f(a_1) = 0$, while $p(a_2, a_1) = a_2 f(a_1) - f(a_1 a_2) + a_1 f(a_2) = 1$, a contradiction. Thus, p is a non-cobounding cocycle. There must then exist a singular m.u. extension of defect 1 by the field R, with a radical of dimension 1 over R (or of dimension 4 over F). To this end, define $g(x, y)$ as $g(x, y) = p(x, y)n$, where $n \neq 0$ is an element of a trivially two-sided vector space over R. Consider the subspace $N = Rn$ spanned by n, and define $N^2 = \{0\}$. Using the fact that $g(x, y) = A_x A_y - A_{xy}$ where $\{A_x\}$ is a residue system of A/N for the A which is yet to be produced, we see that A must have the multiplication table

1	a_1	a_2	$a_1 a_2$	n	$a_1 n$	$a_2 n$	$a_1 a_2 n$
a_1	α_1	$a_1 a_2$	$\alpha_1 a_2$	$a_1 n$	$\alpha_1 n$	$a_1 a_2 n$	$\alpha_1 a_2 n$
a_2	$a_1 a_2 + n$	α_2	$\alpha_2 a_1 + a_2 n$	$a_2 n$	$a_1 a_2 n$	$\alpha_2 n$	$\alpha_2 a_1 n$
$a_1 a_2$	$\alpha_1 a_2 + a_1 n$	$\alpha_2 a_1$	$\alpha_1 \alpha_2 + a_1 a_2 n$	$a_1 a_2 n$	$\alpha_1 a_2 n$	$\alpha_2 a_1 n$	$\alpha_1 \alpha_2 n$,

where we have written a_1 for A_{a_1}, etc. One can verify that this is an m.u. algebra A, hence an m.u. ring because it is not commutative.

III. The product of division algebras

1. In this part our algebras are assumed to have an identity element. We write $Z = A^A$ and N_Z $(= Z \cap N)$ for the center of A and the radical of the center respectively.

2. We first prove some lemmas not concerning products in particular.

LEMMA 12. *Let J be an ideal in A and J_Z an ideal in Z such that $N \supset J \supset N^2$, $N_Z \supset J_Z \supset N_Z^2$, and $Z/J_Z \cong (A/J)^{A/J}$. Then Z/J_Z is cleft if A/J is cleft.*

Proof. Let $A' = A/J$, $Z' = Z/J_Z$, so $Z' \cong (A')^{A'}$. Now assume a cleavage $A' = A^* + N'$ of A'. Identifying Z' with $(A')^{A'}$, we have for any $z' \in Z'$ a unique expression $z' = z'^* + n_z'$, where $z'^* \in A^*$, $n_z' \in N'$. For every $a^* \in A^*$ we have $a^*z' = z'a^*$, hence $a^*z'^* = z'^*a^*$ for all z'^*. Thus $Z'^* = \{z'^*\} \subset (A^*)^{A^*}$. Furthermore, for every $n' \in N'$ we have $n'z' = z'n'$, so $n'z'^* = z'^*n'$ for all z'^*. Thus $Z'^* \subset Z'$. Hence Z'^* is semisimple (a sum of fields), so $Z' = Z'^* + N_{Z'}$ is a cleavage of Z', where $N_{Z'} = \{n_{z'}\}$, Q.E.D.

LEMMA 13. *Let A be such that $Z/N_Z \cong (A/N)^{A/N}$. Then A is cleft if Z is cleft.*

Proof. Consider a cleavage $Z = Z^* + N_Z$. Then $Z^* = \sum\oplus Z_i^*$ with $Z_i^* Z_j^* = \delta_{ij} Z_i^*$ where Z_i^* is a field whose identity element e_i is a central idempotent of A, hence $A = \sum\oplus A_i$ where A_i is a two-sided ideal of A with identity element e_i. Now $Z_i^* \subset A_i$ and $\sum\oplus (A_i/N_i)^{A_i/N_i} \cong (A/N)^{A/N} \cong Z/N_Z \cong \sum\oplus Z_i^*$, where N_i is the radical of A_i. Thus A_i/N_i is central simple over Z_i^*, which implies the separability of A_i, hence its cleavage, as an algebra over Z_i^*. Finally each Z_i^* is an algebra over the ground field K of A, Q.E.D.

LEMMA 14. *There exists a cleavage $A = A^* + N$ with $(A^*)^{A^*} \subset Z$ if and only if there exists a cleavage $Z = Z^* + N_Z$ with $Z^* \cong (A/N)^{A/N}$.*

Proof. Suppose the cleavage of A exists. Then let $Z^* = (A^*)^{A^*}$. Now if K is the base field of A, then $[Z/N_Z:K] \leqq [(A/N)^{A/N}:K] = [Z^*:K] \leqq [Z/N_Z:K]$, because $Z^* \subset Z$, hence $[Z/N_Z:K] = [Z^*:K]$. But then $Z/N_Z \cong Z^*$ and hence $Z = Z^* + N_Z$, the desired cleavage of Z. Conversely, for a cleavage of Z as described, Lemma 13 and its proof imply that $A = \sum\oplus A_i$ is cleft in the manner desired, Q.E.D.

LEMMA 15. *Let $N^2 = \{0\}$ and $(A/N)^{A/N} \cong Z/N_Z$. Then $A = A^* + N$ if and only if $Z = Z^* + N_Z$.*

Proof. This follows from Lemmas 12, 13 with $J = J_Z = \{0\}$.

3. Let L and M be two finite algebraic extension fields of K. We recall some well known facts. If L or M is separable, then $L \times_K M = \sum\oplus C_i$, where $C_i = [LM]_i$ is a field and an ideal of $L \times_K M$, and the collection $[LM]_i$, $i = 1, \cdots$, is the set of all distinct composites of L and M. When L and M are both inseparable, then C_i is a primary commutative algebra whose residue class field is isomorphic to $[LM]_i$. If L and M are both pure inseparable, then $L \times_K M$ is itself primary. (See [11].)

If A is a central simple algebra over K and if B is any algebra (also with identity) over K, then the two-sided ideals of $A \times_K B$ are of the form $A \times B_0$ where B_0 is a two-sided ideal of B. (See [1], p 62.)

LEMMA 16. *Let S and T be simple algebras over K, and let $L = S^S$, $M = T^T$. Every two-sided ideal of $S \times_K T$ is generated by an ideal of $L \times_K M$. Also $S \times_K T = \sum \oplus A_i$, where A_i is an indecomposable two-sided ideal of $S \times_K T$, and $A_i \supset C_i$.*

Proof. The first part follows from the following formulas: $S \times_K T \cong (S \times_K M) \times_M T$, $S \times_K M \cong S \times_L (L \times_K M)$. The second part follows from the fact that the identity element e_i of C_i is a central idempotent, hence defines a two-sided ideal A_i of A. But C_i is primary, and any decomposition of A_i would lead to a central decomposition of the identity element of C_i, which is impossible, Q.E.D.

4. Let D_1 and D_2 be division algebras over the field K. Let $D = D_1 \times_K D_2$, $D_0 = Z_1 \times_K D_2$, $Z = Z_1 \times_K Z_2$, where Z_1, Z_2, and hence Z (see [1], p. 68) are centers of D_1, D_2, and D, respectively. Also we write D_1 for $D_1 \times_K e_2$ and D_2 for $e_1 \times_K D_2$ where no confusion ensues, and similarly for Z_1 and Z_2.

LEMMA 17. *If J is any two-sided ideal of D, then $Z/J \cap Z$ and the center of D/J are isomorphic as algebras over K, Z_1, and Z_2.*

Proof. Let $J_0 = J \cap D_0$ and $J_Z = J \cap Z$. For any fixed J, let $v_i \in D_0$, $i = 1, 2, \cdots$, be representatives of a basis $[v_i]$, $i = 1, 2, \cdots$, for D_0/J_0 over Z_1, hence a basis for D/J over D_1. Similarly let $u_i \in Z$, $i = 1, 2, \cdots$, be representatives of a basis $[u_i]$, $i = 1, 2, \cdots$, for Z/J_Z over Z_2, hence a basis for D_0/J_0 over D_2. Thus

$$b_0 \in D_0 \leftrightarrow b_0 = \sum u_\sigma d_{2\sigma} + j_0 \qquad \text{where } d_{2\sigma} \in D_2,\ j_0 \in J_0,$$

$$b \in D \leftrightarrow b = \sum d_{1\sigma} v_\sigma + j \qquad \text{where } d_{1\sigma} \in D_1,\ j \in J.$$

Then $[b] \in D/J$ is in the center of D/J if and only if $(bx - xb) \in J$ for all $x \in D$; so if d_1 and d_2 are any elements of D_1 and D_2 respectively. we have in particular

$$\begin{aligned} bd_1 - d_1 b &= (\textstyle\sum d_{1\sigma} v_\sigma + j)d_1 - d_1(\textstyle\sum d_{1\sigma} v_\sigma + j) \\ &\equiv \textstyle\sum (d_{1\sigma} d_1 - d_1 d_{1\sigma})v_\sigma \pmod{J} \\ &\equiv 0 \pmod{J}. \end{aligned}$$

Hence $d_{1\sigma} d_1 - d_1 d_{1\sigma} = 0$, that is $d_{1\sigma} \in Z_1$. We therefore write $b = b_0 + j$ where $b_0\ (= \sum d_{1\sigma} v_\sigma) \in D_0$. Now we may represent this same b_0 by $b_0 = \sum u_\sigma d_{2\sigma} + j_0$, where $j_0 \in J_0$, hence $b = b_z + j'$ where $b_z = \sum u_\sigma d_{2\sigma} \in D_0$ and $j' = (j_0 + j) \in J$. So in particular

$$\begin{aligned} bd_2 - d_2 b &= (b_z + j')d_2 - d_2(b_z + j') \\ &\equiv \textstyle\sum (d_{2\sigma} d_2 - d_2 d_{2\sigma})u_\sigma \pmod{J} \\ &\equiv 0 \pmod{J}; \qquad \text{hence} \\ &\equiv 0 \pmod{J_0}, \qquad \text{because } J \cap D_0 = J_0. \end{aligned}$$

Thus $d_{2\sigma} d_2 - d_2 d_{2\sigma} = 0$, or $d_{2\sigma} \in Z_2$, which means that $b_z \in Z$. Finally the correspondence $b \leftrightarrow b_z$ induces the desired isomorphism, Q.E.D.

THEOREM 8. *With respect to K, Z_1, or Z_2, $D = \sum \oplus A_i$ where A_i is a two-sided ideal of D such that $(A_i/N_i)^{A_i/N_i} \cong [Z_1 Z_2]_i$ for each i.*

Proof. By Lemma 16 we have $A_i \supset C_i$, and by Lemma 17 we have $C_i = A_i^{A_i}$ and $[Z_1 Z_2]_i \cong (A_i/N_i)^{A_i/N_i}$, Q.E.D.

THEOREM 9. *$D = A^* + N$ with $(A^*)^{A^*} \subset Z$ if and only if $Z = Z^* + N_Z$.*

Proof. If D cleaves as in the statement of the theorem, then the cleavage of Z follows by Lemma 14. Conversely, if Z cleaves, then by Lemma 17 the condition $Z^* \cong (D/N)^{D/N}$ is automatically fulfilled; hence, by Lemma 14, D cleaves as desired, Q.E.D.

LEMMA 18. *Let J be an ideal such that $N^2 \subset J \subsetneq N$. With respect to K, Z_1, or Z_2, D/J is $\begin{Bmatrix} cleft \\ m.u. \end{Bmatrix}$ if and only if Z/J_Z is $\begin{Bmatrix} cleft \\ m.u. \end{Bmatrix}$.*

Proof. Let $D' = D/J$, $Z' = Z/J_Z$. D/J has radical of index 2, its center is isomorphic to Z/J_Z by Lemma 17, and by the same lemma, D' modulo its radical has a center isomorphic to Z' modulo its radical. Hence Lemma 15 is applicable and implies the mutual cleavage of D/J and Z/J_Z.

Now suppose that D' is not m.u. Then there exists an ideal $J' \subset D'$, $J' \subsetneq N'$, and corresponding ideal $J'_Z \subset Z'$, $J'_Z \subsetneq N_{Z'}$, such that $D'' = D'/J'$ is cleft and with its center isomorphic to $Z'' = Z'/J'_Z$ by Lemma 17. Now Lemma 15 is applicable to D'' and Z''; hence Z'' is cleft. Thus, Z' m.u. implies D' m.u. Conversely, the assumption that Z' is not m.u. leads to the existence of a J'_Z and J' such that Z'' is cleft; hence by the same argument as before D'' is cleft. Thus D' m.u. implies Z' m.u., Q.E.D.

THEOREM 10. *With respect to K, Z_1, or Z_2, D is m.u. if and only if Z is m.u.*

Proof. The theorem follows from a combination of (III) Part I, and Lemma 17.

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