ON FINITE GROUPS CONTAINING AN ELEMENT OF ORDER FOUR WHICH COMMUTES ONLY WITH ITS POWERS

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This paper is a study of a particular case of the following question:

Let K be a subset of a finite group H. Suppose that another finite group G contains H in such a way that the centralizer in G of any element of K is contained in H. Then what can we say about the structure of G? In particular for given H and K, are there infinitely many *simple* groups G satisfying the above condition?

So far no general solution nor general method to attack this problem has been found. The purpose of this paper is to give an answer to this question in the very special case that H is a cyclic group of order 4 and K consists of its generators. The result of this investigation may be stated as follows:

Let G be a finite group containing an element π of order 4. If π commutes only with its powers, then either G contains a normal subgroup of index 2 which does not contain π, or G contains an abelian normal subgroup G_0 of odd order such that the factor group G/G_0 is one of the following groups: $SL(2, 3)$, $SL(2, 5)$, $LF(2, 7)$, or the alternating groups A_6 or A_7.

1. Throughout this paper G stands for a finite group which satisfies the following condition $(*)$:

$(*)$ G contains an element π of order 4 such that π commutes only with its own powers.

We shall use the letter V to denote the subgroup of G generated by π, and T stands for the subgroup of V generated by the involution $\tau = \pi^2$.

PROPOSITION 1. *Let S be a 2-Sylow subgroup of G containing V. Then S is generated by π and another element ρ with one of the following five relations*:

(1) $\rho^2 = 1, \quad \rho\pi\rho^{-1} = \pi^{-1}$;

(2) $\rho^2 = \tau, \quad \rho\pi\rho^{-1} = \pi^{-1}$;

(3) $\rho^{2^m} = \tau \ (m \geqq 2), \quad \pi\rho\pi^{-1} = \rho^{-1}$;

(4) $\rho^{2^m} = \tau \ (m \geqq 2), \quad \pi\rho\pi^{-1} = \rho^{-1}\tau$;

(5) $\rho = 1$.

Proof. By assumption the centralizer of V in S coincides with V. Hence V contains the center of S. In particular T is the only normal subgroup of order 2 in S. If $S = V$, we have the last case (5). We now assume that $S \neq V$. The normalizer of V is therefore of order 8. If $[S:e] = 8$, we have

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either the first or the second case. If $[S:e] > 8$, we may apply Lemma 4 of [8] to the factor group S/T. This group S/T is by the lemma generated by two cosets πT and ρT such that the coset ρT generates a cyclic subgroup of index 2. Hence S contains a subgroup P of index 2 such that P/T is cyclic. P is abelian since it is a cyclic extension of a central subgroup T. If P is not cyclic, P must contain a characteristic subgroup of order 2, $\neq T$. This is, however, impossible since T is the only normal subgroup of order 2 in S. Thus P is cyclic, and we can show easily that we have case (3) or (4) taking ρ as a generator of P.

Proposition 2. *If a 2-Sylow subgroup of G is of type* (3), (4), *or* (5), *then G contains a normal subgroup G_0 of index* 2 *which does not contain π.*

Proof. If G contains a normal subgroup G_1 such that $G_1 S = G$ and $G_1 \cap S = e$, then it is easy to find a normal subgroup G_0 of G satisfying every requirement. In case (5), 2-Sylow subgroups are cyclic, and hence the existence of G_1 is assured by Burnside's theorem ([5], §243). In cases (3) and (4) we denote by P_1 the subgroup of order 4 in $\{\rho\}$, the subgroup generated by ρ. The centralizer of P_1 in G contains ρ and is certainly of order > 4 by the condition $m \geqq 2$. Hence P_1 is not conjugate to V in G. By a theorem of Grün (cf. [10], p. 135) the intersection of S and the commutator subgroup G' of G is generated by the commutator subgroup S' of S and subgroups of the form $S \cap \sigma S' \sigma^{-1}$. In our case $S' \supseteqq P_1$, and P_1 is not conjugate to V in G. Hence $S \cap \sigma S' \sigma^{-1}$ is a part of the subgroup R generated by ρ^2 and $\rho\pi$. Therefore G contains a normal subgroup G_0 of index 2, and $G_0 \cap S = R$. Since $S = \{\pi, R\}$, G_0 does not contain π.

Proposition 3. *If the index of the commutator group of G is even, G contains a normal subgroup G_0 of index* 2 *which does not contain π.*

Proof. If the structure of 2-Sylow subgroups of G is of type (3), (4), or (5), then this proposition follows from the previous one. So we assume that a 2-Sylow subgroup S of G is either a dihedral group or a quaternion group. In either case S is of order 8. From the assumption, G contains a normal subgroup M of index 2. If $M \not\ni \pi$, M satisfies all the requirements. If $M \ni \pi$, V is a 2-Sylow subgroup of M since the order of M is divisible by 4 but not by 8. The normalizer of V in M is the intersection of M and the normalizer of V in G, the latter being S. Hence V is in the center of its normalizer in M. Applying a theorem of Burnside to M, we conclude that G contains a normal subgroup G_1 such that $G = G_1 S$ and $G_1 \cap S = e$. As before we can find a normal subgroup G_0 of G with all the requirements of Proposition 3.

2. In this section we shall assume that the order of the commutator factor group G/G' of G is *odd*. It follows from Proposition 2 that a 2-Sylow subgroup is either of type (1) or (2). We shall denote by N the normalizer of T in G. For the structure of N we have

PROPOSITION 4. *If a 2-Sylow subgroup is a dihedral group (type 1), then N contains a normal subgroup U such that $US = N$ and $U \cap S = e$. On the other hand, if a 2-Sylow subgroup of G is a quaternion group (type 2), then N is isomorphic with the special linear group $SL(2, p)$ with $p = 3$ or 5.*

Proof. Let $\bar{N} = N/T$ and $\bar{S} = S/T$. $\bar{S}$ is an abelian group of order 4 and is a 2-Sylow subgroup of $\bar{N}$. The normalizer of $\bar{S}$ in $\bar{N}$ has the form H/T where H is the normalizer of S in G. H contains a normal subgroup K which is the centralizer of S. Since $V \subseteqq S$, K is a part of the centralizer of V. By the assumption (*) we have $V \supseteqq K$; therefore $K = T$. Thus $H = S$, or else H/T is isomorphic with the alternating group of four letters.

Assume that $H = S$. Then $\bar{S}$ coincides with its own normalizer in $\bar{N}$, and by applying a theorem of Burnside we conclude that N contains a normal subgroup U such that $N = US$ and $U \cap S = e$. If S is a quaternion group, $S \cap \sigma S' \sigma^{-1} \subseteqq T$ for all $\sigma \in G$. Hence again by a theorem of Grün (loc. cit.) G contains a normal subgroup of index 2. This contradicts our assumption that G/G' is of odd order. Hence in this case S must be of dihedral type.

Assume finally that H/T is isomorphic with the alternating group of four letters. In this case all the maximal subgroups of S are conjugate in H. In particular S is a quaternion group. Every involution of $\bar{N}$ is conjugate to the coset πT, and the centralizer of πT is $\bar{S}$. Hence by a theorem of Fowler [6] we have one of two cases, namely: $\bar{S}$ is a normal subgroup of $\bar{N}$, or $\bar{N}$ is isomorphic with the linear fractional group over the field F of which the additive group is isomorphic with $\bar{S}$. Hence in our case $\bar{N}$ is isomorphic with $LF(2, 4) \cong LF(2, 5)$ or with the tetrahedral group. Since T is the only subgroup of order 2 in S, it follows from a theorem of Schur [7] that N itself is isomorphic with $SL(2, 5)$ or $SL(2, 3)$.

PROPOSITION 5. *If a 2-Sylow subgroup of G is a quaternion group and if $[G:G']$ is odd, then G contains a normal subgroup G_0 such that G_0 is an abelian group of odd order and G/G_0 is isomorphic with $SL(2, 5)$ or $SL(2, 3)$.*

This is a direct consequence of Theorem D of [9].

3. The rest of this paper will be devoted to the study of the structure of G in which a 2-Sylow subgroup S is of dihedral type. The normalizer N of T contains by Proposition 4 a subgroup U such that $N = US$ and $U \cap S = e$. From the structure of S we see the existence of two involutions τ_1 and τ_2 such that $\{\tau_1, \tau_2\} = S$ and $\tau_1 \tau_2 = \pi$. We have

PROPOSITION 6. *U is an abelian group and is a direct product of two subgroups U_1 and U_2 such that every element of U_i commutes with τ_i $(i = 1, 2)$.*

Proof. The element π induces an automorphism of order 2 in U. The only element of U which is left invariant is the identity. Hence every element of U is mapped into its inverse by π, and so U is abelian. Since U is abelian, U is a direct product of U_1 and U_2 such that U_1 is the totality of elements which commute with τ_1, and U_2 is the totality of elements satisfying $\tau_1\sigma\tau_1^{-1} = \sigma^{-1}$. Since $\pi\sigma\pi^{-1} = \sigma^{-1}$, $\tau_2 = \tau_1\pi$ commutes with every element of U_2. Actually U_2 is the totality of elements in U which commute with τ_2.

4. In this section we consider a special type of groups satisfying the condition (*). Let G be a group possessing the following properties:

(1) G satisfies the condition (*) of the first section;

(2) a 2-Sylow subgroup S of G is a dihedral group of order 8;

(3) S contains an involution which commutes with every element of U.

Here we have used the same notations as in the preceding section. We may take notations so that $U_1 = U$ (cf. Proposition 6). Then τ_1 is an involution in condition (3), and τ_2 commutes only with the identity of U. The involution τ may or may not be conjugate to τ_1 or τ_2 in G. In this section, however, we shall make the following assumption:

(4) τ is conjugate to τ_2 but not to τ_1.

Denote by D_i the subgroup $\{\tau, \tau_i\}$ generated by τ and τ_i $(i = 1, 2)$. Then each D_i is an abelian subgroup of S of order 4. The centralizer Z_i of D_i is a part of the centralizer N of T. Therefore Z_1 is $U \cup D_1 = U \times D_1$, while $Z_2 = D_2$. Put $Z = Z_1$. Let N_i be the normalizer of D_i $(i = 1, 2)$ in G. N_i contains Z_i as a normal subgroup, and the factor group N_i/Z_i is a subgroup of the symmetric group of three letters. By assumption (4), τ is not conjugate to τ_1. Hence N_1/Z_1 is of order 2; in other words N_1 coincides with N. On the other hand τ is conjugate to τ_2 in G. Then τ is actually conjugate to τ_2 in N_2. Hence N_2/Z_2 is of order 6; i.e., N_2 is isomorphic with the octahedral group.

Since $N_1 = N$, every 2-singular class of $U \cup V$ is special with respect to N in the terminology of [9]; i.e., if σ is a 2-singular element of $U \cup V$, then the centralizer of σ is in N, and if σ' is another element of $U \cup V$ conjugate to σ in G, then they are conjugate in N. Therefore we may apply Lemma 4 of [9].

The irreducible characters of N may be determined without much difficulty. The first 2-block B_0 consists of four linear characters η_0, η_1, η_2, η_3, and a character φ of degree 2. For the theory of blocks of characters see [1], [2], and [4]. The remaining characters of N are distributed in $(u - 1)/2$ blocks of defect 2, each consisting of four characters of degree 2. Here we denote by u the order of the subgroup U. We call a linear combination of irreducible characters of N with integral coefficients a B-character if this linear combination vanishes outside of all the 2-singular classes of $U \cup V$. The first 2-block B_0 contributes two B-characters, namely $\eta_0 + \eta_1 - \eta_2 - \eta_3$ and $\eta_0 + \eta_1 - \varphi$ where we have taken η_0 as the principal character and η_1 as the one which has $U \cup V$ as its kernel. If B is a 2-block of defect 2 consisting of four characters θ_1, θ_2, θ_3, and θ_4, the restriction of any θ_i on U is a sum of two associated characters ξ and ξ' of U. B contributes therefore a B-character $\theta_1 + \theta_2 -$

$\theta_3 - \theta_4$ in suitable notations. We can show that any B-character of N is a linear combination with integral coefficients of the special B-characters mentioned above, which we shall call the *basic* B-characters. The following lemma is easily proved but quite essential in the subsequent argument.

LEMMA. *If a linear combination X of characters of N with integral coefficients is orthogonal to all the B-characters, then X vanishes on all the 2-singular classes of $U \cup V$.*

Now if ζ is a B-character of N, then ζ vanishes outside of special classes of N. Hence we may apply Lemma 4 of [9]. The induced character ζ^* of G satisfies the following properties: ζ^* vanishes on all 2-regular classes of G as well as 2-singular classes of the τ_1-section, and if $\sigma \in U$, $\zeta^*(\tau\sigma) = \zeta(\tau\sigma)$ and $\zeta^*(\pi) = \zeta(\pi)$. Therefore if ζ is one of the basic B-characters, then ζ^* is a linear combination of four (with one exception in case $\zeta = \eta_0 + \eta_1 - \varphi$) irreducible characters of G with coefficients 1 or -1. We see that

$$(\eta_0 + \eta_1 - \eta_2 - \eta_3)^* = 1 + \varepsilon_1 H_1 - \varepsilon_2 H_2 - \varepsilon_3 H_3,$$

$$(\eta_0 + \eta_1 - \varphi)^* = 1 + \varepsilon_1 H_1 - \varepsilon_1 \Phi,$$

where H_1, H_2, H_3, and Φ are all different nonprincipal characters belonging to the first 2-block of G. We want to show that we get all the characters of G in 2-blocks of positive defect $\geqq 2$ by decomposing the induced characters from basic B-characters of N. Suppose that an irreducible character X of G does not appear in ζ^* for all B-characters ζ of N. Then the orthogonality relation yields $\langle X, \zeta^* \rangle_G = 0$, where $\langle X, Y \rangle_G$ is the summation $\sum X(\sigma)Y(\sigma^{-1})$ $(\sigma \in G)$ on G divided by the group order. By the reciprocity law of Frobenius we have $\langle X, \zeta^* \rangle_G = \langle X', \zeta \rangle_H$ where X' is the restriction of X on N. Since ζ is any basic B-character, it follows from the lemma that X vanishes on all the 2-singular classes of $U \cup V$. Hence the degree of X must be divisible by 4. Therefore X belongs to a 2-block of defect $\leqq 1$.

Let B be a 2-block of defect 2 of G. The defect group of B is D_1. Let B correspond to a 2-block B' of defect 2 of N in the sense of [1]. Let X be a character of B. Then X must appear in the decomposition of induced characters ζ^*. Suppose X appears in $\zeta_1^*, \cdots, \zeta_k^*$ but not in others, and let δ_i be the multiplicity of X in ζ_i^*: $\zeta_i^* = \delta_i X + \cdots$. Since X belongs to a block of defect 2, X is not H_i nor Φ. Thus each ζ_i is a basic B-character of a block B_i' of defect 2 of N: $\zeta_i = \theta_i + \theta_i' - \cdots$. If X' is the restriction of X on N, and if $Y = X' - \sum \delta_i \theta_i$, then for any B-characters ζ of H we have

$$\langle Y, \zeta \rangle_N = \langle X' - \sum \delta_i \theta_i, \zeta \rangle_N = \langle X, \zeta^* \rangle_G - \sum \delta_i \langle \theta_i, \zeta \rangle_N.$$

If $\zeta \neq \zeta_j$ for $j = 1, 2, \cdots, k$, then $\langle X, \zeta^* \rangle_G = 0$ and likewise $\langle \theta_i, \zeta \rangle_N = 0$ for all i. If $\zeta = \zeta_j$, then $\langle X, \zeta_j^* \rangle_G = \delta_j$ and $\langle \theta_i, \zeta_j \rangle_N = \delta_{ij}$. Thus we have proved $\langle Y, \zeta \rangle_N = 0$ in any case. Hence by the lemma, Y vanishes on all 2-singular classes of $U \cup V$. Hence $X' = \sum \delta_i \theta_i$ on all the 2-singular classes of $U \cup V$. Let ω be the central character defined by $\omega(\sigma) = g(\sigma)X(\sigma)/f$ where $g(\sigma)$ is the number of conjugate elements in the class containing σ and f is the degree of X. For elements $\sigma \neq 1$ of U we have

$$\omega(\sigma\tau) = g(\sigma\tau)X(\sigma\tau)/f \equiv X(\sigma\tau) = \sum \delta_i \theta_i(\sigma) \qquad (\text{mod } 2).$$

If ω' is the central character of the block B' of N, we have a congruence $\omega(\sigma\tau) \equiv \omega'(\sigma\tau)$ (mod $\mathfrak{p}$) with a suitable prime divisor $\mathfrak{p}$ of 2 in an algebraic number field (cf. [1]). If θ is a character in B', $\omega'(\sigma\tau) = \theta(\sigma\tau) \equiv \theta(\sigma)$ (mod 2). Hence we have $\sum \theta_i(\sigma) \equiv \theta(\sigma)$ (mod $\mathfrak{p}$) for all $\sigma \in U$. Since each θ is a sum of two linear characters of U and two different θ and θ' do not contain any common characters, the orthogonality relations yield that the above congruence is a trivial one. This means that X appears in exactly one ζ^*, and this ζ is the basic B-character of the block B' corresponding to the block containing X.

The same argument may be applied to any character X of G such that X is not 1, H_i, Φ and appears in ζ^* coming from a B-character of blocks of defect 2. If such a character X appears in $\zeta_1^*, \cdots, \zeta_k^*$, then $X = \sum \delta_i \theta_i$ on $\sigma\tau$ ($\sigma \in U$). If the degree of X is divisible by 4, $X(\sigma\tau) \equiv 0$ (mod 2) for $\sigma \neq 1$. Hence for all $\sigma \in U$ we have $\sum \theta_i(\sigma) \equiv 0$ (mod $\mathfrak{p}$) which is impossible unless the summation is empty. Hence a character belonging to a block of defect $\leqq 1$ does not appear in any ζ^*. If X belongs to the first 2-block of G, then the degree of X is not divisible by 4. If the degree is odd, $X(\tau) \equiv 1$ (mod 2), while $X(\tau) = \sum \delta_i \theta_i(\tau) \equiv 0$ (mod 2), a contradiction. If the degree is even, then again we get $X(\sigma\tau) \equiv 0$ (mod 2) for $\sigma \neq 1$. Hence the first 2-block consists of exactly five characters: 1, H_i ($i = 1, 2, 3$), and Φ.

Using a theorem of Grün we conclude that G contains a normal subgroup G_0 of index 2 and $G_0 \cap S = D_2$. D_2 is a 2-Sylow subgroup of G_0. In G_0 every involution is conjugate to τ. Hence G_0 has exactly one 2-block of defect 2 and $(u - 1)/2$ blocks of defect 1. Each block of defect 1 consists of two characters ω and ω' which take the same value on all 2-regular classes. From the orthogonality relations we see that $(\omega - \omega')^*$ contains four characters. This means that ω^* is a sum of two irreducible characters X and Y of G. Hence both X and Y remain irreducible on G_0 and coincide with ω. It follows that characters of G in $(\omega - \omega')^*$ are in the same 2-block of defect 2. Since G has exactly $(u - 1)/2$ blocks of defect 2, we conclude that each 2-block of defect 2 of G consists of four characters of the same degree. This implies in particular that characters of the first 2-block do not appear in any ζ^* coming from B-character ζ of defect 2. Hence by an argument similar to that used before, we see that $H_i(\sigma\tau) = \varepsilon_i$, $H_1(\pi) = \varepsilon_1$, $H_2(\pi) = -\varepsilon_2$, $H_3(\pi) = -\varepsilon_3$, $\Phi(\pi) = 0$, and $\Phi(\sigma\tau) = -2\varepsilon_1$.

One of the nonprincipal characters in the first 2-block must be linear. This character takes the value -1 on π and 1 on $\sigma\tau$. Hence without loss of generality we may assume that H_3 is linear and $\varepsilon_3 = 1$.

Denoting the conjugate class containing ρ by $\langle\rho\rangle$, we consider the coefficient of $\langle\sigma\tau\rangle$ in the expansion of $\langle\tau\rangle^2$ in the group ring. Orthogonality relations yield that this coefficient is

$$(g/n^2)(\textstyle\sum_\mu X_\mu(\tau)^2 X_\mu(\sigma\tau)/f_\mu),$$

where $g = [G:e]$, $n = [N:e]$, f_μ is the degree of X_μ, and the summation ranges over all the irreducible characters of G (cf. [3], §5). On the other hand the same coefficient is equal to the number of pairs of conjugate elements τ' and τ'' of τ such that $\tau'\tau'' = \sigma\tau$. If $\tau'\tau'' = \sigma\tau$, then $\tau'\tau = \tau\tau'$ and $\tau'\sigma = \sigma^{-1}\tau'$. Hence the number of pairs (τ', τ'') satisfying $\tau'\tau'' = \sigma\tau$ is $2u$ where $u = [U:e]$. Hence we get

$$2un^2/g = \textstyle\sum_\mu X_\mu(\tau)^2 X_\mu(\sigma\tau)/f_\mu .$$

The contribution to the right side from the blocks of defect $\leqq 2$ is zero. For each block of defect $\leqq 1$ this is clear since $\sum X(\sigma\tau) = 0$. For blocks of defect 2 this follows from the fact that such blocks consist of four characters of the same degree. Let f be the degree of H_1. Then the degree of H_2 is also f, while the degree of Φ is $f + \varepsilon$, where $\varepsilon = \varepsilon_1 = \varepsilon_2$. Hence the contribution of the first 2-block is $1 + (\varepsilon/f) + (\varepsilon/f) + 1 - (8\varepsilon/(f + \varepsilon)) = 2(f - \varepsilon)^2/f(f + \varepsilon)$. Hence $g = un^2 f(f + \varepsilon)/(f - \varepsilon)^2$.

We apply the same consideration to the coefficient of $\langle\sigma\tau\rangle$ in the expansion of $\langle\tau_1\rangle^2$. In this case we get

$$\begin{aligned}(g/n_1^2)(\textstyle\sum_\mu X_\mu(\tau_1)^2 X_\mu(\sigma\tau)/f_\mu) &= 2 \qquad \text{if} \quad \sigma = 1,\\ &= 0 \qquad \text{if} \quad \sigma \neq 1,\end{aligned}$$

where n_1 is the order of the normalizer of τ_1. Since τ_1 commutes with every element of U, we may write $n_1 = 4uw$. The contributions from blocks of defect $\leqq 1$ vanish since $\sum X_\mu(\sigma\tau) = 0$. Let B_i be a block of defect 2 consisting of X_1, X_2, X_3, and X_4. These four characters have the same degree f_i. Using suitable notations we may assume $X_1(\sigma\tau) = X_2(\sigma\tau) = -X_3(\sigma\tau) = -X_4(\sigma\tau) = \xi_i(\sigma) + \xi_i'(\sigma)$, where ξ_i, ξ_i' are linear characters of U. We have $X_1(\tau_1) = -X_2(\tau_1) = x_i$ and $X_3(\tau_1) = -X_4(\tau_1) = y_i$. Thus the contribution of B_i is $2(x_i^2 - y_i^2)(\xi_i(\sigma) + \xi_i'(\sigma))/f_i$. Consider finally the contribution of the first 2-block. Let $x = H_1(\tau_1)$ and $y = H_2(\tau_1)$. Then $x - y + 2\varepsilon = 0$ since $H_3(\tau_1) = -1$. In order to determine the value x and y, we consider the multiplicity of $\langle\pi\rangle$ in $\langle\tau_1\rangle^2$. This coefficient is zero since $\langle\tau_1\rangle^2$ contains only elements of G_0 while $\pi \notin G_0$. Hence

$$0 = \textstyle\sum_\mu X_\mu(\tau_1)^2 X_\mu(\pi)/f_\mu .$$

Since $X_\mu(\pi) = 0$ except if $X_\mu = 1, H_1, H_2, H_3$, we see that

$$0 = 1 + (x^2\varepsilon/f) - (y^2\varepsilon/f) - 1, \quad \text{or} \quad x^2 = y^2.$$

Hence $2x + 2\varepsilon = 0$, or $x = -\varepsilon$ and $y = \varepsilon$. From the equation

$$1 + \varepsilon H_1(\tau_1) - \varepsilon\Phi(\tau_1) = 0$$

it follows now $\Phi(\tau_1) = 0$. Hence the contribution of the first 2-block is

$1 + (\varepsilon/f) + (\varepsilon/f) + 1 = 2(f + \varepsilon)/f$. We have now

$$2(f + \varepsilon)/f + \sum 2(x_i^2 - y_i^2)(\xi_i(\sigma) + \xi_i'(\sigma))/f_i = 2n_1^2/g \quad \text{or} \quad 0.$$

The orthogonality relation yields

$$(f + \varepsilon)/f = (x_i^2 - y_i^2)/f_i = n_1^2/gu, \qquad g = 16uw^2f/(f + \varepsilon).$$

Comparing the two expressions of the order g we get $w(f - \varepsilon) = 2u(f + \varepsilon)$.

We *assume* that $w \leqq u$. Then $f - \varepsilon \geqq 2(f + \varepsilon)$. Hence $\varepsilon = -1$ and $f \leqq 3$. Since $f + 1 \equiv 0 \pmod 4$, we must have $f = 3$ and $u = w$. The degree of Φ is $f + \varepsilon = 2$. Let G_1 be the kernel of the representation with the character Φ. Then $G_1 \ni \tau$ and τ_2. Hence D_2 is a 2-Sylow subgroup of G_1 since G/G_1 is of even order. Hence $G = G_1 N_2$, and $G/G_1 \cong N_2/N_2 \cap G_1 \sim N_2/D_2$. Hence G/G_1 is isomorphic with the symmetric group of three letters. At the same time we see that $N_2 \cap G_1 = D_2$. Hence by a theorem of Burnside G_1 contains a normal subgroup H such that $G_1 = HD_2$ and $H \cap D_2 = e$. H is also a normal subgroup of G and $G/H \cong N_2$.

Let K be the centralizer of τ_1. Then the order of K is $4u^2$, and K contains a normal subgroup K_0 of order u^2. By the isomorphism theorem $HK/H \cong K/K \cap H$, and hence HK/H is a group containing a normal subgroup of odd order and of index 4. Since G/H is the octahedral group, HK/H is of order 4. Therefore $K \cap H = K_0$; in particular we see that $K_0 \subseteqq H$.

Now we make a *further assumption* that K_0 is abelian. Consider the centralizer Z of U in G. It is clear that the subgroup D_1 is a 2-Sylow subgroup of Z. Burnside's theorem shows the existence of a normal subgroup Z_0 of Z with index 4. Since $K_0 \supseteqq U$, K_0 is a part of Z_0. The conjugate subgroup $\tau_2 K_0 \tau_2^{-1}$ contains $\tau_2 U \tau_2^{-1} = U$ and consists of elements commuting with $\tau_2 \tau_1 \tau_2^{-1} = \tau\tau_1$. Hence $\tau_2 K_0 \tau_2^{-1} \cap K_0 = U$. The element τ induces an automorphism of Z_0/U which leaves only the identity invariant. Hence Z_0/U is abelian. In particular $\tau_2 K_0 \tau_2^{-1} \cup K_0/U$ is an abelian group and so a direct product of K_0/U and $\tau_2 K_0 \tau_2^{-1}/U$. Since $f = 3$ and $\varepsilon = -1$, the order of G is equal to $24u^3$. Thus $[H:U] = u^2 \geqq [Z_0:U] \geqq [\tau_2 K_0 \tau_2^{-1} \cup K_0:U] = u^2$. Hence we conclude $H = Z_0 = \tau_2 K_0 \tau_2^{-1} \cup K_0$: in particular U is in the center of H. If ρ is an element of order 3 which maps τ into τ_2, every element of $\rho U \rho^{-1}$ commutes with τ_2. Hence $\rho U \rho^{-1} \cap K_0 = e$. K_0 is a normal subgroup of H, and $\rho U \rho^{-1}$ is in the center. This implies that $H = K_0 \times \rho U \rho^{-1}$ and H is an abelian group. It follows therefore that H is a direct product of U and another subgroup U_0 consisting of elements of H which are mapped into their inverse by τ. U_0 is on the other hand generated by $\rho U \rho^{-1}$ and $\rho^2 U \rho^{-2}$. We have proved the following proposition which was the aim of this section.

PROPOSITION 7. *Let G be a group satisfying the following conditions:*

(1) *G satisfies the condition $(*)$ of the first section,*

(2) *a 2-Sylow subgroup of G is generated by two involutions τ_1 and τ_2 such that $\tau_1 \tau_2 = \pi$ and $\tau_1 \pi = \pi^3 \tau_1$,*

(3) *if* $\rho \neq 1$ *is an element of odd order in the centralizer of* $\tau = \pi^2$, *then* ρ *commutes with* τ_1 *but not with* τ_2, *and*

(4) τ *is conjugate to* τ_2 *but not to* τ_1.

If the centralizer of τ_1 *contains an abelian normal subgroup* W *of index* 4 *then the order of* W *is not smaller than* u^2, *where* $8u$ *is the order of the normalizer of* τ. *If we have* $[W:e] = u^2$, G *contains an abelian normal subgroup* H *of order* u^3 *such that* G/H *is the octahedral group and* H *is generated by elements of odd order conjugate to an element of the centralizer of* τ.

5. We shall consider in this section a group satisfying the condition (*) of Section 1 such that

(1) 2-Sylow subgroups are a dihedral group of order 8, and

(2) involutions form a single conjugate class.

The second condition is equivalent to saying that there is no normal subgroup of index 2. The purpose of this section is to obtain a formula for the order of such a group. Throughout we use the same notations as in the first three sections.

The subgroup N is the centralizer of τ in G. Then by Propositions 4 and 6, N contains an abelian 2-Sylow complement U such that $U = U_1 \times U_2$. For each irreducible character ξ of U we define a linear combination of characters of N in the following way. If ξ is an irreducible character of U, ξ has either one, two, or four associated characters in N. Precisely if ξ is a nonprincipal character of U/U_i $(i = 1, 2)$, ξ has exactly two associated characters ξ and ξ'. Then N has four irreducible characters $\varphi = \varphi_0, \varphi_1, \varphi_2, \varphi_3$ such that the restriction of each φ_i on U is $\xi + \xi'$. Under suitable notations $\theta(\xi) = \varphi + \varphi_1 - \varphi_2 - \varphi_3$ vanishes on all classes of N except the 2-singular classes of $U \cup V$. If the kernel of the representation with character ξ does not contain U_1 nor U_2, ξ has four associated characters ξ, ξ_1, ξ_2, and ξ_3. This time N has two irreducible characters φ and φ' whose restriction on U is $\xi + \xi_1 + \xi_2 + \xi_3$. In this case $\theta(\xi) = \varphi - \varphi'$ vanishes on N except the 2-singular classes of $U \cup V$. If η_0, η_1, η_2, and η_3 are linear characters of N, $\theta(1) = \eta_0 + \eta_1 - \eta_2 - \eta_3$ vanishes in suitable notations everywhere except on the class $\langle\pi\rangle$. If η is the nonlinear character of N/U, $\theta'(1) = \eta_0 + \eta_1 - \eta$ vanishes outside of 2-singular classes of $U \cup V$. For these θ's we have a lemma similar to the one given in Section 4, namely: if a linear combination of irreducible characters of N with integral coefficients is orthogonal to all the θ's, then it vanishes on all the 2-singular classes of $U \cup V$. Let * indicate the induced character of G. We can apply Lemma 4 of [9] since all the 2-singular classes of $U \cup V$ are special. We see that

$$\theta^*(1) = 1 + \varepsilon_1 H_1 - \varepsilon_2 H_2 - \varepsilon_3 H_3 \quad \text{and} \quad \theta'^*(1) = 1 + \varepsilon_1 H_1 - \varepsilon_1 H_4 .$$

Here H_i are four distinct irreducible characters of G and $\varepsilon_i = \pm 1$. Take another nonprincipal irreducible character $X : X \neq H_i$ $(i = 1, 2, 3, 4)$. Consider all the $\theta^*(\xi)$; here ξ ranges over all representatives from each associated family of nonprincipal characters of U. Suppose X appears in $\theta^*(\xi_i)$ ($i = 1, 2, \cdots, k$) with multiplicity δ_i. There exists a character φ_i of N appearing in $\theta(\xi_i)$ such that $\varphi_i(\sigma\tau) = \sum \xi(\sigma)$ for $\sigma \in U$. The summation here ranges over all the associated characters of ξ. We may assume that the multiplicity of φ_i in $\theta(\xi_i)$ is 1. Consider $Y = X' - \sum \delta_i \varphi_i$ where X' is the restriction of X on N. As before (cf. Section 4) Y is orthogonal to all the $\theta(\xi)$ and hence vanishes on all the 2-singular classes of $U \cup V$. Therefore $X(\pi) = 0$ and $X(\sigma\tau) = \sum \delta_i \varphi_i(\sigma\tau)$. In particular the value $X(\sigma\tau)$ is, as a function of σ, a linear combination of nonprincipal characters of U. It follows also from the equation that X is not in the first 2-block of G. This fact may be proved in a way similar to that used in Section 4 by considering the central character ω.

The same consideration can be applied to H_i. If H_4 appears in $\theta^*(\xi_i)$ with multiplicity δ_i, $H_4(\sigma\tau) = -2\varepsilon_1 + \sum \delta_i \varphi_i(\sigma\tau)$ and $H_4(\pi) = 0$. Since H_4 is in the first 2-block (cf. Theorem 6, [4]), the corresponding central character is congruent to 0 on $\sigma\tau$ modulo a prime divisor $\mathfrak{p}$ of 2 in a suitable algebraic number field. Hence $\sum \varphi_i(\sigma\tau) \equiv 0 \pmod{\mathfrak{p}}$ for all $\sigma \in U$. Since $\varphi_i(\sigma\tau) = \sum \xi(\sigma)$, this is possible only if the summation is empty. This means H_4 does not appear in any $\theta^*(\xi)$ with nonprincipal characters ξ and $H_4(\sigma\tau) = -2\varepsilon_i$. Considering the norm of $\theta'^*(1) \pm \theta^*(\xi)$, we see that no $\theta^*(\xi)$ ($\xi \neq 1$) contains H_1, and $H_1(\pi) = H_1(\sigma\tau) = \varepsilon_1$. Here by a norm we shall mean the average on G of the absolute value square.

If H_2 is contained in some $\theta^*(\xi)$, we may assume $\pm\theta^*(\xi) = \varepsilon_2 H_2 + \cdots$. Considering again the norm of $\theta^*(1) \pm \theta^*(\xi)$ we see that H_3 must appear in $\pm\theta^*(\xi)$ with the coefficient $-\varepsilon_3 : \pm\theta^*(\xi) = \varepsilon_2 H_2 - \varepsilon_3 H_3 + \cdots$. The missing term is a difference (or sum) of two characters in a block of defect < 3. Hence by Theorem 6 of [4], $\varepsilon_2 H_2 = \varepsilon_3 H_3$ on all 2-regular classes. In particular $\varepsilon_2 = \varepsilon_3 = \varepsilon_1$, and the degrees of H_2 and of H_3 are equal.

In any case we have $H_i(\pi) = -\varepsilon_i$ and $H_i(\sigma\tau) = \varepsilon_i + a_i(\sigma)$ for $i = 2$ and 3, where $a_i(\sigma)$ is a linear combination of nonprincipal characters of U. If $a_i(\sigma) \neq 0$, then $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$, and the degrees of H_2 and H_3 are equal.

Let f_i be the degree of H_i and $x_i = H_i(\tau)$. Consider the coefficients $A(\sigma)$ of $\langle\sigma\tau\rangle$ in $\langle\tau\rangle^2$. Orthogonality relations yield that

$$A(\sigma) = (g/64u_1^2 u_2^2) \textstyle\sum_\mu \{X_\mu(\tau)^2 X_\mu(\sigma\tau)/f_\mu\}$$

where $u_i = [U_i : e]$ and f_μ is the degree of X_μ. If $X_\mu \neq 1, H_i$, $X_\mu(\sigma\tau)$ is a sum of nonprincipal characters of U. Hence

$$A(\sigma) = (g/64u_1^2 u_2^2)\{1 + (\varepsilon_1/f_1) + (\varepsilon_2 x_2^2/f_2) + (\varepsilon_3 x_3^2/f_3) - (8\varepsilon_1/f_4) + B(\sigma)\},$$

and $B(\sigma)$ is a linear combination of nonprincipal characters of U. The exact value of $A(\sigma)$ is computed as the number of pairs of involutions (cf. [3]), namely: $A(\sigma) = 2(u_1 + u_2)$ if $\sigma = 1$; $= 2u_i$ if $\sigma \neq 1$ and $\sigma \in U_i$; and $= 0$ if $\sigma \notin U_1$ and $\sigma \notin U_2$. Hence summing over U we get

$$\begin{aligned} 2(u_1 + u_2) + 2u_1(u_1 - 1) + 2u_2(u_2 - 1) &= 2(u_1^2 + u_2^2) \\ &= (g/64u_1 u_2)\{1 + (\varepsilon_1/f_1) + (\varepsilon_2 x_2^2/f_2) + (\varepsilon_3 x_3^2/f_3) - (8\varepsilon_1/f_4)\}. \end{aligned}$$

In a similar way we compute the coefficient of $\langle\pi\rangle$. This time we get

$$4 = (g/64u_1^2 u_2^2)\{1 + (\varepsilon_1/f_1) - (\varepsilon_2 x_2^2/f_2) - (\varepsilon_3 x_3^2/f_3)\}.$$

Hence

$$2(u_1^2 + u_2^2) + 4 u_1 u_2 = 2(u_1 + u_2)^2 = 2(g/64u_1 u_2)\{1 + (\varepsilon_1/f_1) - (4\varepsilon_1/f_4)\}.$$

Using the fact $1 + \varepsilon_1 f_1 = \varepsilon_1 f_4$ and putting $f = f_1$ and $\varepsilon = \varepsilon_1$, we conclude

$$g = 64u_1 u_2(u_1 + u_2)^2 f(f + \varepsilon)/(f - \varepsilon)^2.$$

6. We shall study the case $u_2 = 1$ in this section. Put $u = u_1$. The order formula is now read as $g = 64u(u + 1)^2 f(f + \varepsilon)/(f - \varepsilon)^2$. Let 2^μ be the exact power of 2 dividing $f - \varepsilon$ and 2^ν the similar power for $u + 1$. Since g is not divisible by 16, μ must be larger than 2. Hence $f + \varepsilon$ is not divisible by 4, and we have $\mu = \nu + 2$. Put $f - \varepsilon = 2^\mu v$; then v is an odd integer prime to $f(f + \varepsilon)$. Since g/u is the index of U in G, $g/u = 2^\lambda(u + 1)^2 f(f + \varepsilon)/v^2$ is an integer. This means v divides $u + 1$. We may therefore write $4(u + 1) = w(f - \varepsilon)$ with an odd integer w.

From the orthogonality relation we may write $x_2 = \varepsilon_2(1 + a)$ and $x_3 = \varepsilon_3(1 - a)$. Here a is an integer, and if $a \neq 0$ we must have $\varepsilon_2 = \varepsilon_3 = \varepsilon$ and $f_2 = f_3$, because in this case H_2 does appear in some $\theta^*(\xi)$. In general we have a degree relation $1 + \varepsilon_1 f_1 = \varepsilon_2 f_2 + \varepsilon_3 f_3$. From the formula for the coefficient of $\langle\pi\rangle$ we get

$$g = 256u^2 f_1 f_2 f_3/\{(f_1 + \varepsilon_1)(f_2 - \varepsilon_2)(f_3 - \varepsilon_3) - 2\varepsilon a^2 f_1 f_2\}.$$

We have used here the equalities $f_2 = f_3$ and $\varepsilon_2 = \varepsilon_3$ in case $a \neq 0$. Comparing the two expressions of g we get

$$64uf_2 f_3 = w^2(f + \varepsilon)^2(f_2 - \varepsilon_2)(f_3 - \varepsilon_3) - 2\varepsilon a^2 w^2 f(f + \varepsilon)f_2.$$

Suppose $a \neq 0$. Then $f_2 = f_3$, $\varepsilon_2 = \varepsilon_3 = \varepsilon$, and $f + \varepsilon = 2f_2$. Hence

$$16u = w^2(f_2 - \varepsilon)^2 - \varepsilon a^2 f w^2.$$

Since $4(u + 1) = w(f - \varepsilon)$, w is relatively prime to u. From the above equation it follows that $w = 1$. Hence using the equality $4(u + 1) = w(f - \varepsilon) = 2w(f_2 - \varepsilon)$ we see that $\varepsilon a^2 f = 4(u - 1)^2$. In particular $\varepsilon = 1$. At the same time we see that $f = 4u + 5 = 4(u - 1) + 9$ is a divisor of $(u - 1)^2$ and is a perfect square. The only prime divisor of f is 3, and hence f is a power of 9: $f = 9^n$. Having assumed $a \neq 0$, we get $f > 9$ and hence $u - 1$ is not divisible by 27. Hence $n = 2$ and $u = 19$. The order of G is then $8\cdot 19\cdot 81\cdot 41$. Since U is a normal subgroup of N, the index of the normalizer of U is a divisor of $81\cdot 41$. But no divisor of $81\cdot 41$ is congruent to 1 modulo 19 except 1. By a theorem of Sylow U is a normal subgroup of G. If W is the centralizer of U, W is a normal subgroup of G with index exactly divisible by 2. This implies that G has a normal subgroup of index 2 against our assumption. Thus the possibility of $a \neq 0$ has been ruled out.

We have now $64uf_2f_3 = w^2(f+\varepsilon)^2(f_2-\varepsilon_2)(f_3-\varepsilon_3)$. If

$$(f_2-\varepsilon_2)(f_3-\varepsilon_3) \leqq f_2f_3/2,$$

we get

$$2(f_2-1)(f_3-1) = 2f_2f_3 - 2f_2 - 2f_3 + 2 \leqq f_2f_3.$$

Hence $(f_2-2)(f_3-2) \leqq 2$ which implies that $f_2 = f_3 = 3$ and $\varepsilon_2 = \varepsilon_3 = 1$. Hence we get $\varepsilon = 1$ and $f = 5$ which contradicts the congruence $f \equiv \varepsilon \pmod 8$. Hence $(f_2-\varepsilon_2)(f_3-\varepsilon_3) > f_2f_3/2$, and $128u > w^2(f+\varepsilon)^2$. Since $4(u+1) = w(f-\varepsilon)$, we obtain $w(f+\varepsilon) = 4(u+1) + 2\varepsilon w$, and hence $128u > [4(u+1)+2\varepsilon w]^2$. If $\varepsilon = 1$, we get $128u > [4(u+1)+2]^2$ or $5 > u$. If $\varepsilon = -1$, then $4(u+1) - 2w \geqq 3(u+1)$ since $4(u+1) = w(f+1)$ and $f+1 \equiv 0 \pmod 8$. Hence $128u > 9(u+1)^2$ or $12 \geqq u$. In this case if $f + 1 \neq 8$, we can improve the inequality: i.e.,

$$4(u+1) - 2w \geqq 3\cdot 5(u+1)$$

and $u \leqq 8$. In any case u is an odd integer $\leqq 12$.

$u = 11$. Then $48 = 4(u+1) = w(f+1)$ and $f+1 = 8$. Hence $w = 6$, which is impossible.

$u = 9$. Then $f+1 = 8$ and $w = 5$. In general we have $64uf_2f_3 = w^2(f+\varepsilon)^2(f_2f_3 - \delta f)$ where $\delta = \varepsilon\varepsilon_2\varepsilon_3 = \pm 1$. In this case we have then $16f_2f_3 = 25(f_2f_3 - 7\delta)$ or $9f_2f_3 = 7\cdot 25\delta$, impossible.

$u = 7$. We get $32 = w(f+1)$, $w = 1$ and $f = 31$. Hence

$$64u - w^2(f-1)^2 = -452.$$

Thus $452f_2f_3 = 30^2\cdot 31\delta$, impossible.

$u = 5$. $24 = w(f+1)$. We have two cases: $w = 1$, $f = 23$; or $w = 3$, $f = 7$. The first case can be treated as before. If $w = 3$ and $f = 7$, then $g = 4\cdot 5\cdot 6\cdot 7\cdot 9 = 7560$. U is a 5-Sylow subgroup of G. Since G does not contain a normal subgroup of index 2, U is not a normal subgroup of G. The index of the normalizer of U is a divisor of $27\cdot 7$ and $\equiv 1 \pmod 5$. Hence U has 21 conjugate subgroups. If Z is the centralizer of U, the order of Z is $5\cdot 4\cdot 9$. In Z/U the centralizer of any involution is in a 2-Sylow subgroup. Hence a 2-Sylow subgroup of Z/U is a normal subgroup. This is, however, impossible.

$u = 3$. If $\varepsilon = 1$, we get $16 = w(f-1)$, and hence $f = 17$, $w = 1$. As before, we get $64\cdot 3\cdot f_2f_3 = 18^2(f_2f_3 - 17\delta)$, $11f_2f_3 = 27\cdot 17\delta$, impossible. If $\varepsilon = -1$, then $16 = w(f+1)$, $w = 1$, and $f = 15$. $g = 3\cdot 4\cdot 14\cdot 15 = 2520$. We can show that G is isomorphic with the alternating group of seven letters. The detail of the argument is given in the final section.

$u = 1$. If $\varepsilon = 1$, then $8 = w(f-1)$. Hence $w = 1$ and $f = 9$. Thus $g = 4\cdot 9\cdot 10 = 360$. In this case we get $f_2 = f_3 = 5$, $f_4 = 10$, and $\varepsilon_2 = \varepsilon_3 = 1$. The remaining characters are of degree 8 and there are exactly two such characters. We have the alternating group of six letters.

If $\varepsilon = -1$, then $8 = w(f+1)$. Hence $w = 1$, $f = 7$, and $g = 4\cdot 6\cdot 7 =$

168. We have $f_2 = f_3 = 3$, $f_4 = 6$, and $\varepsilon_2 = \varepsilon_3 = -1$. G has one more character of degree 8. In this case G is isomorphic with $LF(2, 7)$.

7. We return to the general case. Without loss of generality we may assume $u_1 = [U_1:e] \geqq u_2 = [U_2:e]$. Assume moreover $u_2 > 1$, since the case $u_2 = 1$ has been treated. Consider the normalizer H of U_2 in G and consider the factor group $\bar{G} = H/U_2$. Clearly $\bar{G}$ satisfies the condition $(*)$. SU_2/U_2 is a 2-Sylow subgroup of $\bar{G}$ and is generated by the cosets $\bar{\tau}_i$ $(i = 1, 2)$ containing τ_i. If $\bar{\tau}$ is the coset containing τ, the centralizer of $\bar{\tau}$ is N/U_2. Since $U_2 \cup \{\tau, \tau_2\}$ is the centralizer of $\{\tau, \tau_2\}$ in G, τ and τ_2 are conjugate in H. This implies that $\bar{\tau}_2$ is conjugate to $\bar{\tau}$ in $\bar{G}$. On the other hand $\bar{\tau}_1$ is not conjugate to $\bar{\tau}$ in $\bar{G}$, because τ_1 maps every element of U_2 into its inverse. If $\bar{\sigma}$ is any element $\neq 1$ of U/U_2, $\bar{\sigma}$ contains an element $\sigma \neq 1$ of U_1. If $\bar{\sigma}\bar{\tau}_2 = \bar{\tau}_2\bar{\sigma}$, then $\sigma\tau_2\sigma^{-1}\tau_2 \in U_2$. Hence $\sigma^2 \in U_2$, or $\sigma \in U_2$. This is impossible since $U_1 \cap U_2 = e$. Thus $\bar{\sigma} \neq 1$ of U/U_2 does not commute with $\bar{\tau}_2$. Let K/U_2 be the centralizer of $\bar{\tau}_1$ in $\bar{G}$. Then K is the normalizer in G of the subgroup $\{U_2, \tau_1\}$ generated by U_2 and τ_1. Since $\{\tau_1\}$ is a 2-Sylow subgroup of $\{U_2, \tau_1\}$, K is a join of $\{U_2, \tau_1\}$ and the normalizer K_0 of τ_1 in K. Hence $K = U_2 \cup K_0$. K_0 contains a normal subgroup U_0 of index 4, and U_0 is conjugate to a subgroup of U since τ_1 and τ are conjugate in G. By Proposition 6, U is abelian, and so is U_0. From the definition of K_0 we have $U_2 \cap K_0 = e$, and hence $K/U_2 \cong K_0$. Therefore $U_0 U_2/U_2 \cong U_0$ is an abelian subgroup of K/U_2 with index 4. Now the order of $U_0 U_2/U_2$ is equal to the order of U_0. Since U_0 is conjugate to a subgroup of U, the order of U_0 is at most $u_1 u_2$. Since the order of U/U_2 is u_1, we conclude $[U_0 U_2 : U_2] \leqq u_1 u_2 \leqq u_1^2 = [U:U_2]^2$ because we have assumed $u_2 \leqq u_1$. The group $\bar{G}$ satisfies all the assumptions of Proposition 7. Hence we have first of all $[U_0 U_2 : U_2] = [U:U_2]^2$, which means $u_2 = u_1$. Set $u = u_1 = u_2$. The second conclusion is that H contains a normal subgroup H_0 such that $[H_0 : U_2] = u^3$ and H_0/U_2 is an abelian group generated by conjugate subgroups of U/U_2. Since U is abelian, U_2 is in the center of H_0. Consider the normalizer of U in G. This subgroup contains a normal subgroup H_1 of index 8. The element τ induces an automorphism of H_1/U which leaves only the identity fixed. Hence H_1/U is an abelian group. Hence H_1/U is a direct product of W/U and W'/U such that every element of W/U is fixed by τ_1 and W'/U is the totality of cosets commuting with $\tau_1\tau$. As before, we have $[W:U] \leqq u$ and $[W':U] \leqq u$. Hence $[H_1 : U] \leqq u^2 = [H_0 : U]$. Since $H_0 \subseteqq H_1$, we must have $H_0 = H_1$. Now we can apply the same consideration to U_1. Since $u_1 = u_2$ we may apply Proposition 7 to the factor group H'/U_1 of the normalizer H' of U_1. H' contains a normal subgroup H_0' of order u^4. As before H_0' contains U as a normal subgroup. Hence $H_0' \subseteqq H_1 = H_0$. This means $H_0' = H_0$ and U_1 is in the center of H_0. H_0 is generated by subgroups conjugate to $U = U_1 \cup U_2$ in H, and hence H_0 is abelian. If $H = H'$, both U_1 and U_2 are normal subgroups of H. Hence $U = U_1 \cup U_2$ is a normal subgroup of H. We have shown that H_0 is generated by the conjugate subgroups of U. H_0 coincides therefore with U. Considering the order we get $u^4 = u^2$, which gives a contradiction $u = 1$. Hence we see that $H \neq H'$. Hence the normalizer G_1 of H_0 contains both H and H'. All involutions of the factor group G_1/H_0 are conjugate to each other. This may be proved as follows. From Proposition 7 the factor group H/H_0 is isomorphic with the octahedral group. Hence the coset containing τ is conjugate to the coset containing τ_2. Similarly in H'/H_0 the coset containing τ is conjugate to the coset containing τ_1. The centralizer of any involution in G_1/H_0 is a 2-group. Hence the result of the preceding section shows that G_1/H_0 is of order 168 or 360. On the other hand it follows from the order formula at the end of Section 5 that $g = 64 \cdot 4 \cdot u^4 f(f + \varepsilon)/(f - \varepsilon)^2$. Let $f = \varepsilon + 8v$. Then

$$g = 8u^4(8v + \varepsilon)(4v + \varepsilon)/v^2.$$

Since $g/8u^4$ is the index of a subgroup, $(8v + \varepsilon)(4v + \varepsilon)/v^2$ must be an integer. Hence $v = 1$. This implies that the index g/u^4 of H_0 in G is either 168 ($\varepsilon = -1$) or 360 ($\varepsilon = 1$). Hence G_1 must be equal to G. Thus we have shown the validity of the following proposition.

PROPOSITION 8. *Let G be a group satisfying the condition $(*)$. If a 2-Sylow subgroup of G is a dihedral group, and if the index of the commutator subgroup is odd, then G contains an abelian normal subgroup G_0 such that the factor G/G_0 is either $LF(2, 7)$ or $LF(2, 9)$ except in the case that G is the alternating group of seven letters.*

8. This section is devoted to the determination of the structure of the group left at the end of Section 6.

We have $u = 3$ and $\varepsilon = -1$. The order g of G is 2520. The values of f_2, f_3, and f_4 are computed by using the relations $1 + \varepsilon_1 f_1 = \varepsilon_2 f_2 + \varepsilon_3 f_3 = \varepsilon_1 f_4$ and $64uf_2 f_3 = (f + \varepsilon)^2(f_2 f_3 - \delta f)$. This gives $f_2 = 35$, $\varepsilon_2 = -1$, $f_3 = 21$, $\varepsilon_3 = 1$, and $f_4 = 14$.

We shall return to the situation of Section 5. The subgroup U is of order 3. Hence U has exactly one family of associated characters $\xi \neq 1$. Since we have assumed $U_1 = U$, ξ has two associators ξ and ξ'. Let φ be the irreducible character of N such that $\varphi(\sigma\tau) = \xi(\sigma) + \xi'(\sigma)$. The induced character $\theta^*(\xi)$ has four distinct irreducible components with multiplicity ± 1. Let it be $\theta^*(\xi) = \delta_1 Y_1 + \delta_2 Y_2 + \delta_3 Y_3 + \delta_4 Y_4$. In Section 6 we have shown $a = 0$. Hence Y_i are different from H_j. The values of Y_i have been computed (cf. Section 5) as $Y_i(\pi) = 0$ and $Y_i(\sigma\tau) = \delta_i \varphi(\sigma\tau)$ $(i = 1, 2, 3, 4)$. The missing characters, if any, of G vanish on all the 2-singular classes. Hence their degrees are divisible by 8. Let y_i be the degree of Y_i. Then summing Y_i over S we get $y_i + 10\delta_i \equiv 0 \pmod 8$ or $y_i \equiv -2\delta_i \pmod 8$. If $y_1 = 2$, then $\delta_1 = -1$ and $Y_1(\tau) = -2$. This means that the kernel G_1 of the representation with character Y_1 is of odd order, and every involution of the factor group G/G_1 is in the center. This is impossible since 2-Sylow subgroups are not abelian and are generated by involutions. Hence the possible values of y_i are 6, 10, 14, $\cdots$. Now we have $g = 1 + \sum f_i^2 + \sum y_j^2 + \cdots$, and the missing term is a multiple of 64. Hence $\sum y_j^2 + \cdots = 432$. Suppose $y_j \geqq 10$ for all j. Then $\sum y_j^2 \geqq 400$, and hence there is no missing term. Thus at least one of y_j is greater than 10, and we get a contradiction that $\sum y_j^2 \geqq 496$. Hence one of the y_j, say y_4, is 6 and $\delta_4 = 1$.

We have a degree relation $\sum \delta_i y_i = 0$. Another relation is coming from the formula for $A(\sigma)$ (cf. Section 5). Using the values of f_i and the facts $A(1) = 8$ and $\varphi(\tau) = 2$, we get $\sum (\delta_i/y_i) = 4/105$. The relation $y_1^2 + y_2^2 + y_3^2 \leqq 396$ implies that $y_i \leqq 18$. If $y_1 = 18$, then $y_2 = y_3 = 6$. These values however do not satisfy the second relation for degrees. Hence $y_i < 18$. From the first degree relation we see that some of the δ_i must be negative. Hence we may assume that $\delta_3 = -1$. Then the only possible value of y_3 is 10. The first relation is now read as $\delta_1 y_1 + \delta_2 y_2 = 4$. Since $y_i > 2$, we must have $\delta_1 \delta_2 = -1$. We may therefore assume $\delta_1 = 1$ and $\delta_2 = -1$. Again the only possible value for y_2 is 10. Hence y_1 must be 14. Then $\sum y_i^2 = 432$ which implies that G has exactly nine irreducible characters.

We denote by $\langle \sigma \rangle$ the class of conjugate elements containing σ. Sometimes it is convenient to denote it by $\langle n \rangle$ if the order of σ is exactly n. G has classes $\langle 1 \rangle$, $\langle \tau \rangle$, $\langle \pi \rangle$ and one class $\langle \sigma\tau \rangle$ of order 6. Besides we have one class $\langle \sigma \rangle$ containing $\sigma \in U$. Since the order of G is 2520, G has at least two more classes, $\langle 5 \rangle$ and $\langle 7 \rangle$. There are two classes missing.

The values of characters on $\langle 5 \rangle$ have been determined except for Y_1 and Y_4. Let the value of Y_4 on an element ρ in the class $\langle 5 \rangle$ be x. Then x is rational since Y_4 is the only character of degree 6. Hence the eigenvalues $\neq 1$ of the matrix corresponding to ρ are contained with the same multiplicity, say b. If a is the multiplicity of the eigenvalue 1, we have $a + 4b = 6$ and $a - b = x$. If $b = 0$, then $x = 6$. The orthogonality relations yield that $3 + 2x^2 = 75$ is the order of the centralizer of ρ. This is impossible. Hence $b = 1$, $a = 2$, and $x = 1$. This implies in particular that the centralizer of ρ is the cyclic group generated by ρ. The orthogonality relation for columns $\langle 5 \rangle$ and $\langle 7 \rangle$ shows that Y_4 takes the value -1 on elements of order 7.

Let M be the representation module of the representation with character $Y = Y_4$. The dimension of the submodule of M consisting of elements which are left invariant by all the elements in a subgroup of G may be computed by summing Y over this subgroup. Since $Y(\tau) = 2$, the dimension of the subspace of invariant vectors by τ is 4. Hence any subgroup of G generated by two involutions has a space of invariant elements with dimension not smaller than 2. Summation of Y over a subgroup of order 7 is 0, which implies that any element ρ of order 7 has no invariant vector $\neq 0$ and that ρ is not conjugate to ρ^{-1}. Hence one of the missing classes is $\langle 7 \rangle$.

If σ is a generator of U, there is an involution τ' such that $\tau'\sigma = \sigma^2\tau'$.

Hence the dimension of the subspace of invariant vectors by σ is not smaller than 2. This implies that $Y(\sigma) = 0$ or 3. But $Y(\sigma) \equiv Y(\sigma\tau) = -1 \pmod 2$, and hence $Y(\sigma) = 3$. The order of a 3-Sylow subgroup S_3 of G is 9. If S_3 is cyclic, G has at least three classes of elements of order 9. This is impossible since there is only one class missing. Hence S_3 contains only elements of order $\leqq 3$. If $Y = 3$ for all elements of order 3, we get $6 + 24 = 30 \equiv 0 \pmod 9$ by summing Y over S_3. Hence there is an element of order 3 on which $Y \not\equiv 3$. This implies that the missing class is $\langle 3 \rangle$ and $Y = 0$ on this class. We have obtained all the classes and the values of Y.

The centralizer W of U is therefore of order 36, and the normalizer has order 72. There is an involution τ' in the normalizer of U and a 2-Sylow subgroup S' of G such that τ' is in the center of S' and $W \cap S' = e$. The subspace of invariant vectors for $\{U, \tau'\}$ is of dimension 3, and the space for S' has dimension 2. These spaces are subspaces of the space of invariant elements of τ', which is of dimension 4. There must be a vector m left invariant by both U and S'. Let H be the totality of elements of G which leave m invariant. H is a proper subgroup of G with order a multiple of 24 and an index divisible by 7. Since $\tau' \notin W$, U is not a normal subgroup of H. Let R be the normalizer of U in H. If $[H:e] = 24$, then R is of index 4, and hence H is the octahedral group. This is not the case since a central involution τ' of S' maps U into itself. If $[H:e] = 72$, R is again of index 4. If R is not normal, H has a normal subgroup H' of order 3 such that H/H' is the octahedral group. This is again impossible. If R is normal, the 3-Sylow subgroup of R is a normal subgroup of H. Since U has exactly four conjugate subgroups, every element of order 3 in R is conjugate to each other. This is not the case. If $[H:e] = 120$, then H contains a 5-Sylow subgroup T such that $[H:T] = 6$. Hence H is the symmetric group of five letters. The symmetric group of five letters satisfies the following property: if σ is an element of order 3, then for any element π of order 4 such that $\pi^2\sigma\pi^2 = \sigma^{-1}$, a 2-Sylow subgroup S containing π has an involution which commutes with σ. Hence from the condition $S' \cap W = e$ we see that this case can not happen. The only possibility is therefore $[H:e] = 360$ and $[G:H] = 7$. The permutation representation on the cosets of H has degree 7. Its character is therefore the sum of Y and the principal character. The kernel of this representation coincides with the kernel of the representation with character Y. Hence it must be faithful. G is therefore a subgroup of the symmetric group of seven letters. Since G has no normal subgroup of index 2, G is a subgroup of the alternating group A_7 Comparing the order we conclude that $G = A_7$.

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