A THEOREM ON AFFINE CONNEXIONS[1]

BY

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Introduction

The object of this study is to generalize a theorem of W. Ambrose (see [2]) to the non-Riemannian case.[2] The outline of our proof is the same as that of the theorem of Ambrose, but many features must be modified and are more complicated. We refer the reader to the bibliography for all definitions of fundamental concepts on C^∞ manifolds and connexions.

Our theorem characterizes a simply connected C^∞ manifold, on which is defined a complete affine connexion, by the behavior of the curvature and torsion forms under parallel translation along finitely broken geodesics emanating from some fixed point. In the analytic case, one need only consider unbroken geodesics. As an immediate consequence of our result, we obtain the (known) theorem which states that a simply connected manifold, on which is defined a complete connexion having zero curvature and torsion invariant under parallel translation, is a Lie group. Relaxing the simply connected hypothesis to just connected and adding an assumption that the holonomy group be the identity, we can prove the manifold is a homogeneous space, and an example shows we cannot hope to prove M is a Lie group under these hypotheses.

1. Notation and statement of the main theorem

Let M be a C^∞ manifold and m a point in M; then we denote the tangent space at m by M_m. Let $B(M)$ denote the bundle of bases over M, and π the map of $B(M)$ onto M. Given a connexion on $B(M)$, thus an "affine" connexion, we denote its 1-form by ω, i.e., ω is a C^∞ 1-form with values in $\mathrm{gl}(d, R)$, the Lie algebra of the general linear group $\mathrm{GL}(d, R)$, where d is the dimension of M and R is the field of real numbers. By a complete connexion we mean one in which our geodesics can be indefinitely extended (indefinitely in terms of the parameter, for the geodesic may be closed). A complete connexion on $B(M)$ allows us to define a map $\exp_m : M_m \to M$, for any $m \in M$, and a map $\mathrm{Exp}_b : M_m \to B(M)$, for any $b \in B(M)$. The latter allows us to carry information in $B(M)$ back to the simpler space M_m. We define these maps and denote the tangent field to a curve σ by T_σ.

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DEFINITION. Take $p \in M_m$ and let σ_p denote the unique geodesic such that $\sigma_p(0) = m$ and $T_{\sigma_p}(0) = p$. Then $\exp_m p = \sigma_p(1)$. Now let γ_p be the unique horizontal curve in $B(M)$ with $\gamma_p(0) = b$ and $\pi \circ \gamma_p = \sigma_p$. Then $\mathrm{Exp}_b\, p = \gamma_p(1)$.

Our completeness assumption assures us that both $\exp_m$ and Exp_b are defined on all of M_m. These mappings enjoy the following properties:

(1) $\pi \circ \mathrm{Exp}_b = \exp_{\pi(b)}$.

(2) $\exp_m$ and Exp_b are C^∞. (See [2] for proof.)

(3) $d\exp_m$ and $d\mathrm{Exp}_b$ are nonsingular at $0 \in M_m$.

Here we denote by $d\phi$, where ϕ is any C^∞ map, the induced mapping on tangent vectors.

DEFINITION. A connexion-preserving diffeo of a C^∞ manifold M, with connexion ω given on $B(M)$, onto a C^∞ manifold $'M$, with connexion $'\omega$ on $B('M)$, is a diffeo ϕ of M onto $'M$ such that $\bar{\phi}$, the induced diffeo of $B(M)$ onto $B('M)$, carries $'\omega$ into ω, i.e., $\omega = {}'\omega \circ d\bar{\phi}$.

We now define a space X, independent of any manifold. If we choose a manifold M, a complete connexion on M, and a "key" base $b \in B(M)$, then the curvature and torsion forms on $B(M)$ will induce certain real valued functions on X. An element of the space X can be thought of intuitively as a finitely broken geodesic σ emanating from the fixed point $\pi(b) = m$, plus a pair of tangent vectors at the final point of σ. In the following we let $u_1, \cdots, u_d$ denote the canonical coordinates on R^d, and $\delta_i = (\delta_{i1}, \cdots, \delta_{id})$ denote the unit points in R^d (δ_{ij} is the Kronecker symbol).

DEFINITION. We denote by X^d the set of all finite sequences $(r_1, \cdots, r_n; v, w)$—with n variable—where the r_i are any points in R^d and v, w are each arbitrary tangent vectors to R^d at o. We use X^d to characterize d-dimensional manifolds and we henceforth fix the dimension and drop the superscript.

We let Y be the set of all finite sequences $(r_1, \cdots, r_n)$—with n variable—and $r_i \in R^d$. Hence $X = Y \times R^d_o \times R^d_o$, where o is the origin in R^d. We call Y the space of broken segments.

Given M as above and a fixed key base $b = (m, e_1, \cdots, e_d)$ in $B(M)$, where $e_1, \cdots, e_d$ is a base of M_m, we define a mapping, denoted by m, of Y into M, and a mapping b of Y into $B(M)$. First let I_c, where $c = (p, f_1, \cdots, f_d)$ is any point of $B(M)$, be the linear transformation of $R^d \to M_p$ carrying δ_i into f_i. Then define

$$\exp_0 = \exp_m \circ I_b, \qquad \mathrm{Exp}_0 = \mathrm{Exp}_b \circ I_b.$$

If $(r_1, \cdots, r_n)$ is any point in Y, we define by induction on n

$$\exp_{r_1} = \exp_{\exp_0 r_1} \circ I_{\mathrm{Exp}_0 r_1}, \qquad \mathrm{Exp}_{r_1} = \mathrm{Exp}_{\mathrm{Exp}_0 r_1} \circ I_{\mathrm{Exp}_0 r_1},$$

$$\exp_{(r_1,\cdots,r_n)} = \exp_{\exp_{(r_1,\cdots,r_{n-1})} r_n} \circ I_{\mathrm{Exp}_{(r_1,\cdots,r_{n-1})} r_n},$$

$$\mathrm{Exp}_{(r_1,\cdots,r_n)} = \mathrm{Exp}_{\mathrm{Exp}_{(r_1,\cdots,r_{n-1})} r_n} \circ I_{\mathrm{Exp}_{(r_1,\cdots,r_{n-1})} r_n}.$$

Then define, again by induction on n

$$m(r_1) = \exp_0 r_1, \qquad b(r_1) = \operatorname{Exp}_0 r_1,$$
$$m(r_1, \cdots, r_n) = \exp_{(r_1,\cdots,r_{n-1})} r_n,$$
$$b(r_1, \cdots, r_n) = \operatorname{Exp}_{(r_1,\cdots,r_{n-1})} r_n.$$

The effect of $\exp_{(r_1,\cdots,r_n)}$ is to map R^d into M by first mapping it into M_m, then parallel-translating M_m (and thus R^d) along the geodesic into which $\exp_m$ carries the ray from o to $I_b r_1$ in M_m, namely to $\exp_m I_b r_1 = m(r_1)$, then parallel-translating $M_{m(r_1)}$ along the geodesic into which $\exp_{m(r_1)}$ carries the ray from o to $I_{b(r_1)} r_2$ in $M_{m(r_1)}$, namely to $m(r_1, r_2)$, $\cdots$, etc., until we reach $m(r_1, \cdots, r_n)$ which is the last point on this succession of broken geodesics, then spray R^d into M via the geodesics emanating from $m(r_1, \cdots, r_n)$.

We note that

$$\exp_{(r_1,\cdots,r_n)} = \pi \circ \operatorname{Exp}_{(r_1,\cdots,r_n)}, \qquad m(r_1, \cdots, r_n) = \pi \circ b(r_1, \cdots, r_n).$$

Hence we define $e_i(r_1, \cdots, r_n)$ for $i = 1, \cdots, d$ (a standard domain for i and j) by

$$b(r_1, \cdots, r_n) = (m(r_1, \cdots, r_n), e_1(r_1, \cdots, r_n), \cdots, e_d(r_1, \cdots, r_n)).$$

Recall the connexion on $B(M)$ gives rise to d^2 C^∞ real valued 2-forms Ω_{ij} and to d C^∞ real valued 2-forms Ω_i called the curvature forms and torsion forms, respectively (see [1]). Furthermore the connexion gives rise to d C^∞ vector fields E^i on $B(M)$ such that $E^1(b), \cdots, E^d(b)$ span the horizontal subspace of B_b at any point b in $B(M)$.

We now define $d^2 + d$ real valued functions on X, denoted by K_{ij} and K_i for $i, j = 1, \cdots, d$. Let $x = (r_1, \cdots, r_n; v, w)$ be any point in X. Let $v = \sum v_K(\partial/\partial u_K)(0)$, $w = \sum w_a(\partial/\partial u_a)(0)$; then define

$$K_{ij}(x) = \Omega_{ij}(\sum v_K E^K(b(r_1, \cdots, r_n)), \sum w_a E^a(b(r_1, \cdots, r_n))),$$
$$K_i(x) = \Omega_i(\sum v_K E^K(b(r_1, \cdots, r_n)), \sum w_a E^a(b(r_1, \cdots, r_n))).$$

The effect of $K_{ij}(x)$ is to determine the Ω_{ij} value of the pair of horizontal vectors at $b(r_1, \cdots, r_n)$ which lie over the parallel translates of $\sum v_K e_K$ and $\sum w_a e_a$ along the broken geodesic determined by $(r_1, \cdots, r_n)$. These functions K_{ij}, K_i are well defined for any triple consisting of (1) a C^∞ manifold M, (2) a complete connexion on $B(M)$, and (3) a key base $b \in B(M)$.

Our main theorem can now be stated precisely:

THEOREM 1. *Let M and $'M$ be two d-dimensional simply connected C^∞ manifolds each carrying a complete affine connexion. If after choosing $b \in B(M)$ and $'b \in B('M)$, the functions $K_{ij} = {}'K_{ij}$ and $K_i = {}'K_i$ on X, then there is a connexion-preserving diffeo of M onto $'M$.*

We prove this theorem in §3.

2. Induced structure on the tangent space

In this section we assume we have given a complete connexion on $B(M)$ for some fixed M. For any $b = (m, e_1, \cdots, e_d) \in B(M)$ we define forms θ_i^b, θ_{ij}^b, Θ_i^b, Θ_{ij}^b on M_m as follows: $\theta_i^b = \omega_i \circ d\text{Exp}_b$, $\theta_{ij}^b = \omega_{ij} \circ d\text{Exp}_b$, $\Theta_i^b = \Omega_i \circ d\text{Exp}_b$, $\Theta_{ij}^b = \Omega_{ij} \circ d\text{Exp}_b$.

Here ω_i are the natural 1-forms on $B(M)$ and $\omega_{ij} = y_{ij} \circ \omega$ where y_{ij} are the natural (linear) coordinate functions on $\text{gl}(d, R)$. The above forms are all C^∞. Henceforth in this section we fix b and drop it as a superscript on the above forms. We also sometimes use the symbols $e_1, \cdots, e_d$ as functions on M_m defined by $\text{Exp}_b\, p = (\exp_m p, e_1(p), \cdots, e_d(p))$ for $p \in M_m$. The following two lemmas are proved in [2].

LEMMA 1. *If s is a tangent vector to M_m at p, then* $d\exp s = \sum \theta_i(s)e_i(s)$.

LEMMA 2. *Let p be any point in M. We define the curve $\rho(t) = tp$ for $t \geqq 0$ and call ρ the ray through p. Then*

(1) $\sigma = \exp_m \circ \rho$ *is a geodesic emanating from m.*

(2) $\theta_i(T_\rho) = p_i$, *where* $p = \sum p_i e_i$.

(3) $\theta_{ij}(T_\rho) = 0$.

Carrying the Cartan structural equations back to M_m yields

$$\text{(i)} \qquad d\theta_i = -\sum \theta_{i\alpha}\, \theta_\alpha + \Theta_i,$$

$$\text{(ii)} \qquad d\theta_{ij} = -\sum \theta_{i\alpha}\, \theta_{\alpha j} + \Theta_{ij}.$$

These are the main tools in the next theorem.

THEOREM 2. *The 2-forms Θ_i and Θ_{ij} determine the 1-forms θ_i and θ_{ij}.*

This will follow from Theorem 2′ below. Let $z_1, \cdots, z_d$ be the dual linear functionals to $e_1, \cdots, e_d$; thus the z_i give us global coordinate functions on M_m.

THEOREM 2′. *Let $p = \sum p_i e_i$ be any point in M_m such that $\sum p_i^2 = 1$. Let $\rho(t) = tp$ for $t \geqq 0$ be the ray through p. Let T be the tangent to ρ. Let $A = \sum a_i\, \partial/\partial z_i$ be a constant field, i.e., the a_i are real numbers. Then along ρ we have the following functions of t: $\theta_i(tA)$, $\theta_{ij}(tA)$, $\Theta_i(T, tA)$, $\Theta_{ij}(T, tA)$ with*

(a) $\theta_{ij}(tA)(0) = 0, \qquad T\theta_{ij}(tA) = \Theta_{ij}(T, tA) \quad \text{for} \quad t \geqq 0.$

(b) $\theta_i(tA)(0) = 0, \qquad (T\theta_i\, tA)(0) = a_i,$

$T^2\theta_i(tA) = \sum p_\alpha\, \Theta_{i\alpha}(T, tA) + T\Theta_i(T, tA) \quad \text{for} \quad t \geqq 0.$

Theorem 2 follows immediately from Theorem 2′, for having Θ_i and Θ_{ij} allows us to solve the above differential equations for any ray and any A, and thus determine θ_i and θ_{ij} at any point of M_m.

Proof of Theorem 2′. Define the function $r = (\sum z_i^2)^{1/2}$, for $q \neq 0$ in M_m. We denote the "radial" vector field by $R = r^{-1} \sum z_i\, \partial/\partial z_i$. Computing will show that

$$\text{(1)} \qquad [R, rA] = r^{-1}(\textstyle\sum a_j z_j)R.$$

Lemmas 1 and 2 add

$$\theta_{ij}(R) = 0, \tag{2}$$

$$\theta_i(R) = \frac{z_i}{r}, \tag{3}$$

$$R\theta_i(R) = 0. \tag{4}$$

We use the exterior differentiation formula and the structural equation (i) to evaluate $d\theta_i$ in two ways:

$$\begin{aligned} d\theta_i(R, rA) &= R\theta_i(rA) - rA\theta_i(R) - \theta_i[R, rA] \\ &= -\sum \theta_{i\alpha}(R)\theta_\alpha(rA) + \sum \theta_{i\alpha}(rA)\theta_\alpha(R) + \Theta_i(R, rA). \end{aligned}$$

Hence by (1), (2), and (3),

$$\begin{aligned} R\theta_i(rA) &= rA\theta_i(R) + r^{-1}(\textstyle\sum a_j z_j)\theta_i(R) + \sum \theta_{i\alpha}(rA)\theta_\alpha(R) + \Theta_i(R, rA) \\ &= a_i - r^{-2}z_i(\textstyle\sum a_j z_j) + r^{-2}z_i(\sum a_j z_j) + \sum \theta_{i\alpha}(rA)\theta_\alpha(R) \qquad (5) \\ &\quad + \Theta_i(R, rA), \end{aligned}$$

and on ρ,

$$T\theta_i(tA) = a_i + \sum \theta_{i\alpha}(tA)\theta_\alpha(T) + \Theta_i(T, tA). \tag{6}$$

Thus $T\theta_i(tA)(0) = a_i$. By applying T again to (6), and using (4),

$$T^2\theta_i(tA) = \sum p_\alpha T\theta_{i\alpha}(tA) + T\Theta_i(T, tA). \tag{7}$$

We similarly treat $d\theta_{ij}$:

$$\begin{aligned} d\theta_{ij}(R, rA) &= R(\theta_{ij}\, rA) - rA(\theta_{ij}\, R) - \theta_{ij}[R, rA] \\ &= -\sum \theta_{i\alpha}(R)\theta_{\alpha j}(rA) + \sum \theta_{i\alpha}(rA)\theta_{\alpha j}(R) + \Theta_{ij}(R, rA). \end{aligned}$$

Using (1) and (2), we obtain

$$R(\theta_{ij}\, rA) = \Theta_{ij}(R, rA). \tag{8}$$

Evaluating (8) on ρ gives (a), and substituting this into (7) gives (b), Q.E.D.

3. Proof of Theorem 1

We assume the hypothesis of Theorem 1. Let $B(o, \delta)$ denote the open ball of radius δ about the origin in R^d, i.e., $B(o, \delta) = [r \in R^d : |r| < \delta]$. The key bases, b and $'b$, allow us to define a real valued function Δ on Y as follows: let $y \in Y$; then $\Delta(y) = \max[\delta \in R:\ \exp_y$ maps $B(o, \delta)$ diffemorphically onto a neighborhood of $m(y)$, and $'\exp_y$ maps $B(o, \delta)$ diffemorphically onto a neighborhood of $'m(y)]$. We define a set $Z \subset Y$ by

$$Z = [(r_1, \cdots, r_K) \in Y : |r_K| < \Delta(r_1, \cdots, r_{K-1})].$$

Now define an equivalence relation, $\backsim$, on the points of Z as follows: $z_1 \backsim z_2$ if all three of the following hold: (1) $m(z_1) = m(z_2)$, (2) $'m(z_1) = {}'m(z_2)$, and

(3) $b(z_1)g = b(z_2)$ implies $'b(z_1)g = 'b(z_2)$. Let W be the set of equivalence classes of this equivalence relation. Let I denote the natural mapping of $Z \to W$ by $I(z) =$ the equivalence class containing z, and let I_y be the mapping of the ball $B(o, \Delta(y))$ in R^d into W defined by $I_y(r) = I(y, r)$. We then define $e: W \to M$ by $e(w) = m(z)$ for any $z \in w$; and we define $'e: W \to 'M$ in a similar way. From the definition of equivalence these mappings are independent of the representative z.

A topology is defined on W by requiring that each I_y, for all $y \in Y$, be an open mapping of $B(o, \Delta(y))$ into W. Thus the topology is generated by all sets of the form $I_y O$ where $y \in Y$ and O is any open subset of $B(o, \Delta(y))$. We define $P_y = I_y B(o, \Delta(y))$.

LEMMA 3. *e maps P_y 1:1 onto $B_y = \exp_y B(o, \Delta(y))$, $'e$ maps P_y 1:1 onto $'B_y$, and I_y maps $B(o, \Delta(y))$ 1:1 onto P_y.*

Proof. By definition of Δ we know $\exp_y$ maps $B(o, \Delta(y))$ 1:1 onto B_y, but $\exp_y = e \circ I_y$. Hence since I_y maps $B(o, \Delta(y))$ onto P_y, both e and I_y must be 1:1 onto.

LEMMA 4. *e and $'e$ are continuous.*

Proof. The proof is similar to that of Lemma 3 in [2], p. 355.

For each $y \in Y$ we define the spray S_y, which is a mapping of B_y onto $'B_y$, by $S_y = '\exp_y \circ (\exp_y^{-1} \mid B_y)$. We will show eventually that each S_y is a connexion-preserving diffeo of B_y onto $'B_y$. Again for each $y \in Y$ let $J^y: M_{m(y)} \to 'M_{m(y)}$ by $J^y e_i(y) = 'e_i(y)$. We abbreviate by letting $\theta_i^y = \theta_i^{b(y)}$, etc.

LEMMA 5. *If $K_i = 'K_i$ and $K_{ij} = 'K_{ij}$, then $\theta_i^y = '\theta_i^y \circ dJ^y$ and $\theta_{ij}^y = '\theta_{ij}^y \circ dJ^y$ for each $y \in Y$.*

Proof. Choose $p = \sum p_i e_i(y) \in M_{m(y)}$. Let v, w be in $(M_{m(y)})_p$, and let $c = \mathrm{Exp}_{b(y)}\, p$. Then

$$\Theta_i(v, w) = \Omega_i \circ d\mathrm{Exp}_{b(y)}(v, w) = \Omega_i\left(\sum \theta_K(v)E^K(c), \sum \theta_a(w)E^a(c)\right)$$
$$= \sum\nolimits_{K,a} \theta_K(v)\theta_a(w)\, \Omega_i(E^K(c), E^a(c)).$$

Let $r = (p_1, \cdots, p_d) \in R^d$ and $x_{Ka} = (y, r; \partial/\partial u_K, \partial/\partial u_a) \in X$. Then $b(y, r) = c$ and thus $K_i(x_{Ka}) = \Omega_i(E^K(c), E^a(c))$. Hence

$$(*) \qquad \Theta_i(v, w) = \sum \theta_K(v)\theta_a(w)K_i(x_{Ka}).$$

This allows us to substitute (*) into the differential equations of Theorem 2′ and thus implies that knowing the functions K_i and K_{ij} allows one to integrate and obtain the 1-forms θ_i^y and θ_{ij}^y (on any $M_{m(y)}$).

The hypothesis of our lemma implies that θ_i^y and $'\theta_i^y \circ dJ^y$ satisfy the same differential equation and initial conditions on $M_{m(y)}$ and hence are equal. Similarly $\theta_{ij}^y = '\theta_{ij}^y \circ dJ^y$, Q.E.D.

LEMMA 6. *Each S_y is a connexion-preserving diffeo of B_y onto $'B_y$ such that if $|r| < \Delta(y)$ then $S_y m(y, r) = 'm(y, r)$, and $dS_y e_i(y, r) = 'e_i(y, r)$.*

Proof. By definition of the mappings S_y, m, and $'m$ we have $S_y m(y, r) = 'm(y, r)$. The last equation in the lemma follows by a proof similar to the proof of Lemma 4, part (d), p. 356 in [2].

It remains to show S_y is connexion-preserving, i e., letting $\bar{S}:B(B_y) \to B('B_y)$ by $(n, f_1, \cdots, f_d) \to (S_y n, dS_y f_1, \cdots, dS_y f_d)$, we must show $\omega = '\omega \circ d\bar{S}$. It is equivalent to show $d\bar{S}\, E^{ij} = 'E^{ij}$ and $d\bar{S}\, E^i = 'E^i$.

That $d\bar{S}\, E^{ij} = 'E^{ij}$ is immediate, for if $\lambda: GL \to B(M)$ is an admissible map for defining E^{ij} on a fiber in $B(M)$, then $\bar{S} \circ \lambda: GL \to B('M)$ is admissible for defining $'E^{ij}$ since $\bar{S} \circ R_g = R_g \circ \bar{S}$.

We show $d\bar{S}\, E^i = 'E^i$ for points on the cross section $\mathrm{Exp}_{b(y)} \circ \exp_{m(y)}^{-1}(B_y)$, which is sufficient since $\bar{S} \circ R_g = R_g \circ \bar{S}$. Henceforth in this proof we let $S = S_y$, $J = J^y$, $\exp = \exp_{m(y)}$, and $\mathrm{Exp} = \mathrm{Exp}_{b(y)}$. Let $U = \exp^{-1} B_y$, and we show $\bar{S} \circ \mathrm{Exp} = '\mathrm{Exp} \circ J$ on U. Take $p = \sum p_i e_i(y) \in U$, and let $r = (p_1, \cdots, p_d)$; then

$$\bar{S} \circ \mathrm{Exp}(p) = \bar{S}(b(y, r)) = (S \circ m(y, r), dSe_1(y, r), \cdots, dSe_d(y, r))$$

$$= 'b(y, r) = '\mathrm{Exp} \circ J(p).$$

Let $c = \mathrm{Exp}(p) = b(y, r)$, and let $v = d\exp^{-1} e_i(y, r)$. Thus $d(\pi \circ \mathrm{Exp})v = e_i(y, r)$, so $d\mathrm{Exp}(v) = E^i(c) + \sum \theta_{jK}(v) E^{jK}(c)$. Hence $d\bar{S}\, E^i(c) = d\bar{S}\, d\mathrm{Exp}(v) - \sum \theta_{jK}(v) d\bar{S}\, E^{jK}(c) = d'\mathrm{Exp}(dJv) - \sum '\theta_{jK}(dJv)' E^{jK}(\bar{S}c) = 'E^i(\bar{S}c)$, Q.E.D.

For convenience we will write $\exp_y^{-1}$ for $(\exp_y \mid B(o, \Delta(y)))^{-1}$ in the next lemma.

LEMMA 7. *Let $z_1 = (r_1, \cdots, r_K)$ and $z_2 = (s_1, \cdots, s_j)$ be in Z, and let $y_1 = (r_1, \cdots, r_{K-1})$, $y_2 = (s_1, \cdots, s_{j-1})$. If $z_1 \backsim z_2$, then there are a neighborhood O_1 of r_K and a neighborhood O_2 of s_j, with $O_i \subset B(o, \Delta(y_i))$, such that all the following hold:*

(1) $\exp_{y_2}^{-1} \circ \exp_{y_1}$ *maps O_1 diffemorphically onto O_2,*

(2) $'\exp_{y_2}^{-1} \circ '\exp_{y_1}$ *is the same as* $\exp_{y_2}^{-1} \circ \exp_{y_1}$ *on O_1,*

(3) *if one takes $p_i \in O_i$, then $\exp_{y_1} p_1 = \exp_{y_2} p_2$ implies*

$$(y_1, p_1) \backsim (y_2, p_2).$$

Proof. The proof is similar to that of Lemma 5 in [2], p. 357.

LEMMA 8. *Each I_y is continuous.*

Proof. The proof is similar to that of Lemma 6 in [2], p. 358.

LEMMA 9. *For any y_1 and y_2 in Y, the mappings I_{y_1} and I_{y_2} are C^∞-related, i.e., $(I_{y_2}^{-1} \mid (P_{y_2} \cap P_{y_1})) \circ I_{y_1} \in C^\infty$.*

Proof. The proof is similar to that of Lemma 9 in [2], p. 359.

LEMMA 10. *W is a Hausdorff space.*

Proof. The proof is similar to that of Lemma 7 in [2], p. 359.

LEMMA 11. *W is arcwise connected.*

Proof. Let $w_0 = I_o 0$, where 0 and o are the respective origins in R and R^d. We show we can connect any point of W to w_0 by a curve. Take any $w \in W$ and let $(r_1, \cdots, r_K) \in w$. Let ρ_1 be the ray from o to r_1 in R^d, ρ_2 the ray from o to r_2 in R^d, etc. Then let $\sigma_1 = I_o \circ \rho_1$, $\sigma_2 = I_{r_1} \circ \rho_2$, $\sigma_3 = I_{(r_1, r_2)} \circ \rho_3, \cdots, \sigma_K = I_{(r_1, \cdots, r_{K-1})} \circ \rho_K$. Then the curve $\sigma_K \cdots \sigma_2 \sigma_1$ joins w_0 to w, Q.E.D.

The above lemmas imply that W is an arcwise connected C^∞ manifold, and we proceed to define a connexion on $B(W)$.

DEFINITION. Let $\bar{\omega}$ denote the 1-form of a connexion on $B(W)$ defined as follows: if v is any tangent vector to $B(W)$, then $\bar{\omega}(v) = \omega \circ de(v) = '\omega \circ d'e(v)$.

The last equality holds since $'e = S_y \circ e$ on P_y and S_y is connexion-preserving.

LEMMA 12. *W is complete.*

Proof. Let $w \in W$ and let $z \in w$. Let $\bar{p} \in W_w$ and let $de(\bar{p}) = p$ in $M_{m(z)}$. We know there exists an indefinitely extendable geodesic σ in M such that $T_\sigma(0) = p$. Let $p = \sum p_i e_i(z)$ and let $r = (p_1, \cdots, p_d) \in R^d$. We define a curve $\bar{\sigma}$ in W by $\bar{\sigma}(t) = I(z, tr, 0)$ for all t. Then $e \circ \bar{\sigma}(t) = e \circ I_{(z,tr)}(0) = \exp_{(z,tr)}(0) = \exp_z tr = \exp_{m(z)} tp = \sigma(t)$. Since e is a connexion-preserving local diffeo it follows that $\bar{\sigma}$ is an indefinitely extendable geodesic with $T\bar{\sigma}(0) = \bar{p}$, Q.E.D.

We now generalize a theorem proved in [2]. Following a suggestion by R. Palais, we give a direct proof.

THEOREM 3. *Let M and N be d-dimensional connected C^∞ manifolds each carrying affine connexions. Let N be complete, and let ϕ be a connexion-preserving local diffeo of N into M. Then N is a covering space of M.*

Proof. To show ϕ is onto, we show $\phi(N)$ is both open (which is trivial since ϕ is a local diffeo) and closed. Let $m \in \overline{\phi(N)}$. Though M is not assumed to be complete, the map $\exp_m$ is defined and nonsingular in a neighborhood U of $0 \in M_m$. We may further assume that U is an open ball with respect to some base at m as metric in M_m. Let $V = \exp_m U$ be the corresponding neighborhood of m. By our assumption there is an $m_1 \in (V \cap \phi(N))$. Let $p = (\exp_m \mid U)^{-1}(m_1)$. Then $\sigma(t) = \exp_m tp$, for $t \in [0, 1]$, is a geodesic from m to m_1 with $T_\sigma(0) = p$. Let $\alpha(t) = \sigma(1-t)$; thus α is a geodesic from m_1 to m. Choose any $n \in N$ such that $\phi(n) = m_1$. Let $q = d\phi^{-1} T_\alpha(0) \in N_n$. Let $\gamma(t) = \exp_n tq$ for all t. Then γ is a geodesic in N; hence $\phi \circ \gamma$ is a geodesic in M since ϕ is connexion-preserving. Moreover,

$\phi \circ \gamma(0) = \alpha(0) = m$ and $T_{\phi\circ\gamma}(0) = d\phi(q) = T_\alpha(0)$, which implies $\phi \circ \gamma = \alpha$. Hence $\phi \circ \gamma(1) = m$, and ϕ is onto. Note we have also proved that M is complete.

We next show ϕ evenly covers any $m \in M$. Let U and V be associated with m as in the first paragraph. We show V is evenly covered by ϕ. Let $n \in N$ and $\phi(n) = m$. Since ϕ is a local diffeo, $d\phi^{-1}$ maps M_m isomorphically onto N_n. Define $f: V \to N$ by $f = \exp_n \circ d\phi^{-1} \circ (\exp_m \mid U)^{-1}$. Let $f(V) = V'$. Then (1) f is C^∞ by definition; (2) $\phi \circ f =$ identity map on V for f lifts geodesics in V that emanate from m into geodesics in V' that emanate from n; moreover, since ϕ is connexion-preserving, ϕ projects these geodesics back into geodesics that have the same tangent vectors at m, and hence for such geodesics σ, $\phi \circ f \circ \sigma = \sigma$; (3) similarly $f \circ (\phi \mid V') =$ identity. Thus ϕ is a diffeo of V' onto V, and with this fact it is trivial to show V' is the connected component of n in $\phi^{-1}(V)$, Q.E.D.

Proof of Theorem 1. The mappings e and $'e$ are connexion-preserving local diffeos of the complete manifold W into M and $'M$, respectively. By Theorem 3, W is a covering space for both M and $'M$. By the hypothesis of Theorem 1, M and $'M$ are simply connected; hence e and $'e$ are connexion-preserving diffeos (in the large) of W onto M and $'M$ respectively. Thus $'e \circ e^{-1}$ is a connexion-preserving diffeo of M onto $'M$ which maps the key base b into the key base $'b$, Q.E.D.

4. The analytic case

Let $Y_k = R^d \times \cdots \times R^d$ (k times), and let $X_k = Y_k \times R_o^d \times R_o^d$. For an admissible triple (M, ω, b) let $K_i^k = K_i \mid X_k$, and $K_{ij}^k = K_{ij} \mid X_k$.

THEOREM 4. *Let M and $'M$ be two d-dimensional simply connected analytic manifolds each carrying a complete (analytic) affine connexion. If for key bases $b \in B(M)$ and $'b \in B('M)$, $K_i = {}'K_i$ and $K_{ij} = {}'K_{ij}$ on X_1, then there exists a connexion-preserving analytic homeomorphism of M onto $'M$ which maps b into $'b$.*

Proof. We first show $K_i^k = {}'K_i^k$ and $K_{ij}^k = {}'K_{ij}^k$ for any $k \geqq 1$. These functions are analytic on the finite-dimensional space X_k, where the analytic structure on X is induced by the natural coordinate functions on R^d. Hence by analyticity it is sufficient to show $K_i^k = {}'K_i^k$ and $K_{ij}^k = {}'K_{ij}^k$ on some neighborhood of "zero" in X_k.

Let $b = (m, e_1, \cdots, e_d)$ be the key base of M. Let $U(m, \delta) = [p \in M_m : p = \sum p_i e_i$ and $\sum p_i^2 < \delta^2]$. Let $U = U(m, \delta)$ for some δ (fixed) such that $\exp_m$ maps $U(m, \delta)$ diffemorphically onto a neighborhood of m. Let $V = \exp_m U$, and for $n \in V$ let $b(n)$ be the base at n obtained by parallel-translating b along the unique geodesic in V from m to n. Then for $n \in V$ and real $\varepsilon > 0$ let $U(n, \varepsilon) =$ the open ball, in M_n, of radius ε with respect to $b(n)$; let $V(n, \varepsilon) = \exp_n U(n, \varepsilon)$; and let $S(n, \varepsilon) = \exp_n$ [the boundary of $U(n, \varepsilon)$].

LEMMA. *Take* $k \geqq 1$, m, *and* δ *as above. For each* $j = 1, \cdots, k$ *there is a real number* $\varepsilon(j) > 0$ *such that if* $n \in \bar{V}(m, (j-1)\delta/k)$ *then* $V(n, \varepsilon(j)) \subset V(m, j\delta/k)$.

Proof. For $j = 1$ this is trivial. Consider any $j \neq 1$. Then certainly for every $n \in \bar{V}(m, (j-1)\delta/k)$ there is a real $\varepsilon(n)$ such that $V(n, \varepsilon(n)) \subset V(m, j\delta/k)$ since the latter is open. We show $\varepsilon(n)$ are bounded away from zero, i.e. $\varepsilon(n) > \varepsilon > 0$ and then $\varepsilon(j) = \varepsilon$.

Suppose no such ε exists. Then for every $\varepsilon_h = 1/h$ there is a point $n_h \in \bar{V}(m, (j-1)\delta/k)$ and a point $m_h \in S(m, j\delta/k)$ with $m_h \in V(n_h, \varepsilon_h)$. Compactness of $\bar{V}(m, (j-1)\delta/k)$ implies $n_{\nu h} \to n \in \bar{V}(m, (j-1)\delta/k)$ and we reorder to have $n_h \to n$. Consider any real $\rho > 0$. There is an integer H_0 such that for $h \geqq H_0$, $n_h \in V(n, \rho)$. Moreover, since the $V(n_h, \varepsilon_h)$ become arbitrarily small (i.e., $\varepsilon_h \to 0$), we may choose $H \geqq H_0$ such that $V(n_H, \varepsilon_H) \subset V(n, \rho)$. Hence $m_H \in V(n, \rho)$. Hence $m_h \to n$, and since $S(m, j\delta/k)$ is closed, $n \in S(m, j\delta/k)$ which contradicts the fact that $\exp_m$ is a diffeo on U. This completes the proof of the lemma.

We may now define our neighborhood in X. Let δ be chosen so that both $U(m, \delta)$ and $'U('m, \delta)$ are mapped in a 1:1 way. Let $\varepsilon(1), \cdots, \varepsilon(k)$ be the sequence of ε's provided by the lemma applied to k, m, and δ; and let $'\varepsilon(1), \cdots, '\varepsilon(k)$ belong similarly to k, $'m$, and δ. Let $\delta(j) = \min[\varepsilon(j), '\varepsilon(j)]$ for $j = 1, \cdots, k$. Then let $O = B(o, \delta(1)) \times \cdots \times B(o, \delta(k))$, and we prove $K_i = 'K_i$, etc., on $O \times R_o^d \times R_o^d$.

Let $y = (r_1, \cdots, r_k) \in O$. Then for $t \in [0, 1]$ the points $m(tr_1)$, $m(r_1, tr_2), \cdots, m(r_1, \cdots, r_{k-1}, tr_k)$ lie in V by the lemma. Thus we may use $(\exp_m | U)^{-1}$ on V to lift the broken geodesic σ determined by y to a broken curve γ in U. Similarly we obtain $'\gamma$ in $'U$. Let $\bar{\sigma}$ be the unique horizontal curve in $B(M)$ lying over σ with $\bar{\sigma}(0) = b$ and let $\tilde{\sigma} = \mathrm{Exp}_b\, \gamma$. We define the curve g in GL by $\tilde{\sigma} = R_g \circ \bar{\sigma}$. Similarly define $'\bar{\sigma}$, $'\tilde{\sigma}$, and $'g$. The assumption $K_i = 'K_i$ and $K_{ij} = 'K_{ij}$ on X_1 implies (by Lemma 5 with $y = o$) that $\theta_i = '\theta_i \circ dJ$ and $\theta_{ij} = '\theta_{ij} \circ dJ$ on M_m. Then by Lemma 6, $S = '\exp_{'m} \circ J \circ \exp_m^{-1}$ is a connexion-preserving analytic homeomorphism of V onto $'V$, hence $'\bar{\sigma} = \bar{S} \circ \bar{\sigma}$, $'\tilde{\sigma} = \bar{S} \circ \tilde{\sigma}$, and $g = 'g$.

Let $v = \partial/\partial u_j$, $w = \partial/\partial u_h$, and $x = (y; v, w)$. Then

$$K_i(x) = \Omega_i(E^j(b(y)), E^h(b(y))) = \Omega_i(E^j(\bar{\sigma}(k)), E^h(\bar{\sigma}(k)))$$

$$= \Omega_i\left(\sum g_{\alpha j}^{-1}\, dR_{g^{-1}}\, E^\alpha(\tilde{\sigma}), \sum g_{\beta h}^{-1}\, dR_{g^{-1}}\, E^\beta(\tilde{\sigma})\right) = \sum g_{\alpha j}^{-1}\, g_{\beta h}^{-1} g_{is}\, \Omega_s(E^\alpha(\tilde{\sigma}), E^\beta(\tilde{\sigma})),$$

where all curves are to be evaluated at $t = k$. If we let $\gamma(k) = \sum p_i e_i$, $r = (p_1, \cdots, p_d) \in R^d$, and $x_1 = (r; \partial/\partial u_\alpha, \partial/\partial u_\beta)$, then $\Omega_s(E^\alpha(\tilde{\sigma}), E^\beta(\tilde{\sigma})) = K_s(x_1) = 'K_s(x_1)$ by hypothesis, since $x_1 \in X_1$. Hence by substitution into the above equation $K_i(x) = 'K_i(x)$, and similarly $K_{ij}(x) = 'K_{ij}(x)$ for all $x \in O \times R_o^d \times R_o^d$.

Thus $K_i^k = 'K_i^k$ and $K_{ij}^k = 'K_{ij}^k$ on any X_k which implies the equality of the K's and $'K$'s on X since any x is in some X_k. Then by Theorem 1 we have a connexion-preserving diffeo ϕ of M onto $'M$. Locally, in any B_y, $\phi = {'\exp_y} \circ \exp_y^{-1}$, and thus both ϕ and ϕ^{-1} are analytic, Q.E.D.

5. Two applications

THEOREM 5. *Let M be a simply connected manifold on which is defined a complete affine connexion with zero curvature and torsion invariant under parallel translation. Then M admits a Lie group structure such that left translations induce the original connexion.*

Proof. Simple connectedness implies the global holonomy group equals the local holonomy group, and zero curvature then implies the latter is the identity (see [1]). Thus choosing a key base $b \in B(M)$ we obtain a unique horizontal cross section through b. The vector field form of the Cartan structural equations is (see [1])

$$[E^i, E^j] = -\sum \Omega_k(E^i, E^j)E^k - \sum \Omega_{kh}(E^i, E^j)E^{kh},$$

which in this case becomes $[E^i, E^j] = \sum c_{ijk} E^k$, where $c_{ijk} = -\Omega_k(E^i, E^j)$ is constant on the horizontal section through b. The constants c_{ijk} give us a set of structural constants for a Lie algebra L over the real field. Let G be the corresponding simply connected Lie group. Taking the affine connexion on G induced by left translation we obtain a complete connexion. Let $Y^1, \cdots, Y^d$ be a basis of L such that $[Y^i, Y^j] = \sum c_{ijk} Y^k$, and let $'b = (e, Y^1(e), \cdots, Y^d(e))$ be the key base in $B(G)$. Then $K_{ij} = {'K_{ij}} = 0$ and trivially $K_i = {'K_i}$ at $O \times R_o^d \times R_o^d$ and hence everywhere. By Theorem 1 we obtain a connexion-preserving diffeo of M onto G which induces the group structure on M, Q.E.D.

THEOREM 6. *Let M be a connected manifold on which is defined a complete affine connexion with the identity as holonomy group and torsion invariant under parallel translation. Then M is diffemorphic to a homogeneous space. Indeed, there is a connexion-preserving diffeo of M onto a homogeneous space G/K where G is a simply connected Lie group, K is a discrete subgroup of G, G/K is the space of right cosets, and the connexion on G/K is induced by the left invariant vector fields on G.*

Proof. Let N be a simply connected covering of M, and let $\pi: N \to M$ be the projection map. Since π is a local diffeo, we have an induced map $\bar{\pi}: B(N) \to B(M)$. Letting ω be the connexion form on $B(M)$, we define a connexion form on $B(N)$ by $\bar{\omega} = \omega \circ d\bar{\pi}$. Thus π becomes a connexion-preserving local diffeo, and hence the connexion $\bar{\omega}$ has zero curvature and torsion invariant under parallel translation. By Theorem 5 we may define a group structure on N such that N becomes a simply connected Lie group G and $\bar{\omega}$ becomes the connexion induced by the left invariant vector fields on G.

Let e be the identity of G; let $m_0 = \pi(e)$; and let $K = \pi^{-1}(m_0)$. Then K, as a point set, is in 1:1 correspondence with the group of deck transformations of G, i.e., each element $k \in K$ gives rise to a diffeo $\bar{k}:G \to G$ such that $\bar{k}(e) = k$ and $\pi \circ \bar{k} = \pi$. With this transformation group structure K is isomorphic to the fundamental group of M. We next prove three lemmas concerning K.

LEMMA 1. *Each deck transformation $\bar{k}$ is connexion-preserving.*

Proof. Let $k_*:B(G) \to B(G)$ be induced by $\bar{k}$. Then $\bar{\omega} = \omega \circ d\bar{\pi} = \omega \circ d(\bar{\pi} \circ k_*) = \bar{\omega} \circ dk_*$, Q.E.D.

LEMMA 2. *Each deck transformation $\bar{k}$ is equal to left translation by k, which we denote by ϕ_k.*

Proof. We use the fact that a connexion-preserving diffeo is completely determined by its action on one base, and thus we need only show $\bar{k}$ and ϕ_k are the same for the base $(e, X^1, \cdots, X^d)$ where $X^1, \cdots, X^d$ is any base of G. First note $\bar{k}(e) = k$ and $\phi_k(e) = ke = k$. Let $d\phi_k X_e^i = X_k^i$ (thus making the X into left invariant vector fields), and we show that $d\bar{k}X_e^i = X_k^i$.

Since G is connected, let σ be any (broken) C^∞ curve with $\sigma(0) = e$ and $\sigma(1) = k$. Let $\bar{\sigma}(t) = \big(\sigma(t), X^1(\sigma(t)), \cdots, X^d(\sigma(t))\big)$; thus $\bar{\sigma}$ is a horizontal curve in $B(G)$ lying over σ. Let $d\pi X^i = e^i$ define a parallel base along the closed curve $\alpha = \pi \circ \sigma$. Thus $d\pi X_k^i = e^i(\alpha(1))$, but zero holonomy implies $d\pi X_k^i = e^i(\alpha(0)) = d\pi X_e^i$. On the other hand, $d\pi X_e^i = d\pi(d\bar{k}X_e^i)$. Hence $d\pi(d\bar{k}X_e^i) = d\pi X_k^i$ and since $d\pi$ is an isomorphism, $d\bar{k}X_e^i = X_k^i = d\phi_k X_e^i$. Hence $\bar{k} = \phi_k$, Q.E.D.

LEMMA 3. *The set K is a discrete subgroup of G, and the group structure on K induced by the deck transformations is isomorphic to the subgroup structure.*

Proof. If $k_1, k_2 \in K$ then $\pi(k_1 k_2^{-1}) = \pi \circ \phi_{k_2} \circ \phi_{k_1}^{-1}(k_1 k_2^{-1}) = \pi(e) = m_0$ implies $k_1 k_2^{-1} \in K$, and thus K is a subgroup. The discreteness of K follows from the fact that π is a local diffeo. Finally, $\bar{k}_1 \circ \bar{k}_2 = \phi_{k_1} \circ \phi_{k_2} = \phi_{k_1 k_2}$, Q.E.D.

Now let $'M = [Kg]$ be the right coset space G/K. With the usual manifold structure on $'M$, the projection $\eta:G \to {'M}:g \to Kg$ is a local diffeo, the kernel of η is K, and $\eta \circ \phi_k = \eta$ for all $k \in K$. We define a connexion on $'M$ by giving a section in $B('M)$. Let $X^1, \cdots, X^d$ be, as above, a base of L. Then $d\eta X^1, \cdots, d\eta X^d$ defines a base at every point of $'M$. This follows since if $g_2 = kg_1$, then $d\eta X_{g_1}^i = d\eta \circ d\phi_k X_{g_1}^i = d\eta X_{g_2}^i$ at the point $Kg_1 = Kg_2$ in $'M$. By definition, η is connexion-preserving for $\bar{\eta}$ maps a horizontal section in $B(G)$ onto a horizontal section in $B('M)$. This implies the torsion of $'\omega$ is invariant under parallel translation, and the existence of a horizontal section implies the holonomy group is the identity.

The map $\psi:{'M} \to M$ by $\psi(Kg) = \pi(g)$ is a well defined diffeo. Since $\psi \circ \eta = \pi$ we have $\omega \circ d\bar{\psi} \circ d\bar{\eta} = \omega \circ d\bar{\pi} = \bar{\omega} = {'\omega} \circ d\bar{\eta}$. Since η is a local diffeo, $d\bar{\eta}$ is an isomorphism; thus $\omega \circ d\bar{\psi} = {'\omega}$, Q.E.D.

Example.[3] The following example indicates that under the hypothesis of Theorem 6 we cannot hope to prove M is a Lie group, even if we assume further that the fundamental group of M be abelian.

We define M. Let S^3 be the 3-sphere considered as the set of unit quaternions. Let K be the subgroup of S^3 consisting of the elements $[e, i, -e, -i]$. The subgroup K is abelian but not normal in S^3. Let M be the space of right cosets of S^3 modulo K. With the connexion on M defined as in the proof of Theorem 6, M satisfies the hypothesis of that theorem. But M is a compact 3-dimensional manifold with a fundamental group equal to the cyclic group of order 4. By the Cartan-Killing classification, the only compact 3-dimensional Lie groups are S^3 and R_3, the rotation group on R^3. Neither S^3 nor R_3 has the same fundamental group as M. Hence M cannot be a Lie group.

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