THE ENGEL ELEMENTS OF A SOLUBLE GROUP

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An ordered pair of subsets (A, B) of a given group G shall be said to satisfy the Engel condition if to each $a \in A$ and $b \in B$ there corresponds an integer $k = k(a, b)$ such that

$$[a, \underbrace{b, \cdots, b}_{k}] = 1.$$

(Here $[x_1, x_2]$ stands for the element $x_1^{-1}x_2^{-1}x_1 x_2$ and, for $k > 1$, $[x_1, \cdots, x_{k+1}] = [[x_1, \cdots, x_k], x_{k+1}]$.) The fact that (A, B) satisfies the Engel condition will be denoted by the symbol $A \mathfrak{e} B$. If, moreover, the integer k can be chosen independent of the element a in A, or of b in B, then this will be written $A \mid\mathfrak{e} B$, or $A \mathfrak{e}\mid B$, as the case may be.

We shall be mainly concerned with situations in which one of the sets A, B consists of a single element and the other of the whole group. It will be convenient to have a terminology specially adapted to these cases. Let us therefore call an element g a *left Engel element*,[1] or *bounded left Engel element* according as $G \mathfrak{e} g$, or $G \mid\mathfrak{e} g$; and a *right Engel element*, or *bounded right Engel element* according as $g \mathfrak{e} G$, or $g \mathfrak{e}\mid G$. Thus, for example, an element in a locally nilpotent normal subgroup is a left Engel element, and one in a nilpotent normal subgroup is a bounded left Engel element; while an element in any term of the ascending central series is a right Engel element, and one in a finite term of this series is a bounded right Engel element. Just how typical these examples really are is at present unknown. The best that can be said is the following: in a group, satisfying the maximal condition on subgroups, the set of all left Engel elements coincides with the set of all bounded left Engel elements and forms the maximal nilpotent normal subgroup; while the set of all right Engel elements coincides with the set of all bounded right Engel elements and forms the hypercentre of the group. These exceedingly elegant results are due to Reinhold Baer [3].

Our main aim in this paper is to show that, besides the groups with maximal condition, there exists another class of groups in which the Engel elements are well behaved: we shall, in fact, prove that in every soluble group the sets consisting of the four types of Engel elements all form subgroups. However, unlike the situation in the groups studied by Baer, it is here quite possible for no two of these subgroups to coincide: we shall construct a soluble group in which the four subgroups are distinct from each other and from the hypercentre. We begin, in §1, by considering four conditions analogous to, but considerably stronger than, the four Engel conditions, and we show that, in

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[1] Plotkin calls a left Engel element a nilelement.

any group, the set of elements satisfying any one of these conditions forms a subgroup. It then turns out (in §2) that in a *soluble* group these four subgroups coincide precisely with the four different sets of Engel elements.

1. Radicals

The strengthened versions of the four Engel conditions to be studied in this section all depend on Wielandt's notion of "Nachinvarianz".

By a *series* from a subgroup H to the group G we shall always mean a well-ordered ascending normal system in the sense of Kurosh [7], p. 171. If H can be linked to G by a series, it will be called *serial*, and we shall write $H \infty\triangleleft G$. If, in particular, there exists a finite series linking H to G, we shall say that H is *finitely-serial*[2] and (following Wielandt) write $H \triangleleft\triangleleft G$. (If H is normal, we write $H \triangleleft G$.) An element g will be called serial or finitely-serial according as the subgroup generated by g is serial or finitely-serial.[3]

It is rather easy to see that $g \infty\triangleleft G$ implies $G \mathfrak{e} g$ and $g \triangleleft\triangleleft G$ implies $G \,|\mathfrak{e}\, g$. We are therefore dealing with conditions which genuinely are restrictions of the two left Engel conditions.

Let us denote the set of all serial elements in G by $\sigma(G)$ and of all finitely-serial elements by $\bar{\sigma}(G)$. It was shown by Baer [2] that $\bar{\sigma}(G)$ is always a characteristic subgroup, and it will be proved below that the same is also always true of $\sigma(G)$. Our proof of this fact is closely modelled on Baer's argument, but first we must establish that a group generated by serial elements is locally nilpotent. This turns out to be a very easy consequence of the theorem of Hirsch [6] and Plotkin [8] that local nilpotence is a multiproperty. (A property $\mathcal{P}$ is called a multiproperty if, whenever two normal subgroups possess $\mathcal{P}$, then their product also possesses $\mathcal{P}$.)

The proposition that follows below is not really needed until §2, but since it leads to a new way of looking at the Hirsch-Plotkin theorem, we have thought it worth while to place it here. Our tools in the following lemmas are the basic commutators.

Let X be a finite subset of a given group, and define $X_0 = X$; if subsets $X_0, \cdots, X_{k-1}$ and elements $b_1, \cdots, b_{k-1}$ have already been defined, then choose b_k to be any element in X_{k-1} of lowest possible weight (in the elements of X), and define X_k to be the set consisting of all elements

$$[b, \underbrace{b_k, \cdots, b_k}_{r}]$$

for $r \geqq 0$ and $b \in X_{k-1}$, $b \neq b_k$.[4] The resulting sequence of elements $b_1, b_2, b_3, \cdots$ is called a basic sequence in X, and we shall call $X_0, X_1, X_2, \cdots$ the corresponding *generating sequence.*

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[2] Some authors use the term "accessible", others the German "nachinvariant."

[3] Baer calls a finitely-serial element a nilelement.

[4] $[x, \underbrace{y, \cdots, y}_{0}]$ is interpreted to be x.

Suppose F is a free group of finite rank and $b_1, b_2, \cdots$ a basic sequence in a free set of generators of F. If b_N is the last basic commutator of weight $< n$, and K is the normal closure of $b_{N+1}, b_{N+2}, \cdots$, then one can show, by an argument very similar to that used to prove Lemma 5.1 in [5], that every element of F can be written in the form

$$b_1^{k_1} \cdots b_N^{k_N} c$$

with integers $k_1, \cdots, k_N$ and $c \in K$. From this, and Theorem 5.6 in [5], it follows immediately that K is actually the n^{th} term of the descending central series of F. This fact is due to P. Hall.

Throughout this paper we shall write the descending central series of a group G as

$$G = \gamma_1(G) \geqq \gamma_2(G) \geqq \cdots$$

and, for a subset S of G, denote the subgroup that S generates by $\mathrm{Gp}\{S\}$. Thus

$$\gamma_n(G) = \mathrm{Gp}\{[g_1, \cdots, g_n], \text{all } g_i \in G\}.$$

We shall also adopt the notation

$$[x, \underbrace{y, \cdots, y}_{r}] = [x, {}_r y].$$

LEMMA 1. *Let F be the free group on a finite set X and $b_1, b_2, \cdots$ a basic sequence in X. Define E to be the normal closure in F of $b_2, b_3, \cdots$, and choose any positive integer k. Then any b_s of weight* wt $b_s \geqq k^2 - k + 1$ *either satisfies $b_s \in \gamma_k(E)$ or involves $[b_j, {}_r b_1]$, where $r \geqq k$ and $b_j \in X$.*

Proof. Let wt $b_s \geqq k^2 - k + 1$, and suppose the only simple basic commutators with b_1 involved that occur in b_s are, with correct multiplicities,

$$[b_{j_i}, {}_{r_i} b_1], \qquad i = 1, \cdots, \rho,$$

where $0 < r_i < k$ for each i. (To assert that, for example, $c = [b_j, {}_r b_1]$ occurs with correct multiplicity in the above set of ρ commutators simply means that the number of integers i for which $j_i = j$ and $r_i = r$ is precisely the number of times c occurs in b_s.) We shall prove $b_s \in \gamma_k(E)$.

Suppose to the contrary that $b_s \notin \gamma_k(E)$. We choose any ρ distinct nonunit elements lying in E but not in X, say $b_{-1}, \cdots, b_{-\rho}$, replace the occurrence (in b_s) of

$$[b_{j_i}, {}_{r_i} b_1]$$

by b_{-i} ($i = 1, \cdots, \rho$), and denote the resulting element of E by $\tilde{b}_s$. Our hypothesis that $b_s \notin \gamma_k(E)$ implies that wt $\tilde{b}_s < k$ (where wt $\tilde{b}_s$ represents the weight of $\tilde{b}_s$ as commutator in $b_{-1}, \cdots, b_{-\rho}, X$). Consequently

$$\text{(1)} \qquad \mathrm{wt}\, b_s = \mathrm{wt}\, \tilde{b}_s + (r_1 + \cdots + r_\rho) < k + (r_1 + \cdots + r_\rho),$$

and since, clearly,

$$\text{wt } b_s \geqq \rho + (r_1 + \cdots + r_\rho),$$

we conclude

$$(2) \qquad \rho \leqq k - 1.$$

Now, by hypothesis, wt $b_s \geqq k^2 - k + 1$, and so (1) implies

$$k^2 - k + 1 < k + (r_1 + \cdots + r_\rho) \leqq k + \rho(k - 1),$$

which, by (2), yields

$$k^2 - k + 1 < k + (k - 1)^2 = k^2 - k + 1,$$

a visible contradiction. Hence $b_s \in \gamma_k(E)$, as required.

PROPOSITION 1. *Let H be a normal subgroup in $G = \text{Gp}\{H, g\}$, and suppose H has a generating set Y such that $[y, {}_kg] = 1$ for all $y \in Y$. If H is nilpotent of class c, then G is nilpotent of class $\leqq m(m - 1)$, where $m = \max(k, c + 1)$.*

Proof. It is clear that we need only prove every *finitely generated* subgroup of G to be nilpotent of class $\leqq m(m - 1)$. Now every such subgroup is contained in a subgroup generated by a set of the form $\{g, y_1, \cdots, y_q\}$, where $y_i \in Y$. Put $K = \text{Gp}\{g, y_1, \cdots, y_q\}$.

Let F be a free group of rank $q + 1$ and $b_1, b_2, \cdots$ a basic sequence in a free set of generators of F. The mapping

$$\begin{cases} b_1 \to g \\ b_{i+1} \to y_i \quad \text{for} \quad i = 1, \cdots, q \end{cases}$$

extends to a homomorphism φ of F onto K, and the image under φ of E, the normal closure of $b_2, b_3, \cdots$, is contained in $K \cap H$. If $m = \max(k, c + 1)$ and wt $b_s \geqq m^2 - m + 1$, then, by Lemma 1, either $b_s \in \gamma_m(E)$, or b_s involves $[b_j, {}_rb_1]$ with $r \geqq m$. Hence either $b_s^\varphi \in \gamma_m(H)$, or b_s^φ involves $[y_{j-1}, {}_rg]$ with $r \geqq k$, so that anyway $b_s^\varphi = 1$. But, as was noted above, the normal closure in F of all b_s of weight $\geqq m^2 - m + 1$ is precisely $\gamma_{m^2-m+1}(F)$, and hence

$$\gamma_{m^2-m+1}(F)^\varphi = \gamma_{m^2-m+1}(K) = 1,$$

as required.

LEMMA 2. *Let the subset Y of G generate the subgroup H. If the element g is such that $g^{-1}Yg \leqq H$ and $Y \mathfrak{e}\, g$, then $g^{-1}Hg = H$.*

Proof. Since $g^{-1}Yg \leqq H$ implies $g^{-1}Hg \leqq H$, it only remains to show that $gHg^{-1} \leqq H$. It is therefore sufficient to prove $gYg^{-1} \leqq$ H.

Writing $y_r = [y, {}_rg]$, we see, by a simple induction on m, that, for all $m \geqq 0$,

$$g^{-m}yg^m = w(m)y_m,$$

where $w(m)$ is a word in $y_0, y_1, \cdots, y_{m-1}$. Now $y \in H$ implies $g^{-m}yg^m \in H$

for all $m \geqq 0$, and so, if we already know that $y_0, \cdots, y_{m-1}$ all lie in H, we obtain $y_m \in H$. Thus $y_i \in H$ for all $i \geqq 0$.

We observe next that $[y, g^{-1}] = [y_1, g^{-1}]^{-1}y_1^{-1}$ and, by repeated application of this identity, deduce that, for any given positive integer k, $[y, g^{-1}]$ is a word in $y_1, \cdots, y_k, [y_k, g^{-1}]$. Choose $y \in Y$ and k such that $y_{k+1} = 1$. Then $[y_k, g^{-1}] = 1$, and hence $[y, g^{-1}]$ is a word in $y_1, \cdots, y_k$, whence $[y, g^{-1}] \in H$. Thus $gyg^{-1} \in H$ for all $y \in Y$.

THEOREM 1. *Let G be a group generated by a finite set X, and $b_1, b_2, \cdots$ a basic sequence in X with generating sequence $X_0, X_1, \cdots$. If there exists $k \geqq 0$ such that* $\mathrm{Gp}\{X_k\}$ *is nilpotent and $X_{i-1} \mathfrak{e} b_i$ for each $i = 1, \cdots, k$, then G is nilpotent.*

Proof. Putting $H_i = \mathrm{Gp}\{X_i\}$, we see that $H_{i-1} = \mathrm{Gp}\{X_i, b_i\}$, $b_i^{-1}X_i b_i \leqq H_i$ and, for $i \leqq k$, $X_i \mathfrak{e} b_i$ (because $X_{i-1} \mathfrak{e} b_i$). Hence, by Lemma 2, $H_i \lhd H_{i-1}$ for all $i \leqq k$.

If X_{i-1} is finite and $X_{i-1} \mathfrak{e} b_i$, then X_i is finite. But $X_0 = X$ is given to be finite, and hence X_i is finite for all $i \leqq k$, whence $X_{i-1} |\mathfrak{e}\, b_i$ for all $i \leqq k$. It now follows from Proposition 1 that if H_i is nilpotent, then H_{i-1} is also nilpotent. Since H_k is given to be nilpotent, it follows that $H_0 = G$ is nilpotent.

COROLLARY (Hirsch-Plotkin). *If U and V are locally nilpotent normal subgroups of a given group, then so is UV.*

Proof. We must prove the nilpotence of every subgroup of the form $G = \mathrm{Gp}\{S, T\}$ where S is a finite subset of U, and T a finite subset of V. Let $b_1, b_2, \cdots$ be a basic sequence in $\{S, T\}$ with generating sequence $X_0, X_1, \cdots$. If $\gamma_c(\mathrm{Gp}\{S\}) = \gamma_c(\mathrm{Gp}\{T\}) = 1$, then every b_s of weight $\geqq c$ lies in $U \cap V$, and hence there exists k such that $X_k \leqq U \cap V$. Obviously $X_{i-1} \mathfrak{e} b_i$ for all $i \geqq 1$, and therefore X_k is finite, and so $\mathrm{Gp}\{X_k\}$ is nilpotent. Thus G is nilpotent by Theorem 1.

The fact that local nilpotence is a multiproperty implies that the union of all locally nilpotent normal subgroups in any group G is itself locally nilpotent. We shall denote this subgroup by $\varphi(G)$ and call it the *Fitting radical* of G.[5]

Let us recall that a local system of subgroups of a group G is a set of subgroups such that (i) every element of G is contained in some subgroup of the set, and (ii) any two subgroups in the set are contained in at least one other subgroup of the set (see [7], p. 166). If $\mathcal{P}$ is a given group property and G has a local system all of whose members have $\mathcal{P}$, then G is called *locally* $\mathcal{P}$; while if it should happen that all the subgroups of the local system are also serial in G, then we shall say G is *σ-locally* $\mathcal{P}$. The property $\mathcal{P}$ is called σ-local if σ-locally $\mathcal{P}$ is the same thing as $\mathcal{P}$.

The reader should observe that the class of σ-locally $\mathcal{P}$ groups *is contained in*

[5] Plotkin calls this the nilradical.

the class of locally $\mathcal{P}$ groups, but that, on the other hand, the class of σ-local properties *contains* the class of local properties.

LEMMA 3. *If $\mathcal{P}$ is a multi-, and a σ-local, property and K is a serial subgroup of G possessing $\mathcal{P}$, then $\bar{K}$, the normal closure of K in G, also possesses $\mathcal{P}$.*

Proof. Let $K = K_0 < K_1 < \cdots < K_\alpha = G$ be a series from K to G and, for each λ, define H_λ to be the normal closure of K in K_λ. Thus $K = H_0 = H_1$ and $H_\alpha = \bar{K}$. It is clear that $H_\lambda \lhd H_{\lambda+1}$, and we easily verify that, for each limit ordinal λ, $H_\lambda = \bigcup_{\mu<\lambda} H_\mu$. Hence

$$K = H_1 \leqq H_2 \leqq \cdots \leqq H_\alpha = \bar{K} \leqq G$$

is a series, and so each H_λ is serial in G.

We prove the lemma by an induction on λ. Suppose we already know that H_μ has $\mathcal{P}$ for all $\mu < \lambda$. If λ is a limit ordinal, then the set of all H_μ with $\mu < \lambda$ provides a σ-local system of H_λ all of whose members have $\mathcal{P}$. Thus H_λ is σ-locally $\mathcal{P}$ and hence is $\mathcal{P}$. If, however, λ is not a limit ordinal, then $x^{-1}H_{\lambda-1}\,x = H_{\lambda-1}^x$ is normal in $K_{\lambda-1}$ for every $x \in K_\lambda$, and so

$$H_\lambda = \prod_{x \in K_\lambda} H_{\lambda-1}^x .$$

The product of any finite number of conjugates of $H_{\lambda-1}$ by elements in K_λ again has $\mathcal{P}$ since $\mathcal{P}$ is a multiproperty and also, of course, is a normal subgroup of H_λ. Thus the set of all such products is a σ-local system of H_λ whose elements all possess $\mathcal{P}$, and so H_λ is σ-locally $\mathcal{P}$. Thus, whatever the nature of λ, H_λ is $\mathcal{P}$, and the induction is complete.

Since a local property is necessarily σ-local, we may apply Lemma 3 with $\mathcal{P}$ the property of local nilpotence and K the subgroup generated by a serial element to obtain

PROPOSITION 2. *The normal closure of a serial element is locally nilpotent.*

We are now in a position to begin the proof of the fact that $\sigma(G)$ is a subgroup.

LEMMA 4. *Let H be a subgroup of G and g an element such that $H \mathfrak{e}\, g$. If $K = \bigcap_{m=-\infty}^{+\infty} g^{-m}Hg^m$, then $x \in K$ if, and only if, $[x, {}_rg] \in H$ for all $r \geqq 0$.*

Proof. Clearly $x \in K$ implies $x^{g^m} = g^{-m}xg^m \in H$ for all integers m. As in the proof of Lemma 2, we may conclude that $x_r = [x, {}_rg] \in H$ for all $r \geqq 0$.

Conversely, let $x_r \in H$ for all $r \geqq 0$, and let $X = \mathrm{Gp}\{x_r\, , r \geqq 0\}$. Then $X^g \leqq X$ (because $x_r^g = x_r\, x_{r+1}$), and $X \mathfrak{e}\, g$ (because $X \leqq H$ and $H \mathfrak{e}\, g$), whence, by Lemma 2, g lies in the normaliser of X. Hence $x^{g^m} \in X$ for all values of m satisfying $-\infty < m < +\infty$, and so $x \in H^{g^m}$ for all m, i.e., $x \in K$.

For our next lemma we need some further terminology. If S is a given subset of G and K can be linked to G by a series all of whose terms are invariant under the inner automorphisms determined by S, we shall say that K is S-

serial. If $\mathcal{P}$ is a given group property and there exists a series from K to G in which quotients of all successive terms possess $\mathcal{P}$, we shall say that K is *serially-$\mathcal{P}$ in G*; and if, in particular, 1 is serially-$\mathcal{P}$ in G, then G will be said to be *serially-$\mathcal{P}$*. Finally, we recall that a variety of groups is the class of all groups which satisfy a given set of identities, and that $\mathcal{P}$ is called a *varietal property* if the class of all groups having $\mathcal{P}$ forms a variety.

LEMMA 5. *If K is serial in G and the left Engel element g lies in the normaliser of K, then K is g-serial.*

Further, if $\mathcal{P}$ is a varietal property, and K is serially-$\mathcal{P}$ in G, then K is g-serially-$\mathcal{P}$ in G.

Proof. Let $K = K_0 < K_1 < \cdots < K_\alpha = G$ be a series from K to G, and, for each λ, define

$$L_\lambda = \bigcap_{m=-\infty}^{+\infty} K_\lambda^{g^m}.$$

Then $L_0 = K$, because g is in the normaliser of K, and $L_\alpha = G$. Moreover, for each λ, $L_\lambda^g = L_\lambda$, and $L_\lambda \lhd L_{\lambda+1}$. Hence $K = L_0 \leqq L_1 \leqq \cdots \leqq L_\alpha = G$ is a series provided only that, for every limit ordinal λ,

$$L_\lambda = \bigcup_{\mu<\lambda} L_\mu .$$

Clearly $\bigcup_{\mu<\lambda} L_\mu \leqq L_\lambda$, so that it only remains to prove the converse inequality. Let $x \in L_\lambda$ and suppose $[x, {}_kg] = 1$. By Lemma 4, $[x, {}_rg] \in K_\lambda$ whenever $0 \leqq r < k$, and so there exists $\mu < \lambda$ such that $[x, {}_rg] \in K_\mu$ for $0 \leqq r < k$, whence, again by Lemma 4, $x \in L_\mu$, as required.

Finally, assume that for every λ, $K_{\lambda+1}/K_\lambda$ has the varietal property $\mathcal{P}$, and let $w(x_1, \cdots, x_n) = 1$ be one of the identities holding in the variety determined by $\mathcal{P}$. Choose any $a_1, \cdots, a_n$ in $L_{\lambda+1}$ and any integer m. Then we can find elements $b_1, \cdots, b_n$ in $K_{\lambda+1}$ such that $a_i = b_i^{g^m}$, for all i, and hence $w(a_1, \cdots, a_n) \in K_\lambda^{g^m}$. This is true for every m, and so $w(a_1, \cdots, a_n) \in L_\lambda$, whence $w(x_1, \cdots, x_n) = 1$ holds identically in $L_{\lambda+1}/L_\lambda$. Consequently $L_{\lambda+1}/L_\lambda$ has $\mathcal{P}$, and the lemma is completely proved.

The next three lemmas do for serial elements what Baer's Lemmas 3 and 4 and Theorem 1 in [2], pp. 416–417, do for finitely-serial elements. The reader will note that Baer had no need to state explicitly the analogue for finitely-serial elements of our Lemma 5 simply because this analogue is completely trivial. We have worded the whole of the following argument leading to Theorem 2 in such a way that it remains true when "serial" is replaced by "finitely-serial".

LEMMA 6. *If K is serial in G and the serial element g lies in the normaliser of K, then* $\mathrm{Gp}\{K, g\}$ *is serial in G.*

Proof. Since $g \;\infty\!\lhd\; G$ we have $G \,\mathfrak{e}\, g$, and so, by Lemma 5, K is g-serial in G. So let $K = K_0 < \cdots < K_\alpha = G$ be a g-series from K to G, and define $H_\lambda = \mathrm{Gp}\{K_\lambda, g\}$. Now $g \;\infty\!\lhd\; G$ implies $g \;\infty\!\lhd\; H_{\lambda+1}$, and so, since K_λ

is normal in $H_{\lambda+1}$, $gK_\lambda \;\infty\!\lhd\; H_{\lambda+1}/K_\lambda$, i.e., $H_\lambda/K_\lambda \;\infty\!\lhd\; H_{\lambda+1}/K_\lambda$, whence $H_\lambda \;\infty\!\lhd\; H_{\lambda+1}$. Since this holds for every λ and, as we easily verify, $H_\lambda = \bigcup_{\mu<\lambda} H_\mu$ for every limit ordinal λ, we deduce that $H_0 \;\infty\!\lhd\; G$, as required.

LEMMA 7. *If K is normal in $G = \mathrm{Gp}\{K, g\}$ and every element of the set $\{K, g\}$ is serial in G, then every element of G is serial.*

Proof. Choose any x in G and write it in the form $x = yg^m$ with $y \in K$. So $X = \mathrm{Gp}\{x, g\}$ is the same as $\mathrm{Gp}\{y, g\}$, and since y and g are both serial in X, it follows, from Proposition 2, that X is nilpotent. Hence $x \lhd\lhd X$, and it only remains to prove that $X \;\infty\!\lhd\; G$.

Since $X \cap K$ is a finitely generated nilpotent group, there exists a series

$$1 = X_0 < X_1 < \cdots < X_n = X \cap K$$

where $X_i \lhd X_{i+1} = \mathrm{Gp}\{X_i, x_i\}$ for all i. If $X_i \;\infty\!\lhd\; G$, then, since $x_i \;\infty\!\lhd\; G$ and $x_i^{-1}X_i x_i = X_i$, it follows from Lemma 6 that $X_{i+1} \;\infty\!\lhd\; G$. But X_0 is obviously serial in G, and so we have $X \cap K \;\infty\!\lhd\; G$. Since however $g \;\infty\!\lhd\; G$ and $g^{-1}(X \cap K)g = X \cap K$, we again use Lemma 6 to conclude

$$X = \mathrm{Gp}\{X \cap K, g\} \;\infty\!\lhd\; G,$$

as had to be shown.

LEMMA 8. *If U and V are normal subgroups all of whose elements are serial, then the same is true of UV.*

Proof. Choose any $x \in UV$, and write x in the form $x = uv$ with $u \in U$, $v \in V$. Now $v \;\infty\!\lhd\; UV$ implies $vU \;\infty\!\lhd\; UV/U$, and this, in turn, that $\mathrm{Gp}\{x, U\} = \mathrm{Gp}\{v, U\} \;\infty\!\lhd\; UV$. But $x \;\infty\!\lhd\; \mathrm{Gp}\{x, U\}$, by Lemma 7 (with $v = g$ and $U = K$), and hence $x \;\infty\!\lhd\; UV$.

Let $\mathcal{P}$ be the property defined by the statement that a group G has $\mathcal{P}$ if, and only if, every element of G is serial. Then Lemma 8 asserts that $\mathcal{P}$ is a multi-property, and we easily verify that $\mathcal{P}$ is also σ-local.[6] Hence by Lemma 3, the normal closure of every serial element has $\mathcal{P}$, and consequently the union of all such normal closures has $\mathcal{P}$. This union contains every serial element and thus is precisely $\sigma(G)$. This fact, together with Proposition 2, yields

THEOREM 2. *The set of all serial elements in any group forms a characteristic subgroup and is contained in the Fitting radical of the group.*

It is not known whether $\sigma(G)$ can ever be strictly less than $\varphi(G)$. This is certainly not the case when $\varphi(G)$ is countably generated, nor, as we shall see in §2, when G is soluble (or even serially-cyclic).

We shall now introduce conditions that relate to the right Engel conditions. If a is a given element of a group G and x another element, then let us write a_x

[6] This paragraph remains true when "serial" is replaced by "finitely-serial" provided that also σ is replaced by $\bar{\sigma}$ and the reference to Lemma 3 by the corresponding result for $\bar{\sigma}$-local properties and finitely-serial subgroups.

for the subgroup generated by x and the normal closure of a in G:

$$a_x = \mathrm{Gp}\{x \text{ and } g^{-1}ag \text{ for all } g \in G\}.$$

We shall denote by $\rho(G)$ the set of all a such that x is serial in a_x, for all $x \in G$. So if $a \in \rho(G)$, then, for any x, $x \in \sigma(a_x)$ and hence, in particular, $a \mathrel{e} x$, from which it follows that a is a right Engel element in G. It should also be observed that $a \in \rho(G)$ implies $a \infty\lhd a_a$ and so, since a_a is normal in G, $a \infty\lhd G$. Thus $\rho(G) \leqq \sigma(G)$.

Next suppose b is a finitely-serial element in G. By the *defect* of b we understand (following Baer) the length of the shortest series linking $\mathrm{Gp}\{b\}$ to G, and we shall write this integer as $d(b)$. The notation $b \lhd\lhd_{\leqq n} G$ will mean that $d(b) \leqq n$. We now introduce the set $\bar{\rho}(G)$ of all elements a such that, for all x in G, x is finitely-serial in a_x and of defect $\leqq m$, where m is independent of x (but may depend on a). Thus $x \lhd\lhd_{\leqq m} a_x$ for all x and so $[a, {}_{(m+1)}x] = 1$ for all x, whence $a \mathrel{e|} G$. Moreover $a \in \bar{\sigma}(a_a)$, and so $\bar{\rho}(G) \leqq \bar{\sigma}(G)$.

The conditions that define $\rho(G)$ and $\bar{\rho}(G)$ are rather strong, and it should therefore come as no surprise that one can prove very easily both these sets to be subgroups. (They are then actually *characteristic* subgroups, as is immediately seen if one observes that $(a_x)^\varphi = (a^\varphi)_{x\varphi}$ for any automorphism φ.)

LEMMA 9. *The sets $\rho(G)$, $\bar{\rho}(G)$ are subgroups.*

Proof. Take any a, b in $\rho(G)$ and any x in G. Then $x \infty\lhd b_x$ implies $a_x/a_a \infty\lhd b_x a_a/a_a$, and this yields $a_x \infty\lhd b_x a_a$. Now $(ab)_x \leqq b_x a_a$, and hence $a_x \cap (ab)_x \infty\lhd (ab)_x$. But $x \infty\lhd a_x$, so $x \infty\lhd a_x \cap (ab)_x$ and therefore $x \infty\lhd (ab)_x$. Thus $ab \in \rho(G)$.

Next, suppose a, b in $\bar{\rho}(G)$ and $x \lhd\lhd_{\leqq m} a_x$, $x \lhd\lhd_{\leqq n} b_x$, for all x. Then, as above, $x \lhd\lhd_{\leqq n} b_x$ implies $a_x \cap (ab)_x \lhd\lhd_{\leqq n} (ab)_x$, and this, with $x \lhd\lhd_{\leqq m} a_x$, gives $x \lhd\lhd_{\leqq m+n} (ab)_x$, so that $ab \in \bar{\rho}(G)$.

In view of Baer's result that in a group with maximal condition on subgroups the set of right Engel elements coincides with the hypercentre, we wish to discuss the relation of the subgroups $\rho(G)$, $\bar{\rho}(G)$ of an arbitrary group G to the hypercentre of G. We shall write the ascending central series as

$$1 = \alpha_0(G) \leqq \alpha_1(G) \leqq \cdots$$

and denote the hypercentre (which is the limit of this series) by $\alpha(G)$. It is well-known that an element g lies in $\alpha(G)$ if, and only if, to every sequence $\{x_1, x_2, \cdots\}$ of elements in G there corresponds an integer k such that $[g, x_1, x_2, \cdots, x_k] = 1$.

LEMMA 10. $\alpha(G) \leqq \rho(G)$ *and* $\alpha_\omega(G) \leqq \bar{\rho}(G)$.

Proof. If $a \in \alpha(G)$, then a_a is contained in $\alpha(G)$, and hence $a_a \leqq \alpha(a_x)$ for any $x \in G$. If we put $X_\lambda = \mathrm{Gp}\{\alpha_\lambda(a_x), x\}$ and assume that $\alpha_\nu(a_x) = \alpha(a_x)$, we see that

$$\mathrm{Gp}\{x\} = X_0 \leqq X_1 \leqq \cdots \leqq X_\nu = a_x$$

is a series and hence that x is serial in a_x. Thus $a \in \rho(G)$.

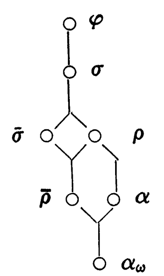
To establish the second inequality it is sufficient to prove $\alpha_n(G) \leqq \bar{\rho}(G)$ for every integer n. So let $a \in \alpha_n(G)$ and choose any x in G. Then $a_a \leqq \alpha_n(a_x)$, and so $a_x/\alpha_n(a_x)$ is cyclic, whence $\alpha_n(a_x) = a_x$. Thus $x \lhd\lhd_{\leqq n} a_x$ and, since this holds for all x in G, we conclude that $a \in \bar{\rho}(G)$.

The inequalities of Lemma 10 are, in a sense, the best that are possible. In §3 we shall meet a group in which $\bar{\rho}(G) < \alpha_{\omega+1}(G)$ and another in which $\alpha(G) < \bar{\rho}(G)$.

We sum up our conclusions concerning $\rho(G)$ and $\bar{\rho}(G)$ in

THEOREM 3. *The sets $\rho(G)$, $\bar{\rho}(G)$ in any group G form characteristic subgroups satisfying $\alpha(G) \leqq \rho(G) \leqq \sigma(G)$ and $\alpha_\omega(G) \leqq \bar{\rho}(G) \leqq \bar{\sigma}(G)$.*

The accompanying diagram illustrates the inclusion relationships satisfied by the groups discussed in this section.



2. Soluble groups

The main object in this section is the proof of the following result.

THEOREM 4. *In a soluble group G, the sets of all* (i) *left Engel elements,* (ii) *bounded left Engel elements,* (iii) *right Engel elements, and* (iv) *bounded right Engel elements coincide respectively with* (i) $\sigma(G)$, (ii) $\bar{\sigma}(G)$, (iii) $\rho(G)$, *and* (iv) $\bar{\rho}(G)$.

Parts of this theorem remain true in more general classes of groups than the soluble groups. Accordingly, we shall present the proofs of the four parts in the most general form in which they remain valid. The reader will note that a consequence of (i) is that, in a soluble group, the set of left Engel elements coincides with the Fitting radical. This fact has also been found by Plotkin [9].

There are two classes of generalised soluble groups that enter our discussion. The first is the class of *serially-cyclic groups*. It will be recalled that, by the definition given in §1, a group is serially-cyclic if it has a series from 1 to the whole group in which quotients of successive terms are cyclic.[7] The second class consists of the groups that we shall call *hyperabelian*: If $\mathcal{P}$ is a subgroup

[7] Serially-cyclic groups are called SN^*-groups in the book of Kurosh [7].

property (i.e., a property which a subgroup of a group may, or may not possess), then we call a group G *hyper-*$\mathcal{P}$ if every homomorphic image $G^{\theta} > 1$ contains a nontrivial normal subgroup K^{θ} having $\mathcal{P}$ in G^{θ}. Thus the hyperabelian groups are precisely the SI^*-groups of Kurosh and the soluble groups of Baer. Note also that the hypercyclic groups are the ones often referred to as supersoluble groups, and that the hypercentral groups are the ZA-groups of Kurosh and the upper nilpotent groups of Baer.

LEMMA 11. *Let A be an abelian normal subgroup of $G = \mathrm{Gp}\{A, g\}$.*
(i) *If g is a left Engel element, then g is serial in G;*[8]
(ii) *if g is a bounded left Engel element and $[x, {}_k g] = 1$ for all $x \in G$, then g is finitely-serial in G and $d(g) \leqq k$; and*
(iii) *if g is a right Engel element, then g is a left Engel element.*[9]

Proof. (i) Choose any nonunit $a \in A$, and suppose $[a, {}_{(m+1)}g] = 1$ but $[a, {}_m g] \neq 1$. Then $[a, {}_m g]$ is in the centre of G. This argument may be applied to any homomorphic image of G, and so G is hypercentral, whence $g \infty\triangleleft G$.

(ii) We show G is nilpotent of class $\leqq k$. So consider $c = [x_0, \cdots, x_k]$ where each $x_i \in A$ or $x_i = g$. If $x_0 = g$, then $x_1 \in A$, whence

$$c = [x_1^{-1}, x_0, x_2, \cdots, x_k]$$

and so, without loss of generality, we may assume that $x_0 \in A$. Then either one of $x_1, \cdots, x_k$ lies in A, or $x_1 = \cdots = x_k = g$, and in either case $c = 1$.

(iii) For any $a \in A$ we clearly have

$$[g, {}_k ga] = [g, a, {}_{(k-1)}g] = [a, {}_k g]^{-1},$$

and from this our assertion follows.

PROPOSITION 3. *An element in a serially-cyclic group is a left Engel element if, and only if, it is serial.*

Proof. Let G be our serially-cyclic group and $G \ni g$. From Lemma 5 (with $\mathcal{P}$ the property of being an abelian group and $K = 1$) we deduce that 1 is g-serially-abelian in G. So let

$$1 = A_0 < A_1 < \cdots < A_\alpha = G$$

be a g-series such that $A_{\lambda+1}/A_\lambda$ is abelian for all λ and define $B_\lambda = \mathrm{Gp}\{A_\lambda, g\}$. When λ is a limit ordinal, it is clear that $B_\lambda = \bigcup_{\mu<\lambda} B_\mu$ because $A_\lambda = \bigcup_{\mu<\lambda} A_\mu$. On the other hand, when $\lambda - 1$ exists, $A_{\lambda-1} \triangleleft B_\lambda$, and $A_\lambda/A_{\lambda-1}$ is abelian, whence $gA_{\lambda-1} \infty\triangleleft B_\lambda/A_{\lambda-1}$, by Lemma 11 (i), i.e., $B_{\lambda-1}/A_{\lambda-1} \infty\triangleleft B_\lambda/A_{\lambda-1}$, and consequently $B_{\lambda-1} \infty\triangleleft B_\lambda$. We conclude $B_0 = \mathrm{Gp}\{g\} \infty\triangleleft G$, as required.

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[8] This fact is probably well known: cf. Lemma 2, p. 224 of [7].
[9] See Baer [3], p. 259.

LEMMA 12. *If G is soluble and $[x, {}_kg] = 1$ for all $x \in G$, then $g \lhd\lhd_{\leqq tk} G$, where t is the derived length of G.*

Proof. We may use induction on t since the lemma is trivially true for abelian groups (i.e., groups of derived length one). So we assume that $gA \lhd\lhd_{\leqq (t-1)k} G/A$ where A is the last nontrivial term of the derived series of G. Then by Lemma 11 (ii), $g \lhd\lhd_{\leqq k} \mathrm{Gp}\{A, g\}$, and hence $g \lhd\lhd G$ with $d(g) \leqq k + (t-1)k$, so that the induction is complete.

The following result may already be found in Baer [3], p. 258.

LEMMA 13. *If G is hyperabelian, then $g \mathfrak{e} G$ implies $G \mathfrak{e} g$.*

Proof. Since the assertion that G is hyperabelian is clearly the same as saying that it is G-serially-abelian, we know that there exists a series $1 = A_0 < \cdots < A_\alpha = G$, in which, for all λ, A_λ is normal in G and $A_{\lambda+1}/A_\lambda$ is abelian. Put $B_\lambda = \mathrm{Gp}\{A_\lambda, g\}$ and assume we already know that $B_\mu \mathfrak{e} g$ for all $\mu < \lambda$. Then obviously also $B_\lambda \mathfrak{e} g$ if λ is a limit ordinal, while if $\lambda - 1$ exists, $B_\lambda/A_{\lambda-1} \mathfrak{e} gA_{\lambda-1}$ by Lemma 11 (iii), and so $B_\lambda \mathfrak{e} g$. Hence, by an induction on λ, we conclude $B_\alpha \mathfrak{e} g$, as required.

LEMMA 14. *In any group G, the set of all right Engel elements contained in $\varphi(G)$ forms a subgroup.*

Proof. Take any right Engel elements a, b lying in $\varphi(G)$, any x in G, and define $X = \mathrm{Gp}\{a, b, x\}$ and $Y = \{[a, {}_rx], [b, {}_rx], r \geqq 0\}$. Then $Y \mathfrak{e} x$ and $Y^x \leqq \mathrm{Gp}\{Y\}$, whence $\mathrm{Gp}\{Y\} \lhd X$ by Lemma 2. Also $\mathrm{Gp}\{Y\}$ is nilpotent because Y is a finite subset of $\varphi(G)$, and hence, by Proposition 1, X is nilpotent. Thus, in particular, $[ab, {}_kx] = 1$ and $[a^{-1}, {}_kx] = 1$ for a suitable integer k.

It should be remarked that Lemma 14 can also be deduced from Baer's result in [3]: In the notation of the above argument, we have $\mathrm{Gp}\{Y\}$ is nilpotent and finitely generated, and $X/\mathrm{Gp}\{Y\}$ is cyclic, whence X satisfies the maximal condition on subgroups, and therefore, by Baer's theorem, a and b belong to the hypercentre of X. Thus $\alpha(X) \geqq \mathrm{Gp}\{Y\}$, and so X is nilpotent because $X/\mathrm{Gp}\{Y\}$ is cyclic.

LEMMA 15. *If G is any group such that, for every subgroup H, the set of all left Engel elements in H coincides with $\sigma(H)$, then $\rho(G)$ coincides with the set of right Engel elements lying in $\varphi(G)$.*

Proof. Suppose $a \mathfrak{e} G$ and $a \in \varphi(G)$. Then the same holds for all conjugates of a, and so, by Lemma 14, a_a (the normal closure of a in G) consists entirely of right Engel elements. Choose any $x \in G$, and observe that every element of $a_x = \mathrm{Gp}\{a_a, x\}$ can be written in the form $b = x^k u$ with $u \in a_a$. Now $[b, {}_rx] = [u, {}_rx]$, and consequently $a_x \mathfrak{e} x$ because $a_a \mathfrak{e} G$. Thus x is a left Engel element in a_x, and so, by our hypothesis concerning G, $x \infty\lhd a_x$. We conclude that $a \in \rho(G)$.

Proposition 3 and Lemmas 13 and 15 combine to yield

PROPOSITION 4. *An element in the hyperabelian group G is a right Engel element if, and only if, it belongs to $\rho(G)$.*

It now only remains to establish the last part of Theorem 4. If a is a bounded right Engel element, we shall write $r(a)$ for the smallest of the integers k such that $[a, {}_k x] = 1$ for all x in the group.

LEMMA 16. *If G is a soluble group and a, b are bounded right Engel elements, then ab and a^{-1} are also bounded right Engel elements, and there exists an increasing integer-valued function $g(n_1, n_2, n_3)$ such that*

$$r(ab) \leqq g(r(a), r(b), t(G)) \quad \textit{and} \quad r(a^{-1}) \leqq g(r(a), r(1), t(G)),$$

where $t(G)$ is the derived length of G.

Proof. Choose any x in G and define

$$X = \mathrm{Gp}\{a, b, x\}, \qquad Y = \{[a, {}_r x], [b, {}_r x], r \geqq 0\}, \qquad H = \mathrm{Gp}\{Y\}.$$

Then, as in the proof of Lemma 14, H is a nilpotent normal subgroup of X. Now

$$S = \{\text{all } x^{-m}ax^m \text{ with } 0 \leqq m < r(a) \text{ and all } x^{-n}bx^n \text{ with } 0 \leqq n < r(b)\}$$

is another generating set of H (cf. the second paragraph of the proof of Lemma 2), and every element $s \in S$ is a bounded right Engel element with $r(s)$ equal to $r(a)$ or $r(b)$. Hence H has a set of $r(a) + r(b)$ generators S such that $[s, {}_m h] = 1$, for all $s \in S$ and all $h \in H$, where $m = \max(r(a), r(b))$.

Next, choose any three nonnegative integers n_1, n_2, n_3, and let F be the free group on some set Z of $n_1 + n_2$ elements. Define N to be the least normal subgroup of F containing (i) the elements

$$[z, {}_{\max(n_1,n_2)} v]$$

for all $z \in Z$ and all $v \in F$, and (ii) the n_3^{th} term of the derived series of F. Then F/N is a soluble group generated by a finite number of right Engel elements and so, by Proposition 4, is nilpotent. We shall denote by f the function

$$(n_1, n_2, n_3) \to \mathrm{class}(F/N),$$

so that $\mathrm{class}(F/N) = f(n_1, n_2, n_3)$. (Observe that $f(n_1, n_2, n_3) = 0$ if, and only if, (n_1, n_2, n_3) equals $(0, 0, n_3)$ or $(n_1, n_2, 0)$.) It is clear from the definition that f is an increasing (but not necessarily strictly increasing!) function of each of its three arguments.

If now $n_1 = r(a)$, $n_2 = r(b)$, and $n_3 = t(G)$, then any mapping of Z onto S extends to a homomorphism of F onto H and, in view of what we know about H, the kernel of this homomorphism contains N, so that, in fact, H is a homomorphic image of F/N. Hence H is nilpotent of class $\leqq f(r(a), r(b), t(G)) = f_0$, say, whence, by Proposition 1, $X = \mathrm{Gp}\{Y, x\}$ is nilpotent of class $\leqq m(m-1)$, where $m = \max(r(a), r(b), f_0 + 1)$. Then

$$[ab, {}_{m(m-1)} x] = 1,$$

and since m is independent of x, this is true for all x, whence $r(ab) \leqq m(m-1)$. Further, if $b = 1$ in the above argument, then we conclude that $\mathrm{Gp}\{a, x\}$ has class $\leqq \bar{m}(\bar{m}-1)$ where

$$\bar{m} = \max(r(a), r(1), f(r(a), r(1), t(G)) + 1),$$

and so

$$r(a^{-1}) \leqq \bar{m}(\bar{m}-1).$$

Thus $g(n_1, n_2, n_3) = m(m-1)$, where $m = \max(n_1, n_2, f(n_1, n_2, n_3) + 1)$ is a function with the required properties.

LEMMA 17. *If G is a soluble group, then $a \,\mathfrak{e}|\, G$ implies $a_a \,|\mathfrak{e}|\, G$.*

Proof. It is a consequence of Lemma 16 that the set of all bounded right Engel elements in G forms a subgroup, which is then obviously characteristic. So, in particular, every element of a_a is a bounded right Engel element, i.e., in our notation, $a_a \,\mathfrak{e}|\, G$. From this alone, however, it certainly does not follow that $a_a \,|\mathfrak{e}|\, G$.

We shall prove the lemma by an induction on $t(a_a)$, the derived length of a_a. Since a_a is a normal subgroup of G, the same is true of every term of the derived series of a_a. If $t(a_a) = 0$, the lemma is trivially true. Assume it proved when $t(a_a) \leqq m$, and consider the case $t(a_a) = m + 1$. Thus there exists an integer p such that $[u, {}_px] \in A$, for all $u \in a_a$ and all $x \in G$, where A is the last nontrivial term of the derived series of a_a. If we can prove that there exists q such that $[v, {}_qx] = 1$ for all $v \in A$ and all $x \in G$, then $[u, {}_{(p+q)}x] = 1$ for all $u \in a_a$ and all $x \in G$, thus completing the induction argument.

Let $S^{(0)}$ be the set of all conjugates of a in G and, if $S^{(k)}$ has already been defined, let $S^{(k+1)}$ be the set of all elements $[u, v]^x$ for all $u, v \in S^{(k)}$ and all $x \in G$. Thus $S^{(t)}$ generates the t^{th} term of the derived series of a_a and, in particular, $\mathrm{Gp}\{S^{(m)}\} = A$. We shall prove that to each k there corresponds an integer k^* such that $r(u) \leqq k^*$ for all $u \in S^{(k)}$. Then, in particular, $[u, {}_{m^*}x] = 1$ for all $u \in S^{(m)}$ and all $x \in G$. Since A is abelian and normal in G, we have, for any v_1, v_2 in A and any x in G,

$$[v_1 v_2, {}_ix] = [v_1, {}_ix][v_2, {}_ix],$$

and

$$[v_1^{-1}, {}_ix] = [v_1, {}_ix]^{-1},$$

so that $[v, {}_{m^*}x] = 1$ for all $v \in A$ and all x.

The lemma will therefore be proved if we can establish the existence of the integers k^*. We do this by an induction on k. Since $r(a^x) = r(a)$ for every x, we may put $0^* = r(a)$. Assume k^* has already been proved to exist, and consider a typical element $w = [u, v]^x$ of $S^{(k+1)}$. Since $[u, v] = u^{-1}u^v$ and $r(u^v) = r(u)$, we know from Lemma 16, that

$$r(w) = r([u, v]) \leqq g(r(u^{-1}), r(u), t(G)),$$

and

$$r(u^{-1}) \leqq g(r(u), r(1), t(G)).$$

By the induction hypothesis, $r(u) \leqq k^*$, whence, because g is an increasing function of each of its arguments,

$$r(u^{-1}) \leqq g(k^*, r(1), t(G)),$$

and so, finally,

$$r(w) \leqq g(g(k^*, r(1), t(G)), k^*, t(G)).$$

We complete the induction by defining $(k + 1)^*$ to be the integer on the right-hand side of this last inequality.

LEMMA 18. *If G is soluble and $a \, \mathfrak{e}| \, G$, then $a \in \bar{\rho}(G)$.*

Proof. By Lemma 17, there exists k such that $[u, {}_kx] = 1$ for all $u \in a_a$ and all $x \in G$.

Choose any x and consider the group $a_x = \mathrm{Gp}\{a_a, x\}$. If t is the derived length of a_a, then that of a_x is $\leqq t + 1$. Moreover, $[v, {}_kx] = 1$ for all $v \in a_x$ and hence, by Lemma 12, x is finitely-serial in a_x and of defect $\leqq (t + 1)k$. The integer $(t + 1)k$ is independent of x, and we may therefore conclude that $a \in \bar{\rho}(G)$.

3. Examples

In this final section we shall construct a metabelian group (i.e., a group with abelian commutator group) in which the four subgroups of Theorem 4 are distinct and, moreover, none of them coincides with the hypercentre. Our group is the direct product of three groups whose definitions follow.

The first group, U, is the symmetric group on three symbols. It is clear that the unique subgroup C of order 3 is $\sigma(U)$ and that $\rho(U) = 1$. Hence

$$\sigma(U) = \bar{\sigma}(U) = C,$$

and

$$\rho(U) = \bar{\rho}(U) = \alpha(U) = 1.$$

The second group, V, is a group of the type given by Baer on p. 408 of [2]. More explicitly, $V = \mathrm{Gp}\{A, b\}$, where A is normal in V and abelian of type 2^∞ and $b^{-1}ab = a^3$ for all $a \in A$. Suppose A is generated by $a_1, a_2, \cdots$, where $a_i^2 = a_{i-1}$ and $a_0 = 1$. Then for any positive integers k, r, we have

$$b^{-k}a_i b^k = a_i^{3^k}$$

and so

$$[a_i, {}_rb^k] = a_i^{(3^k-1)^r}.$$

If 2^m is the exact power of 2 dividing $(3^k - 1)^r$, then

$$[a_{m+1}, {}_rb^k] = a_1 \neq 1,$$

and thus b^k is *not* a bounded left Engel element. Since the inverse of a bounded left Engel element is another of the same type, we conclude that

$$\bar{\sigma}(V) \cap \mathrm{Gp}\{b\} = 1.^{10}$$

---

[10] As a matter of fact, this is true when A is of type p^∞ (and $b^{-1}ab = a^{1+p}$) whatever the prime p.

Clearly $A \leqq \bar{\sigma}(V)$, and hence we have $\bar{\sigma}(V) = A$. In particular, $\alpha_\omega(V) \leqq A$ and, since $a_n \in \alpha_n(V)$, $a_n \notin \alpha_{n-1}(V)$, for all $n \geqq 1$, it follows that $\alpha_\omega(V) = A$ and hence $\alpha_{\omega+1}(V) = V$. In the group V therefore,

$$\alpha_\omega(V) = \bar{\rho}(V) = \bar{\sigma}(V) = A,$$

and

$$\alpha(V) = \rho(V) = \sigma(V) = V.$$

The third group, W, is a group in which both W', the commutator group of W, and W/W' are of exponent 2 and the centre of W is trivial. An example of such a group was given many years ago by Baer [1], p. 412. We shall present here another example, partly for the sake of variety, and partly because the construction of our group is so very simple. If F is the free product of a countable number of cyclic groups, each of order two, then our group is $W = F/(F')^2$, where $(F')^2$ is the subgroup generated by the squares of all elements in F', the commutator group of F.[11] It is clear that W/W' and W' are both of exponent 2, and from this alone it follows that $[x, y, y, y] = 1$ for all x, y in W: for if $c \in W'$, then $[c, y^2] = 1$ because $y^2 \in W'$ and W' is abelian, while, on the other hand, $[c, y^2] = [c, y]^2[c, y, y] = [c, y, y]$ because W' has exponent 2. Thus

$$\bar{\sigma}(W) = \sigma(W) = \bar{\rho}(W) = \rho(W) = W.$$

To show that

$$\alpha(W) = 1$$

amounts to proving that $\alpha_1(W) = 1$. A quick way of doing this is to use Theorem 5.1 of [4]. If $c_1, c_2, \cdots$ are the generators of the free factors of F and $w_1, w_2, \cdots$ are their images under the natural homomorphism $F \to W$, then we know, first, that W/W' is an elementary abelian 2-group with basis $w_1 W', w_2 W', \cdots$ and, secondly, by Theorem 5.1 of [4], p. 47, that W' is an elementary abelian 2-group with basis consisting of all commutators of the form

$$[w_{i_1}, w_{i_2}, \cdots, w_{i_r}],$$

where $i_1, \cdots, i_r$ are distinct, $i_1 > i_2$, $i_2 < \cdots < i_r$, and $r \geqq 2$. If x lies in the centre of W and $x \notin W'$, let $x = w_{j_1} \cdots w_{j_s} c$, where $j_1 < \cdots < j_s$ and $c \in W'$. Then, for all $y \in W'$,

$$[y, x] = [y, w_{j_1} \cdots w_{j_s}] = 1. \tag{3}$$

If $j_1 = 1$ and $s = 1$, then

$$[w_3, w_2, w_1] = [w_2, w_1, w_3]^{-1}[w_3, w_1, w_2],$$

---

[11] More generally, we could take $W = F/(F')^pF''$, where F is a free product of a countable number of cyclic groups, each of order equal to the prime number p. The proof that $\bar{\rho}(W) = W$ and $\alpha(W) = 1$ is similar to that given above for the case $p = 2$.

and this contradicts (3) with $y = [w_3, w_2]$ (because both commutators on the right-hand side are basis elements of W'). If $j_1 > 1$, we take $y = [w_j, w_1]$ where $j > j_s$ and observe that $[w_j, w_1, w_{j_1} \cdots w_{j_s}]$ can be written as a product, of basis elements of W', in which the commutator $[w_j, w_1, w_{j_1}, \cdots, w_{j_s}]$ occurs exactly once. Hence again (3) is contradicted, and there only remains the possibility $j_1 = 1$ and $s > 1$. But $w_1^2 = 1$ and so $[w_j, w_1, w_1] = 1$, whence

$$[w_j, w_1, w_1 w_{j_2} \cdots w_{j_s}] = [w_j, w_1, w_{j_2} \cdots w_{j_s}],$$

and we are back at the case just dismissed. Thus our assumption $x \notin W'$ was wrong, and we know $x = w(b_1, \cdots, b_n)$, a word in elements $b_1, \cdots, b_n$ of our basis of W'. If $b_1, \cdots, b_n$ involve only $w_1, \cdots, w_m$, then

$$[x, w_{m+1}] = w([b_1, w_{m+1}], \cdots, [b_n, w_{m+1}]),$$

and this is not 1 because each $[b_i, w_{m+1}]$ is again a basis element. Hence $x = 1$ as required.

The possibility of obtaining a useful group by forming a direct product rests on

PROPOSITION 5. *If $G = A \times B$, then $\xi(G) = \xi(A) \times \xi(B)$, where ξ is any one of σ, $\bar{\sigma}$, ρ, $\bar{\rho}$, α, α_ω.*

The proof of this Proposition is entirely straightforward and will be omitted.

The group that we are after is $G = U \times V \times W$, where U, V, W are the three groups defined above. In view of Proposition 5 we may now assert:

$$\sigma(G) = C \times V \times W, \qquad \bar{\sigma}(G) = C \times A \times W,$$
$$\rho(G) = 1 \times V \times W, \qquad \bar{\rho}(G) = 1 \times A \times W,$$
$$\alpha(G) = 1 \times V \times 1, \qquad \alpha_\omega(G) = 1 \times A \times 1.$$

This has established our final result:

THEOREM 5. *There exists a countably generated metabelian group in which the subgroups determined by σ, $\bar{\sigma}$, ρ, $\bar{\rho}$, α, α_ω are all distinct.*

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