A NOTE ON SPACES WITH OPERATORS

BY

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1. Introduction

If a group Π operates on a topological space, the singular chain-complex of the space is a graded Π-module with derivation; the cycle, boundary, and homology groups are also Π-modules. It is clear that information about this situation can be extracted from the representation theory of Π.

There are however only scattered examples of such applications of representation theory to topology, the most important perhaps being the fixed-point theorem of P. A. Smith for periodic transformations of spheres ([5], cf. also [3]). In the absence of a comprehensive theory, one more special result may have some interest.

The situation studied in this note is that in which a group $\Pi \approx (Z_p)^r$ operates without fixed points on a connected space $\mathfrak{X}$. Under suitable finiteness conditions on $\mathfrak{X}$, viz., $H_i(\mathfrak{X}; Z_p)$ finite for all i, $H_i(\mathfrak{X}; Z_p)$ and $H_i(\mathfrak{X}/\Pi; Z_p)$ zero for sufficiently large i, inequalities involving the Betti numbers $\mathfrak{b}_i = \dim H_i(\mathfrak{X}; Z_p)$ will be demonstrated. These inequalities generalize results of P. A. Smith and P. E. Conner.

The notation (m, n) will be used for the binomial coefficient $(m + n)!/m!\,n!$

2. The representation theory

Let Π be a group isomorphic to $(Z_p)^r$ and $\mathfrak{k}$ a field of characteristic p (which might as well be the prime field), and denote by Λ the group algebra $\mathfrak{k}(\Pi)$. In the category of left Λ-modules, Λ is indecomposable and injective. Thus by a theorem of Nakayama [4] the free, projective and injective Λ-modules coincide.

A Λ-module A will be called *quasi-finite* if it is a direct sum $A_0 + X$ with A_0 finite-dimensional and X free. It is easy to see that if $0 \to A' \to A \to A'' \to 0$ is a short exact sequence of Λ-modules and any two are quasi-finite, then the third one is too.

The *complete derived sequence* $\hat{H}^n(A) = \hat{H}^n(\Pi, A)$, $n = 0, \pm 1, \cdots$, of a Λ-module A is defined in [1, XII, §2]; the functors $\hat{H}^n$ form an exact connected sequence, so that if $0 \to A' \to A \to A'' \to 0$ is exact then

$$\cdots \to \hat{H}^{n-1}(A'') \to \hat{H}^n(A') \to \hat{H}^n(A) \to \hat{H}^n(A'') \to \hat{H}^{n+1}(A') \to \cdots$$

is exact. The groups $\hat{H}^n(A)$ are in fact vector spaces over $\mathfrak{k}$; if A is quasi-finite, they are finite-dimensional vector spaces, and the integers $s_n A = \dim \hat{H}^n(A)$ are defined. If $0 \to A' \to A \to A'' \to 0$ is an exact sequence of

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quasi-finite modules, then

$$s_n A \leqq s_n A' + s_n A''. \tag{2.1}$$

Moreover, each $\hat{H}^n$ vanishes on free Λ-modules. Thus in the sequence just mentioned

$$s_{n-1} A'' = s_n A' \qquad \text{if } A \text{ is free.} \tag{2.2}$$

For the trivial Λ-module $\mathfrak{k}$ these invariants are easily computed by Künneth's theorem and duality:

$$s_n \mathfrak{k} = \begin{cases} (n, r) & n \geqq 0 \\ (-n - 1, r) & n < 0. \end{cases} \tag{2.3}$$

Since every finite-dimensional Λ-module A contains a submodule isomorphic to $\mathfrak{k}$, induction on the dimension of A gives immediately

$$s_n A \leqq \begin{cases} (n, r)\dim A & n \geqq 0 \\ (-n - 1, r)\dim A & n < 0. \end{cases} \tag{2.4}$$

3. Graded Λ-modules with derivation; topological application

Suppose X is a Λ-module graded by nonnegative degrees with a derivation of degree -1. Suppose further that each X_i is free, that each $H_i = H_i(X)$ is finite-dimensional, and that H_i and $H_i(X \otimes_\Lambda \mathfrak{k})$ both vanish for $i > m$.

Since the modules $B_i = B_i(X)$ and $Z_i = Z_i(X)$ fall into exact sequences $0 \to Z_i \to X_i \to B_{i-1} \to 0$ and $0 \to B_i \to Z_i \to H_i \to 0$, it follows immediately that all the B_i and Z_i are quasi-finite. Since

$$H_i(X \otimes_\Lambda \mathfrak{k}) = \operatorname{Tor}^{\Lambda}_{i-m-1}(\mathfrak{k}, B_m) = 0$$

for $i > m$, all the $B_i = Z_i$ are free for $i > m$.

In addition it follows from (2.2) that $s_n B_{j-1} = s_{n-1} Z_j$, and thus from (2.1) that

$$s_{k-j+1} B_{j-1} - s_{k-j} B_j \leqq s_{k-j} H_j . \tag{3.1}$$

If these equations are added from $j = 1$ to $j = m + 1$, and it is recalled that $Z_0 = X_0$ so that $s_k B_0 = s_{k+1} H_0$, they yield the inequalities

$$s_{k+1} H_0 \leqq \textstyle\sum_{j=1}^{k} s_{k-j} H_j , \tag{3.2}$$

which, together with (2.4), give

$$s_{k+1} H_0 \leqq \textstyle\sum_{j=1}^{k} (k - j, r) b_j + \sum_{j=k+1}^{m} (j - k - 1, r) b_j ,$$

where $b_j = \dim H_j$.

Consider now the topological situation described in the introduction. The singular chain-complex of $\mathfrak{X}$ with coefficients $\mathfrak{k}$ is just such an object as the X described above. In addition, $H_0(\mathfrak{X}; k) = \mathfrak{k}$ since $\mathfrak{X}$ is connected. Thus

$$(k + 1, r) \leqq \textstyle\sum_{j=1}^{k} (k - j, r)\mathfrak{b}_j + \sum_{j=k+1}^{r} (j - k - 1, r)\mathfrak{b}_j . \tag{3.3}$$

4. The theorems of Smith and Conner

P. E. Conner has announced the following result [2], which generalizes an older theorem of P. A. Smith [6]:

If $\mathfrak{X}$ has the cohomology ring, modulo p, of $(S^n)^{r-1}$, then $(Z_p)^r$ cannot operate without fixed points on $\mathfrak{X}$.

For $r = 2, 3$ he shows that the hypothesis can be weakened to the assumption that $\mathfrak{X}$ has the homology groups, modulo p, of $(S^n)^{r-1}$. This latter result may be slightly generalized, using (3.3) above.

THEOREM 1. *If $r \leqq 4$ or if $r \leqq 8$ and n is sufficiently large, and if $\mathfrak{X}$ has the homology groups, modulo p, of $(S^n)^{r-1}$, then $(Z_p)^r$ cannot operate without fixed points on $\mathfrak{X}$.*

The reader will be spared the tedious computation of (3.3) for $r = 4$, $k = 2n$ and $r = 8$, $k = 4n$.

Another mild generalization of Conner's result is the following:

THEOREM 2. *If $\mathfrak{X}$ has the homology groups, modulo p, of $S^n \times S^m$, then $(Z_p)^3$ cannot operate without fixed points on $\mathfrak{X}$.*

If $n \geqq m$, substitution of $r = 3$, $k = n$ into (3.3) leads to a contradiction.

Amateurs of binomial coefficients may of course enlarge upon these very naive applications.

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