ON CASTELNUOVO'S CRITERION OF RATIONALITY $p_a = P_2 = 0$ OF AN ALGEBRAIC SURFACE

To Emil Artin on his sixtieth birthday

BY
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1. Introduction

Let F be a nonsingular (irreducible) algebraic surface over an algebraically closed ground field k. A theorem of Castelnuovo asserts that if the arithmetic genus p_a and the bigenus P_2 of F are both zero then F is a rational surface. This theorem has now been proved for fields k of arbitrary characteristic p, except in the case $(K^2) = 1$, where K is a canonical divisor on F.[2] In our cited paper MM (see footnote 2) we have stated that we have also a proof for the case $(K^2) = 1$, and in the present paper we shall give this proof.

An immediate consequence of Castelnuovo's criterion of rationality is the well-known theorem of Castelnuovo on the rationality of plane involution. This theorem, in the case of arbitrary characteristic, is to be stated as follows:

Let $k(x, y)$ be a purely transcendental extension of an algebraically closed field k, of transcendence degree 2, and let Σ be a field between k and $k(x, y)$, also of transcendence degree 2 over k.[3] If $k(x, y)$ is a separable extension oj Σ, then Σ is a pure transcendental extension of k.

We shall show by an example that the condition of separability of $k(x, y)/\Sigma$ is essential.

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We shall make use of results established in MM for the case of surfaces F for which $P_a = P_2 = 0$ and $(K^2) > 0$. If $(K^2) = 1$, then the Riemann-Roch inequality shows that the dimension of the anticanonical system $| K_a | (= | -K |)$ is $\geqq 1$. If $| K_a |$ is reducible, then F is rational, by Proposition 7.3 of MM. *We shall therefore assume that $| K_a |$ is irreducible.* In that case we have dim $| K_a | = 1$ (MM, Lemma 10.1), i.e., $| K_a |$ is a pencil; it has a single base point O, every member K_a of $| K_a |$ has a simple point at O, and

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[2] See our recent paper *The problem of minimal models in the theory of algebraic surfaces*, Amer. J. Math., vol. 80 (1958), pp. 146–184. This paper will be referred to in the sequel as MM.

[3] The theorem is also true if Σ/k has transcendence degree 1 (without any assumption on separability), but in that case the theorem is an easy consequence of the theorem of of Lüroth.

two distinct K_a's in $|K_a|$ have distinct tangents at O (all this follows from $(K^2) = 1$).

From the irreducibility of $|K_a|$ follows (see MM, Lemma 10.2) that for each $n > 1$ the system $|nK_a|$ is irreducible, is free from base points, and has dimensions $n(n+1)/2$. We shall use the systems $|2K_a|$, $|3K_a|$ and $|6K_a|$. We have

$$\dim |2K_a| = 3, \tag{1}$$

$$\dim |3K_a| = 6, \tag{2}$$

$$\dim |6K_a| = 21. \tag{3}$$

An irreducible curve D on F will be a fundamental curve of $|nK_a|$, $n > 1$, if and only if $(K_a \cdot D) = 0$ (since $|nK_a|$, $n > 1$, has no base points). There will then exist a member K_a^0 of the pencil $|K_a|$ such that D is a component of K_a^0. We cannot have $K_a^0 = D$ since $(K_a^0 \cdot K_a) = 1$ while $(D \cdot K_a) = 0$. Hence K_a^0 is not a prime cycle. Now we prove the following:

PROPOSITION 1. *If K_a^0 is a member of $|K_a|$ which is not a prime cycle, then some prime component of K_a^0 is an exceptional curve of the first kind.*

Proof. For every prime component E of K_a^0 we must have $p(E) = 0$ (MM, Lemma 7.2). Since $|K_a|$ is irreducible (whence E is not a fixed component of $|K_a|$), we have $(K_a \cdot E) \geqq 0$. Since $(K_a^2) = 1$, it follows that there exists one and only one prime component E of K_a^0 such that $(K_a \cdot E) > 0$, and for that component E we must have $(K_a \cdot E) = 1$. Since $p(E) = 0$ and since $(X^2) - 2p(X) + 2 = (K_a \cdot X)$ for every cycle X on F, it follows that $(E^2) = -1$. Thus E is an exceptional curve of the first kind. QED.

The presence of an irreducible exceptional curve E of the first kind implies that F can be transformed birationally into a surface F' which is strictly dominated by F and on which the self-intersection number of a canonical divisor is 2. Hence F' (and therefore also F) is rational, by the case $(K^2) = 2$.

We may therefore assume that the systems $|nK_a|$, $n > 1$, are free from fundamental curves. By Proposition 1, this is equivalent to assuming that *each member of $|K_a|$ is a prime cycle.*

We summarize our assumptions concerning the nonsingular surface F:

(A) $p_a = P_2 = 0$; $(K^2) = 1$.

(B) *Every member of $|K_a|$ is a prime cycle.*

Our proof of the rationality of F will consist in showing that *under the assumptions* (A) *and* (B) *the surface F carries an exceptional curve of the first kind* (whence F is not a relatively minimal model). The rationality of F follows then by the case $(K^2) \geqq 2$. As in the case $(K^2) = 2$ (see MM, §10), so also in the present case, our method of proof will consist in constructing the entire algebraic family of surfaces satisfying assumptions (A) and (B).

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Let t be a parameter of the pencil $|K_a|$:

$$(t) = C - C_0, \tag{4}$$

where $C, C_0 \in |K_a|$. For any $n \geqq 1$ we denote by $\mathfrak{L}(nC_0)$ the space of functions ξ in $k(F)$ such that $(\xi) + nC_0 \geqq 0$. We have $1, t, t^2 \in \mathfrak{L}(2C_0)$, and by (1), we have $\dim \mathfrak{L}(2C_0) = 4$. We choose an element x in $\mathfrak{L}(2C_0)$ such that $\{1, t, t^2, x\}$ is a basis of $\mathfrak{L}(2C_0)$ over k. Since $|2K_a|$ is irreducible (therefore not composite with a pencil), the field $k(x, t)$, generated over k by the homogeneous coordinates of the point $(1, t, t^2, x)$ in S_3, cannot be of transcendence degree 1. *Hence x and t are algebraically independent over k*, and $k(F)$ is an algebraic extension of $k(x, t)$.

The locus, over k, of the point $(z_0, z_1, z_2, z_3) = (1, t, t^2, x)$ is the cone W: $z_0z_2 - z_1^2 = 0$, and this cone is a rational transform of F, the plane sections of W corresponding to the cycles of $|2K_a|$. Since W has order 2 and $|2K_a|$ has degree 4, it follows that[4]

$$[k(F):k(x, t)] = 2. \tag{5}$$

The 6 elements $1, t, t^2, t^3, x, xt$ belong to $\mathfrak{L}(3C_0)$, and by (2) we have $\dim \mathfrak{L}(3C_0) = 7$. We can therefore find an element y in $\mathfrak{L}(3C_0)$ such that $\{1, t, t^2, t^3, x, xt, y\}$ is a basis of $\mathfrak{L}(3C_0)$ over k.

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[4] It is at this stage that a short proof of the rationality of F can be given, provided that the characteristic p of k is different from 2. We shall outline here this proof.

Since $|2K_a|$ has no base points and no fundamental curves and since F is nonsingular (hence normal), F is a normalization of the cone W in the field $k(F)$ (F is a double covering of the cone W). The pencil $|K_a|$ on F corresponds to the pencil of generators of the cone, each K_a being a double covering (*a priori*, not necessarily a normalization) of the corresponding generator. Let D be the branch curve, on W, of the double covering $W \rightarrow F$. It can be shown that D does not pass through the vertex of W. The general generator g of the cone W cannot have a contact P with D such that the intersection multiplicity of D and g at P is > 2, for in the contrary case it is easily seen that the general cycle K_a would not be prime (one must remember that $p(K_a) = 1$). It cannot have a contact with intersection multiplicity 2 since $p \neq 2$. From this it follows that the general K_a is *nonsingular and hence is elliptic*. The general generator g of the cone W must carry 4 branch points of the double covering $g \rightarrow K_a$. It is not difficult to see that one of the branch points is at the vertex O of the cone W (it is an isolated branch point of the covering $W \rightarrow F$, since $O \notin D$). Hence $(g \cdot D) = 3$, and therefore *D is a curve of order 6*. It can be shown that D is in fact complete intersection of W with a cubic surface. By using this fact it is possible to derive the existence of a tritangent plane π of D. The cycle in $|2K_a|$ which corresponds to the plane section $W \cdot \pi$ splits then into two prime cycles D, E, neither one of which is a member of $|K_a|$. This shows that there exist cycles on F which are not linearly equivalent to an integral multiple of K_a, and thus the rationality of F follows from Proposition 9.1 of MM. As a matter of fact, the curves D, E are necessarily exceptional curves of the first kind; this is proved in the beginning of §7. Thus F is not a relatively minimal surface.

If $\mathfrak{p}$ is an (algebraic) place of $k(F)/k$ such that $t\mathfrak{p}$ and $x\mathfrak{p}$ are finite, and if P is the center of $\mathfrak{p}$ on F, then P cannot belong to C_0, for in the contrary case P would have to lie on each of the cycles $(t) + 2C_0$, $(t^2) + 2C_0$, $(x) + 2C_0$, and thus P would be a base point of $| 2K_a |$. Since $P \notin C_0$ it follows that also $y\mathfrak{p} \neq \infty$. *Hence y is integral function of t and x.*

I assert that $y \notin k(x, t)$ (whence y is a primitive element of $k(F)/k(x, t)$; see (5)). For in the contrary case y would be a polynomial in x, t, say $y = f(x, t)$. Let $c_0 t^m + c_1 t^{m-2}x + c_2 t^{m-4}x^2 + \cdots$ be the sum of terms ct^ix^j in $f(x, t)$ for which $m = i + 2j$ is maximum. Were $m > 3$, then there would have to be at least two such terms (since t^ix^j is infinite on C_0 to the order $i + 2j$, while y is infinite to the order 3 on C_0), and the C_0-residue α of x/t^2 would have to satisfy the equation $c_0 + c_1\alpha + c_2\alpha^2 + \cdots = 0$. Thus $\alpha \in k$, and $x - \alpha t^2$ would belong to $\mathfrak{L}(C_0)$, i.e., $x - \alpha t^2$ would be linearly dependent on 1, t, in contradiction with the linear independence of 1, t, t^2, x over k. Hence $m = 3$ and $y = f_3(t) + xf_1(t)$, where f_1 and f_3 are polynomials of degree 1 and 3 respectively. This contradicts the linear independence of 1, t, t^2, t^3, x, xt, y over k.

To find the irreducible equation (of degree 2) for y over $k(x, t)$, we observe that the 23 functions

$$\omega_\nu = t^qx^ry^s, \qquad 0 \leqq q + 2r + 3s \leqq 6 \quad (0 \leqq q, r, s), \tag{6}$$

belong to the vector space $\mathfrak{L}(6C_0)$, and that, by (3), this space has dimension 22. Hence the above 23 functions are linearly dependent. A relation of linear dependence between these functions yields a relation of algebraic dependence between t, x, and y, of degree $\leqq 2$ in y. Since $y \notin k(t, x)$, y^2 must be present in the relation, and thus we find that the equation of algebraic (and integral) dependence for y over $k[x, t]$ has the following form:

$$g = y^2 + [g_3(t) + xg_1(t)]y + [g_6(t) + g_4(t)x + g_2(t)x^2 + g_0x^3] = 0, \tag{7}$$

where $g_i(t)$ is a polynomial of degree $\leqq i$ (with coefficients in k). In particular, the coefficient g_0 of x^3 is a constant. It is important to observe that

$$g_0 \neq 0. \tag{8}$$

To see this we note that if ω is any of the monomials ω_ν in (6) *other than y^2* and x^3, then either C_0 or C is a component of the positive cycle $(\omega) + 6C_0$, and hence the base point O of the pencil $| K_a |$ belongs to this cycle. Were x^3 missing in (7) it would then follow that O belongs also to the cycle $(y) + 3C_0$. Since O also belongs to the cycles $(t^i) + 3C_0$ ($i = 0, 1, 2, 3$), $(x) + 3C_0$, $(xt) + 3C_0$, it would then follow that O belongs to each cycle in $| 3K_a |$, in contradiction with the fact that $| 3K_a |$ has no base points. In the sequel we shall set $g_0 = 1$ (this amounts to replacing x by a constant multiple of x).

Since equation (7) is irreducible, it is the only relation of linear dependence between the monomials ω_ν in (6). It follows that these monomials span the entire space $\mathfrak{L}(6C_0)$. We denote by G the locus, over k, of the point in the projective space S_{22} whose homogeneous coördinates are the ω_ν (actually G lies in S_{21}, since the ω_ν are linearly independent). Thus G is a rational transform of our surface F, and is obtained by having the linear system $|6K_a|$ cut out by hyperplanes. The surface G is a *birational* transform of F, since $k(t, x, y) = k(F)$. Since $|6K_a|$ has no base points and no fundamental curves and since F is nonsingular (hence normal), it follows that *F is a normalization of G*. We shall show, however, that *G i itselj normal*, and from this it will follow that G and F are biregularly equivalent surfaces and that, consequently, G is *nonsingular*. To show the normality of G we shall show that for any $n \geqq 1$ it is true that the linear system L_n cut out on G by the hypersurfaces of order n, in the ambient space S_{21} of G, coincides with the complete system $|6nK_a|$ (and hence G is arithmetically normal; note that $L_1 = |6K_a|$, by the definition of G, and that consequently $L_n \subset |6nK_a|$). If ρ_n is the number of linearly independent monomials ω' of the form $t^\alpha x^\beta y^\gamma$ such that $\alpha + 2\beta + 3\gamma \leqq 6n$ (α, β, γ nonnegative integers), then $\dim L_n = \rho_n - 1$. Since y^2 is linearly dependent on the monomials $t^q x^r y^s$ such that $q + 2r + 3s \leqq 6$ and $s = 0, 1$, it follows that the monomials ω' are linearly dependent on these particular monomials ω' for which γ is either zero or 1. An easy computation shows that the number of these particular monomials ω' is $3n(6n + 1) + 1$, and these monomials are linearly independent since t and x are algebraically independent over k and since $y \notin k(t, x)$. Hence $\dim L_n = 3n(6n + 1)$, i.e., $\dim L_n = \dim |6nK_a|$, and this proves our assertion.

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In this section we shall retrace in reverse the procedure of the preceding section. We shall start with an affine surface g in S_3 defined by an equation of the form (7). We denote by G the projective surface in S_{22} which is the locus, over k, of the point whose homogeneous coördinates are the monomial ω_ν given in (6) (actually G lies in an S_{21}). We make the following two assumptions:

(A′) The coefficient g_0 of x^3 in (7) is different from zero (and we shall assume that $g_0 = 1$).

(B′) The surface G is nonsingular.

Proposition 2. *Under assumptions* (A′) *and* (B′), *the surface G satisfies conditions* (A) *and* (B) *of Section* 2.

Proof. If L_n denotes the system cut out on G by the hypersurfaces of order n, then—as was shown in the preceding section—we have $\dim L_n = 18n^2 + 3n$. Since the constant term of this quadratic polynomial is zero, it follows that *the arithmetic genus of G is zero.*

For the rest of the proof (and also for other applications which will be made in subsequent sections) we shall exhibit a suitable covering of G by three affine representatives.

We number the monomials ω_ν in (6) so as to have $\omega_0 = 1$, $\omega_1 = t^6$, $\omega_2 = y^2$. Let G_j $(j = 0, 1, 2)$ be the affine representative of G on which all the quotients ω_ν/ω_j are regular. It is clear that G_0 can be identified with the given affine surface g in S_3. We set

$$(9) \quad t' = 1/t\ (= t^5/\omega_1), \qquad x' = x/t^2\ (= xt^4/\omega_1), \qquad y' = y/t^3\ (= yt^3/\omega_1).$$

Then t', x', y' are among the quotients ω_ν/ω_1, and we see at once that all the quotients ω_ν/ω_1 are monomials in t', x', and y'. Hence G_1 can be identified with the affine surface g' in S_3 which is the locus, over k, of the point (t', x', y'). The equation of g' has the same form as that of g:

$$g' = y'^2 + [g_3'(t') + x'g_1'(t')]y' + [g_6'(t') + g_4'(t')x' + g_2'(t')x'^2 + x'^3] = 0,$$

where

$$g_i'(t') = t'^i g_i(1/t'), \qquad 1 \leqq i \leqq 6.$$

As to the affine representative G_2 of G, we introduce the functions

$$(10) \quad \tau = xt/y\ (= xty/\omega_2), \qquad \xi = x/y\ (= xy/\omega_2), \qquad \eta = x^3/y^2\ (= x^3/\omega_2),$$

and we denote by g'' the locus of (τ, ξ, η) over k. Since τ, ξ, η are regular on G_2, the affine surface g'' is a regular (and obviously birational) transform of G_2. On the other hand, we have, for all nonnegative integers q, r and s such that $q + 2r + 3s \leqq 6$,

$$t^q x^r y^s/y^2 = \tau^q \xi^{6-q-2r-3s} \eta^{r-2+s}.$$

Hence the functions which are regular on G_2 are also regular at all points of g'' where $\eta \neq 0$. These points of g'' form an open subset which we shall denote by g_0''. Thus g_0'' can be identified with a part of G_2.

We note that g'' has the following equation:

$$(11) \quad \begin{aligned} g'' = \eta^3 &+ [1 + g_1''(\tau, \xi) + g_2''(\tau, \xi)]\eta^2 \\ &+ [g_3''(\tau, \xi) + g_4''(\tau, \xi)]\eta + g_6''(\tau, \xi) = 0, \end{aligned}$$

where

$$g_i''(\tau, \xi) = \xi^i g_i(\tau/\xi), \qquad 1 \leqq i \leqq 6.$$

We shall show that

$$(12) \qquad G = g \cup g' \cup g_0''.$$

In fact, *we shall show that the point* $\xi = \tau = 0$, $\eta = -1$ *is the only point of* g_0'' *which is not covered by* $g \cup g'$.

Let v be any valuation of $k(G)/k$ whose center on G belongs neither to G_0 nor to G_1, i.e., let

$$\min\{v(t), v(x), v(y)\} < 0,$$

$$\min\{v(t'), v(x'), v(y')\} < 0.$$

If $v(x) \geqq 0$, then necessarily $v(t) < 0$, for otherwise we would have $v(y) \geqq 0$ since y is an integral function of x and t. But then, by (9), $v(t') > 0$, $v(x') > 0$, and hence also $v(y') \geqq 0$, in contradiction with our assumption. Thus we have necessarily $v(x) < 0$, and similarly $v(x') < 0$.

Since x is an integral function of t and y, and $v(x) < 0$, we have either $v(t) < 0$ or $v(y) < 0$. Similarly, from $v(x') < 0$ follows that either $v(t') < 0$ or $v(y') < 0$. Were $v(y) \geqq 0$, we would have $v(t) < 0$, and consequently, by (9), $v(t') > 0$, $v(y') > 0$, which is impossible. Hence we have necessarily $v(y) < 0$, and similarly $v(y') < 0$.

Thus our valuation v is necessarily such that

$$v(x), \quad v(y), \quad v(x'), \quad v(y')$$

are all negative.

We note that the expressions of τ, ξ, and η in terms of t', x', and y' are similar to their expressions (10) in terms of t, x, and y, with τ and ξ interchanged, namely

$$\tau = x'/y', \qquad \xi = x't'/y', \qquad \eta = x'^3/y'^2.$$

Hence we may assume that $v(t) \geqq 0$ (if not, then $v(t') \geqq 0$).

It is clear that in equation (7) of the surface g the terms y^2 and x^3 are the only possible terms which have minimum v-value. Hence we must have $v(y^2) = v(x^3)$, $v(y^2/x^3 + 1) > 0$, and $0 < v(y) < v(x)$.

This implies, by (10), that $v(\tau) > 0$, $v(\xi) > 0$, and $v(\eta + 1) > 0$, showing that the center of v on g'' is the point $\tau = \xi = 0$, $\eta = -1$, as asserted. We shall denote this point by A.

Thus the covering (12) of G is established.

We now show that the *pencil* $\{C\}:t =$ const. *consists of anticanonical curves on G and that A is an ordinary simple base point of that pencil.* In fact, consider the differential $\omega = dtdx/(\partial g/\partial y)$. We have $\omega = -dt'dx'/(t'\partial g'/\partial y')$. If we take into account the fact that both affine surfaces g and g' are non-singular and that the part of G which is not covered by $g \cup g'$ reduces to the point A, we conclude that the divisor (ω) of ω on G is equal to $-C_0$, where C_0 is the member of the pencil $\{C\}$ which corresponds to the value ∞ of the parameter t. This shows that the C's are anticanonical cycles. This, of course, implies already that *all the plurigenera of G are zero.* Furthermore, the pencil $\{C\}$ has only one base point on G, namely the point A. Equation (11) shows that τ and ξ are uniformizing parameters at A, and since $t = \tau/\xi$ [by (10)], it follows that any two C's have a simple intersection at A. Consequently $(K_a^2) = 1$ (where K_a denotes any anticanonical cycle on F).

The presence of the terms y^2 and x^3 in the equation (7) of the surface and the fact that the coefficient of y is of degree $\leqq 1$ in x imply that every cycle $t =$ const. of the pencil $\{C\}$ is prime. This completes the proof of Proposition 2.

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From now on we always assume that our surface G is nonsingular.

PROPOSITION 3. *If the equation $g(X, Y, t) = 0$, regarded as an equation in X, Y, over $k(t)$, has a solution of the form*

$$X = \psi_2(t), \qquad Y = \psi_3(t), \tag{12'}$$

where ψ_i is a polynomial in $k[t]$, of degree $\leqq i$, then the surface G is rational. Furthermore, the irreducible curve E defined by (12′) is an exceptional curve on G, of the first kind.

Proof. By assumption, the polynomial $g(t, \psi_2(t), Y)$, of degree 2 in Y, has a root $Y = \psi_3(t)$. Therefore, it has a second root $Y = \varphi_3(t)$, where $\varphi_3(t)$ is also a polynomial of degree $\leqq 3$. We denote by D the irreducible curve defined by

$$X = \psi_2(t), \qquad Y = \varphi_3(t).$$

The replacing of X and Y by $X - \psi_2(t)$ and $Y - \varphi_3(t)$ respectively amounts to a linear transformation of the homogeneous coördinates in the ambient space of G, and, furthermore, this transformation sends g into a polynomial of the same type [see (7)]. Hence we may assume that $\psi_2(t) = \varphi_3(t) = 0$. Then in (7), $g_6(t)$ is zero, and the equations of the surfaces g and g' have the following form:

$$g\colon y(y - \psi_3(t)) + x[g_1(t)y + g_4(t) + g_2(t)x + x^2] = 0,$$

$$g'\colon y'(y' - \psi_3'(t')) + x'[g_1'(t')y' + g_4'(t') + g_2'(t')x' + x'^2] = 0,$$

where $\psi_3'(t') = t'^3\psi_3(1/t')$, $g_i'(t') = t'^i g_i(1/t')$.

Let v and v_1 be the prime divisors of $k(G)$ defined by the curves E and D respectively. We observe that $\psi_3(t)$ is not the constant zero, for if it were and if we denote by t_0 a root of $g_4(t)$, then the point $t = t_0$, $x = y = 0$ would be a singular point of g. (If $g_4(t)$ is a constant, different from zero, then $g_4'(t')$ is a constant multiple of t'^4, and the point $t' = x' = y' = 0$ would be a singular point of g'.) Since $v(x) > 0$ and $v(y - \psi_3(t)) > 0$, it follows from $\psi_3(t) \neq 0$ that $v(y) = 0$. Hence $v(\eta) = v(x^2/y^3) > 0$, showing that the point $A\colon \tau = \xi = 0$, $\eta = -1$ of G does not belong to E. The preceding argument about the nonvanishing of $\psi_3(t)$ shows also that $g_4(t)$ is not zero; in fact, it also shows that $\psi_3(t)$ and $g_4(t)$ have no common factor. The presence of the terms $g_4(t)x - \psi_3(t)y$ in g implies that $v_1(x) = v_1(y)$, since $v_1(x) > 0$ and $v_1(y) > 0$. Hence $v_1(\eta) > 0$, showing that also the curve D does not pass through A. We have therefore proved that

$$E + D \subset g \cup g'.$$

Since $g_6(t)$ is zero, equation (11) of the surface g'' (after division by η) has the form

$$(13) \qquad g'': \eta^2 + [1 + g_1''(\tau, \xi) + g_2''(\tau, \xi)]\eta + g_3''(\tau, \xi) + g_4''(\tau, \xi) = 0.$$

Equations (10) define a birational transformation T of G onto g''. Since $v(x) > 0$ and $v(y - \psi_3(t)) > 0$, while $\psi_3(t) \neq 0$, it follows that $v(y) = v(t) = 0$. Hence $v(t) > 0$, $v(\xi) > 0$, and $v(\eta) > 0$, showing that the center, on g'', of the prime divisor v is the origin O. Thus E is an exceptional curve of T and corresponds to the point O.

We assert that E is the total T^{-1}-transform of O. For let w be any zero-dimensional valuation of $k(G)$ having center O on g''. Since O, $A \in g''$ and $A \neq O$, it follows that the center P of w on G belongs to $g \cup g'$. Because of the symmetric roles played by g and g', we may assume that $P \in g$, i.e., that $w(x) \geqq 0$, $w(y) \geqq 0$, and $w(t) \geqq 0$. Since $\xi = x/y$ and $w(\xi) > 0$, it follows that $w(x) > 0$. Hence $P \in E + D$. Suppose $P \in D$, whence $w(y) > 0$. Since $w(x) > w(y)$, division of the equation $g = 0$ by y shows that $w(\psi_3(t)) = 0$, i.e., P is the point $t = t_0$, $x = y = 0$, where t_0 is a root of $\psi_3(t)$. But then P belongs also to E, and this proves our assertion.

We also assert that T is regular at each point of E. We have only to show that τ, ξ, and η are regular at any point P of G such that $P \in E$. We may assume that $P \in g$ (since $E \subset g \cup g'$). Let $t = t_0$, $x = 0$, $y = y_0 = \psi_3(t_0)$ at P. If $y_0 \neq 0$, then (10) shows that τ, ξ, and η are regular at P. Assume $y_0 = 0$, whence t_0 is a root of $\psi_3(t)$. It was pointed out above that $\psi_3(t)$ and $g_4(t)$ can have no common root t_0 (for otherwise the point $(t_0, 0, 0)$ would be a singular point of g). Hence $g_4(t_0) \neq 0$, and $g_4(t) + g_2(t)x + x^2$ is a unit ε in the local ring $\mathfrak{o}_P$ of P. Therefore $x/y = -(y - \psi_3 + xg_1)/\varepsilon \in \mathfrak{o}_P$, showing that

$$\xi = x/y \in \mathfrak{o}_P, \qquad \tau = \xi t \in \mathfrak{o}_P, \quad \text{and} \quad \eta = \xi^2 x \in \mathfrak{o}_P,$$

as asserted.

Since it is obvious from (13) that O is a simple point of g'', we have therefore established the second part of our proposition: *E is an irreducible exceptional curve, of the first kind.* It follows that G is not a relatively minimal model, and thus the rationality of G follows from the case $(K^2) \geqq 2$ and from Proposition 2.

Note. Clearly, also, D is an exceptional curve of the first kind. Furthermore, assuming, as we may, that $\psi_3(t)$ is exactly of degree 3 in t (if not, pass to g' and $\psi_3'(t')$), we see at once that the common points of E and D are the points $P(t_0, 0, 0)$, where t_0 is any root of $\psi_3(t)$, and that the intersection multiplicity of E and D at P is the multiplicity of t_0 as a root of $\psi_3(t)$. Hence $(E \cdot D) = 3$.

6

If, in (7), we allow the coefficients of the polynomials $g_i(t)$ $(0 \leqq i \leqq 6, i \neq 5)$ to vary arbitrarily in the universal domain, we obtain an irreducible algebraic system M of surfaces in S_3, of dimension 22. It is easily verified that any *general* member g of M/k is an absolutely irreducible, nonsingular (in the absolute sense) surface, and that the projective surface G, defined in terms of g as in Section 4, satisfies conditions (A) and (B) of Section 2.

PROPOSITION 4. *The general surface g of M/k carries a curve $X = \psi_2(t)$, $Y = \psi_3(t)$, where ψ_2 and ψ_3 are polynomials of degree 2 and 3 respectively (with coefficients in the universal domain).*

Proof. We consider the most general surface h in M whose equation $h(X, Y, t) = 0$ has a rational solution $X = \psi_2(t)$, $Y = \psi_3(t)$ of the above indicated form. Then

$$h(X, Y, t) = [Y - \psi_3(t)][Y - \varphi_3(t)] + \lambda[X - \psi_2(t)][h_1(t)y + x^2 + h_2(t)x + h_4(t)], \tag{14}$$

where λ and the 21 coefficients of the polynomials ψ_3, φ_3, ψ_2, and h_i ($i = 1, 2, 4$) are indeterminates (the subscript indicates in each case the degree of the polynomial). We have to show that the surface h is a general member of M/k. Let N be the irreducible subsystem of M which is the locus of h over k. We have to show that dim $N = 22$. We denote by h^* the point in the affine space A_{22}, of dimension 22, whose coördinates are λ and the coefficients of ψ_3, φ_3, ψ_2, h_i ($i = 1, 2, 4$). The locus, over k, of the pair (h^*, h) is a rational transformation of A_{22} onto N. To prove that dim $N = 22$ we have to show that dim $T^{-1}\{h\} = 0$, or—equivalently—that dim $h^*/K = 0$, where K denotes here the field generated over k by the coefficients of the polynomial $h(X, Y, t)$. Let E and D denote the curves $X = \psi_2(t)$, $Y = \psi_3(t)$ and $X = \psi_2(t)$, $Y = \varphi_3(t)$ respectively. Since both E and D are exceptional curves of the first kind on the surface h, neither one can vary on h in an algebraic system of positive dimension. Hence the coefficients of ψ_2, ψ_3, and φ_3 are algebraic over K. From the identity (14) it follows at once that λ and the coefficients of $h_1(t)$, $h_2(t)$, and $h_4(t)$ belong to the field generated over K by the coefficients of $\psi_2(t)$, $\psi_3(t)$, and $\varphi_3(t)$. This shows the point h^* is algebraic over K, and the proposition is proved.

7

We shall now proceed to the proof of the results which has been announced at the end of Section 2, namely, that the surface F, under the assumptions (A) and (B), carries an exceptional curve of the first kind. We may replace F by the biregularly equivalent surface G which lies in a projective space S_{21} (see Section 3). This surface has order 36, since it is the image of the complete system $|6K_a|$ (which has degree 36). By Proposition 3, it will be sufficient to show that the equation (7) has two rational solutions $X = \psi_2(t)$, $Y = \psi_3(t)$ and $X = \psi_2(t)$, $Y = \varphi_3(t)$ of the type stated in Proposition 3. Let us interpret the existence of such a pair of solutions. They would correspond to two irreducible exceptional curves E and D on G, of the first kind, and these two curves would be the zeros (and the only zeros) on G of the function $x - \psi_2(t)$. Hence $E + D$ is a cycle in $| 2K_a |$.[5] It is a composite cycle, consisting of two (not necessarily distinct) prime cycles E and D. *Moreover, neither one of these two prime cycles is a member of the pencil* $| K_a |$ *defined by* $t = const.$ Conversely, assume that the system $| 2K_a |$ contains a cycle Γ which is not prime but is not a sum of two cycles of $| K_a |$.[6] Since $(\Gamma \cdot K_a) = 2$ and since by assumption (A) of Section 2 we have $(\Delta \cdot K_a) > 0$ for every prime cycle Δ on G, it follows that $\Gamma = E + D$, where E and D are prime cycles, and that $(E \cdot K_a) = (D \cdot K_a) = 1$. Neither E nor D can pass through the base point of the pencil $| K_a |$, for in the contrary case the cycles in $| K_a |$ would have no variable intersections with E (or D), and thus E (or D) would be a component of some cycle of $| K_a |$, and this, in view of assumption (A), is impossible, since $E \notin | K_a |$ and $D \notin | K_a |$. It follows that the trace of $| K_a |$ on E is a linear series of degree 1 and of dimension 1. Hence E is a rational curve, and similarly for D. The curve E has no singular points, since $(E \cdot K_a) = 1$ and since $E \notin | K_a |$. Hence $p(E) = 0$. Similarly $p(D) = 0$. Since $1 = (E \cdot K_a) = (E^2) - 2p(E) + 2$, it follows that $(E^2) = -1$, and thus E is an exceptional curve of the first kind. Similarly D is an exceptional curve of the first kind.

We therefore have only to show that *the system* $| 2K_a |$ *on* G *contains a cycle which is not prime and which is not a sum of two cycles of* $| K_a |$.

Now, let us consider the affine surface h in A_3 which is the general member of the system M/k [see (14)]. This surface h defines a projective surface H in S_{21}, in the same way as G is defined by the affine surface g. From Proposition 2 it follows that also H is a surface of order 36. Since g is a specialization of h over k, it follows that G is a specialization of H over k. By Proposition 3, the surface H carries two prime cycles E and D which are exceptional curves of the first kind and such that $(E \cdot D) = 3$ [see Note at the end of Section 5]. Moreover, $E + D$ is the null cycle, on H, of the function $x - \psi_2(t)$. By the specialization $H \xrightarrow{h} G$ we find on G a composite cycle $\bar{E} + \bar{D}$ which is the null cycle of a function of the form $\lambda x - \psi_2(t)$, whence at any rate $\bar{E} + \bar{D} \in | 2K_a |$. Now, since $(E \cdot D) = 3$, we have also $(\bar{E} \cdot \bar{D}) = 3$, and consequently $\bar{E} \notin | K_a |$, $\bar{D} \notin | K_a |$, since $(K_a^2) = 1$. This completes the proof of our original assertion made in Section 2.

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[5] Just in the way of (redundant) checking, we observe that since $(E^2) = (D^2) = -1$ and $(E \cdot D) = 3$ [see Note at the end of section 5], the self-intersection number of $E + D$ is equal to 4, while $p(E + D) = p(E) + p(D) + (E \cdot D) - 1 = 2$. This checks with the characters of $| 2K_a |$.

[6] Under this assumption it would already follow from Proposition 9.1 of our paper MM that G is either rational or is not a relatively minimal model (hence again rational, by the case $(K^2) \geqq 2$), since G would then carry a cycle which is not linearly equivalent to an integral multiple of K_a. However, we have assigned ourselves the task of proving not only the rationality of G but also the assertion that G is not a relatively minimal model.

8

We now turn to the Castelnuovo theorem of rationality of plane involutions, as formulated in Section 1. We fix a nonsingular projective model F of Σ/k, and we choose a nonsingular model F' of $k(x, y)/k$ such that F' dominates the normalization of F in $k(x, y)$. The rational transformation $f: F' \rightarrow F$ is regular, and the inverse transformation $f^{-1}: F \rightarrow F'$ has only a finite number of fundamental points on F. The multicanonical systems $| nK |$ on F can be defined by regular differential forms $\omega = A(dxdy)^n$, of weight n, on F. The separability of $k(x, y)/\Sigma$ implies that if $\omega \neq 0$ then the inverse image ω' of ω by f is also different from zero, and is of course also regular on F'. Since F' is rational, it follows that we cannot have $\omega \neq 0$. *Thus all the plurigenera of F are zero.*

The rational mapping f of F' onto F defines a rational mapping of the Albanese variety of F' *onto* the Albanese variety of F. Since the former is a point, it follows that also the Albanese variety of F reduces to a point, i.e., the irregularity q of F is zero. Since $p_g = 0$, it follows that $q = -p_a$, whence $p_a = 0$ (see, for instance, Y. NAKAI, *On the characteristic linear systems of algebraic families*, Illinois J. Math., vol. 1 (1957), pp. 552–561). Thus we have shown that $P_2 = p_a = 0$ for F, and consequently F is a rational surface.

Let us call a surface F *unirational* if it is a rational transform of a rational surface F'. The theorem of Castelnuovo asserts that under the separability assumption concerning $k(F')/k(F)$ a unirational surface F is in fact rational. Now consider, for characteristic $p \neq 0$, any surface F in S_3 given by an equation of the form

$$z^p = f(x, y), \tag{15}$$

where $f(x, y)$ is a polynomial. Any such surface F is unirational, for $k(x, y, z) \subset k(x^{1/p}, y^{1/p})$. Now we shall find a surface F of this type such that the geometric genus p_g of F is > 0, and this will show that the condition of separability in Castelnuovo's theorem is essential.

The following is well known, and is true for arbitrary characteristic: if a surface F in S_3 is such that its only singularities are isolated double points, and if each double point of F is no worse than a biplanar point (i.e., the tangent quadric cone at the point is either irreducible or splits into two *distinct* planes), then every surface in S_3 is an adjoint surface of F.[7] Thus, if such a surface F has order $\geqq 4$, the geometric genus of F will be positive. This being so, let the characteristic p be different from 2, and let us consider the surface F defined by the equation

$$F(x, y, z) = z^p + x^{p+1} + y^{p+1} - (x^2 + y^2)/2 = 0, \tag{16}$$

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[7] In other words, if $f(x, y, z) = 0$ is the irreducible equation of the surface F, and if n is the degree of f, then (under the assumption that f is a separable polynomial in z) every double differential of the form $(\phi_{n-4}(x, y, z)/f'_z)\, dx\, dy$, where ϕ is an arbitrary polynomial of degree $\leqq n - 4$, is regular (not only on F but also) on every nonsingular model of the field $k(x, y, z)$.

which is of type (15). One finds at once that the only singular points of the surface are the points $x = m$, $y = n$, $z = \{(n^2 + m^2)/2\}^{1/p}$, where m and n are arbitrary elements of the prime field of characteristic p (there are no singular points at infinity). It is also immediately seen that these p^2 singular points are biplanar double points. Since the order of the surface F is $p + 1 \geqq 4$, its geometric genus is positive.[8]

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[8] An example, of similar nature, could also be given for $p = 2$. It would be similar to (16), but the degree in z would have to be a power 2^n of 2, $n > 1$.