PROJECTIVE TOPOLOGICAL SPACES

BY

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Suppose we have given a category of topological spaces and continuous maps. Let X, Y, and Z be admissible spaces and ϕ and f admissible maps of X into Z and Y into Z respectively. A natural question is whether or not there exists an admissible map ψ of X into Y such that $\phi = f \circ \psi$. One can hardly expect to answer such a question without explicit knowledge of all the data, but it may happen that, for certain spaces X, the answer is always yes provided f satisfies the minimum condition of mapping Y onto Z. Discrete spaces are examples in the category of all spaces and continuous maps. Following the terminology of homological algebra, we shall call such a space projective. In this paper we will determine the projective spaces in the category of compact spaces and continuous maps and discuss the notion of projective resolution for these spaces.

Throughout the paper the word *space* will mean *Hausdorff space*.

1. The necessary condition

We restrict our attention to those categories of spaces and maps for which

(a) All admissible maps are continuous.

(b) If A is an admissible space and $\{p, q\}$ is a two-element space, then $A \times \{p, q\}$ and the projection map of this space onto A are admissible.

(c) If A is an admissible space and B is a closed subspace of A, then B and the inclusion map of B into A are admissible.

These conditions are not stringent and are satisfied by many of the usual categories.

1.1. DEFINITION. A topological space is said to be extremally disconnected if and only if the closure of every open set is again open.

1.2 THEOREM. *In any category of topological spaces and maps satisfying conditions* (a), (b), *and* (c) *above, a projective space is extremally disconnected.*

Proof. Let X be a projective space in such a category. Let G be any open subset of X; we must prove $\bar{G}$ is open.

In $X \times \{p, q\}$ consider the closed set $Y = ((X - G) \times \{p\}) \cup (\bar{G} \times \{q\})$, and its inclusion map i. Let π be the projection of $X \times \{p, q\}$ onto X. Our hypothesis on the category implies that $\pi \circ i$ is an admissible map of Y onto X and that the identity ϕ is an admissible map of X into X. Since X is projective, there is an admissible map ψ of X into Y such that $\phi = \pi \circ i \circ \psi$.

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Because $\pi \circ i$ is one-to-one on $G \times \{q\}$ it is clear that $\psi(x) = \langle x, q \rangle$ for $x \in G$; from the continuity of ψ follows $\psi(x) = \langle x, q \rangle$ for $x \in \bar{G}$. Similarly, for $x \notin \bar{G}$, $\psi(x) = \langle x, p \rangle$. Thus we have proved $\bar{G} = \psi^{-1}(\bar{G} \times \{q\})$. Since ψ is continuous and $\bar{G} \times \{q\}$ is open in Y, $\bar{G}$ is open in X as required.

1.3 THEOREM. *In an extremally disconnected space no sequence is convergent unless it is ultimately constant.*

Proof. Suppose that the sequence $\{x_n\}$ converges to p in the extremally disconnected space X. Assuming this sequence is not ultimately constant, we shall deduce a contradiction.

First we construct inductively a disjoint sequence $\{U_i\}$ of open sets in X such that each U_i contains a member $x_{n(i)}$ of the given sequence, where $\{n(i)\}$ is an increasing sequence of integers. Let $n(1)$ be an index for which $x_{n(1)} \neq p$, and choose an open set U_1 such that $x_{n(1)} \in U_1$ but $p \notin \bar{U}_1$. Suppose we have chosen disjoint open sets $U_1, U_2, \cdots, U_k$ and increasing integers $n(1), n(2), \cdots, n(k)$ such that $x_{n(i)} \in U_i$ and $p \notin \bar{U}_i$ for $i = 1, 2, \cdots, k$. Then $V = X - (\bar{U}_1 \cup \bar{U}_2 \cup \cdots \cup \bar{U}_k)$ is an open neighborhood of p, so $x_q \in V$ for all sufficiently large q. By a suitable choice of $n(k+1)$ we shall have $n(k+1) > n(k)$, $x_{n(k+1)} \in V$ but $x_{n(k+1)} \neq p$ since the original sequence is not ultimately constant. Choose an open set W such that $x_{n(k+1)} \in W$ but $p \notin \overline{W}$, and let $U_{k+1} = W \cap V$. This completes the inductive construction.

Let $G = \bigcup U_{2q}$. Since X is extremally disconnected, $\bar{G}$ is an open set, and $p \in \bar{G}$ being the limit of $\{x_{n(2q)}\}$. Thus $\bar{G}$ is a neighborhood of p, so $x_r \in \bar{G}$ for all large r; in particular, $x_{n(s)} \in \bar{G}$ for some odd integer s. Since U_s is a neighborhood of $x_{n(s)}$, $U_s \cap G$ is not empty, contrary to the definition of G and disjointness of the U's.

1.4 COROLLARY. *In a category in which all spaces satisfy the first axiom of countability and properties* (a), (b), *and* (c) *hold, every projective space is discrete.*

2. Projective spaces in the category of compact spaces and continuous maps

We have seen that in many categories all projective spaces are discrete. Since it is easy to check whether, in a given category, the discrete spaces are projective, we shall discuss sufficient conditions in a more interesting category, that of compact spaces and continuous maps. Of course the discrete spaces are all projective in the category, but there are other projective spaces. A somewhat similar situation prevails in certain categories of modules and homomorphisms; free modules are projective, but there may be other projective modules.

2.1 LEMMA. *Let A and E be spaces. Suppose ρ is a continuous map of E onto A such that $\rho(E_0) \neq A$ for any proper closed subset E_0 of E. Then, for any open set $G \subset E$, $\rho(G) \subset \overline{A - \rho(E - G)}$.*

Proof. There is nothing to prove if G is empty. Supposing otherwise, let a be any point of $\rho(G)$, and let N be any open neighborhood of a. The lemma will follow if we prove that $N \cap (A - \rho(E - G))$ is not void.

Because $G \cap \rho^{-1}(N)$ is a nonempty open subset of E, $\rho(E - (G \cap \rho^{-1}(N))) \neq A$. Take $x \in A - \rho(E - (G \cap \rho^{-1}(N)))$; a fortiori, $x \in A - \rho(E - G)$. Since ρ is onto, $x = \rho(y)$ where evidently $y \in G \cap \rho^{-1}(N)$. Therefore $x = \rho(y) \in \rho(\rho^{-1}(N)) = N$, so $x \in N \cap (A - \rho(E - G))$, and the latter set is not void.

2.2 LEMMA. *In an extremally disconnected space, if U_1 and U_2 are disjoint open sets, then $\bar{U}_1$ and $\bar{U}_2$ are also disjoint.*

Proof. First, $\bar{U}_1$ and U_2 are disjoint because U_2 is open; then $\bar{U}_1$ and $\bar{U}_2$ are disjoint because $\bar{U}_1$ is open.

2.3 LEMMA. *Let A be an extremally disconnected compact space, and let E be a compact space. Suppose ρ is a continuous map of E onto A such that $\rho(E_0) \neq$ A for any proper closed subset E_0 of E. Then ρ is a homeomorphism.*

Proof. We need only show that ρ is one-to-one. Suppose, on the contrary, that x_1 and x_2 are distinct points of E for which $\rho(x_1) = \rho\ (x_2)$. Let G_1 and G_2 be disjoint open neighborhoods of x_1 and x_2 respectively. Both the sets $E - G_1$ and $E - G_2$ are compact, so $A - \rho(E - G_1)$ and $A - \rho(E - G_2)$ are open. The latter sets are disjoint because $E = (E - G_1) \cup (E - G_2)$. By the preceding lemma, $\overline{A - \rho(E - G_1)}$ and $\overline{A - \rho(E - G_2)}$ are disjoint. On the other hand, it follows from Lemma 2.1 that $\rho(x_1) = \rho(x_2)$ is a point common to these sets. This contradiction establishes Lemma 2.3.

2.4 LEMMA. *Let A and D be compact spaces, and let π map D continuously onto A. Then D contains a compact subset E such that $\pi(E) = A$ but $\pi(E_0) \neq A$ for any proper closed subset E_0 of E.*

Proof. This is a well known consequence of Zorn's lemma.

2.5 THEOREM. *In the category of compact spaces and continuous maps, the projective spaces are precisely the extremally disconnected spaces.*

Proof. To prove that all projective spaces in the category are extremally disconnected, we have only to verify the conditions of Theorem 1.2. We turn to the opposite inclusion.

Let A be an extremally disconnected compact space, let B and C be compact spaces, let f be a continuous map of B onto C, and let ϕ be a continuous map of A into C. We must prove that there exists a continuous map ψ of A into B such that $\phi = f \circ \psi$.

In the space $A \times B$ consider $D = \{ \langle a, b \rangle \mid \phi(a) = f(b) \}$. This set is clearly closed and therefore compact. Since f is onto, the projection π_1 of $A \times B$ onto A carries D onto A. By Lemma 2.4 there is a closed subset E of D such that $\pi_1(E) = A$ but $\pi_1(E_0) \neq A$ for any proper closed subset E_0 of

E. Let ρ be the restriction of π_1 to E. Lemma 2.3 asserts that ρ is a homeomorphism. Let $\psi = \pi_2 \circ \rho^{-1}$, where π_2 is the projection of $A \times B$ into B; this is the required map. Say $a \in A$; since $\rho^{-1}(a) \in D$,

$$f(\pi_2(\rho^{-1}(a))) = \phi(\pi_1(\rho^{-1}(a))) = \phi(a).$$

Thus $\phi = f \circ \pi_2 \circ \rho^{-1} = f \circ \psi$; this completes the proof.

3. Projective resolution

In homology theory considerable use is made of the fact that every group is the homomorphic image of a projective group. We now turn our attention to the corresponding question for topological categories. We shall prove that every compact space is the continuous image of an extremally disconnected compact space; otherwise put, in the category of compact spaces and continuous maps every space is the admissible image of a projective space. We shall show, moreover, that this projective space can be selected in a natural way. In the many categories for which projective and discrete are synonymous, the existence or nonexistence of such projective resolutions is trivial.

Since the Stone representation theory for Boolean algebras plays a central role in what follows, it is appropriate to review the main facts of that theory.

THEOREM (Stone [2]). *Every Boolean algebra $\mathcal{B}$ is isomorphic to the set of all open and closed subsets of a certain totally disconnected compact space $\mathcal{S}$.*

The space $\mathcal{S}$ is determined from $\mathcal{B}$ as follows: The points of $\mathcal{S}$ are the maximal subsets of $\mathcal{B}$ which are closed under meets but do not contain 0. For each element $B \in \mathcal{B}$ we define a subset $\xi(B)$ of $\mathcal{S}$ by $\xi(B) = \{p \mid p \in \mathcal{S}, B \in p\}$. The topology of $\mathcal{S}$ is determined by decreeing that each of the sets $\xi(B)$, and all unions of such sets, be open. It then develops that $\mathcal{S}$ is compact and totally disconnected and that the open and closed subsets of $\mathcal{S}$ are precisely those of the form $\xi(B)$ for some $B \in \mathcal{B}$; thus ξ is the required isomorphism of $\mathcal{B}$ onto the Boolean algebra of open and closed subsets of $\mathcal{S}$.

THEOREM (Stone [2]). *Let X be a space. A necessary and sufficient condition that X be homeomorphic to the Stone representation space of some Boolean algebra is that X be totally disconnected and compact.*

In fact X is homeomorphic to the representation space for the Boolean algebra of its own open and closed subsets. The homeomorphism is unique and easily constructed if we require that each open and closed subset of X be carried by the homeomorphism onto its image under the representation isomorphism.

THEOREM (Folk Theorem). *A necessary and sufficient condition that X be homeomorphic to the Stone representation space of some complete Boolean algebra is that X be extremally disconnected and compact.*

Proof. Suppose $\mathcal{S}$ is the Stone representation space for a complete Boolean algebra $\mathcal{B}$. Let G be any open set in $\mathcal{S}$. By the definition of the topology of $\mathcal{S}$, G has the form $\bigcup\{\xi(B_\alpha)\}$ where $\{B_\alpha\}$ is a subset of $\mathcal{B}$. It is easily checked that $\bar{G} = \xi(\bigvee\{B_\alpha\})$. Since the latter is both open and closed, $\mathcal{S}$ is extremally disconnected.

Suppose X is an extremally disconnected and compact space. A fortiori, X is totally disconnected and therefore homeomorphic to the representation space for the Boolean algebra $\mathcal{B}$ of its open and closed subsets; hence we need only prove this latter algebra complete. If $\{B_\alpha\}$ is any subset of $\mathcal{B}$, then $\overline{\bigcup\{B_\alpha\}}$, which is open and closed and therefore in $\mathcal{B}$ by the hypothesis on X, is the least upper bound for $\{B_\alpha\}$ in $\mathcal{B}$.

3.1 Lemma. *Let $\mathfrak{D}(X)$ be the set of all closed domains in a topological space X; that is, subsets D of X satisfying $D = \overline{\operatorname{Int} D}$. Then $\mathfrak{D}(X)$ is a complete Boolean algebra when ordered by inclusion.*

Proof. First, we note that the closure of any open set is a closed domain, since $\overline{\operatorname{Int} \bar{G}} \supset \overline{\operatorname{Int} G} = \bar{G} \supset \overline{\operatorname{Int} \bar{G}}$.

Second, we show that $\mathfrak{D}(X)$ is a complete lattice and obtain formulae for meets and joins. Let $\{D_\alpha\}$ be any collection of closed domains and put $D = \overline{\bigcup\{\operatorname{Int} D_\alpha\}}$. Then $D \in \mathfrak{D}(X)$ and $D \supset \overline{\operatorname{Int} D_\alpha} = D_\alpha$ for all α. Suppose $E \in \mathfrak{D}(X)$ and $E \supset D_\alpha$ for all α. Then $\operatorname{Int} E \supset \operatorname{Int} D_\alpha$ for all α, so $\operatorname{Int} E \supset \bigcup\{\operatorname{Int} D_\alpha\}$ and $E = \overline{\operatorname{Int} E} \supset D$. This proves that D is the least upper bound of $\{D_\alpha\}$. We note also that $D \supset \overline{\bigcup\{D_\alpha\}} \supset \overline{\bigcup\{\operatorname{Int} D_\alpha\}} = D$ and therefore $\bigvee\{D_\alpha\} = \overline{\bigcup\{D_\alpha\}}$; in particular $D_1 \vee D_2 = D_1 \cup D_2 = \overline{\operatorname{Int} D_1 \cup \operatorname{Int} D_2}$. Similarly, we find that $\bigwedge\{D_\alpha\} = \overline{\operatorname{Int} \bigcap\{D_\alpha\}}$, and for finite meets $D_1 \wedge D_2 \wedge \cdots \wedge D_n = \overline{\operatorname{Int}(D_1 \cap D_2 \cap \cdots \cap D_n)}$.

Third, we show that $\mathfrak{D}(X)$ is complemented. Evidently, the null set is in $\mathfrak{D}(X)$ and is the zero element of the lattice, while X itself is the unit element. For any $D \in \mathfrak{D}(X)$ let $D' = \overline{X - D}$. Then $D' \in \mathfrak{D}(X)$ and $D \vee D' = D \cup D' = X$. Since $D \cap \overline{X - D}$ contains no open set, $D \wedge D' = \overline{\operatorname{Int}(D \cap D')} = 0$. This proves that D' is a complement of D.

Finally, we check the distributive law. Let C, D_1, and D_2 be any members of $\mathfrak{D}(X)$. We have seen that $D_1 \vee D_2 = \overline{\operatorname{Int}(D_1 \vee D_2)} = \overline{\operatorname{Int} D_1 \cup \operatorname{Int} D_2}$. In any topological space, if G is open and $\bar{Y} = \bar{Z}$, then $\overline{G \cap Y} = \overline{G \cap Z}$. Applying this formula,

$$\begin{aligned} C \wedge (D_1 \vee D_2) &= \overline{\operatorname{Int} C \cap \operatorname{Int}(D_1 \vee D_2)} = \overline{\operatorname{Int} C \cap (\operatorname{Int} D_1 \cup \operatorname{Int} D_2)} \\ &= \overline{(\operatorname{Int} C \cap \operatorname{Int} D_1) \cup (\operatorname{Int} C \cap \operatorname{Int} D_2)} = \overline{\operatorname{Int}(C \cap D_1)} \cup \overline{\operatorname{Int}(C \cap D_2)} \\ &= (C \wedge D_1) \cup (C \wedge D_2) = (C \wedge D_1) \vee (C \wedge D_2). \end{aligned}$$

This completes the proof that $\mathfrak{D}(X)$ is a complete Boolean algebra.

3.2 Theorem. *Every compact space X is the continuous image of an extremally disconnected compact space. Among the pairs $\langle \mathcal{S}, \phi \rangle$ consisting of* *an extremally disconnected compact space* $\mathcal{S}$ *and a continuous mapping* ϕ *of* $\mathcal{S}$ *onto* X, *there is one for which*

(i) $\phi(\mathcal{S}_0) \neq X$ *for any proper closed subset* $\mathcal{S}_0$ *of* $\mathcal{S}$.

This pair is uniquely determined by X *in the following sense: If* $\langle \mathcal{S}', \phi' \rangle$ *is another such pair satisfying* (i), *then there is a homeomorphism* ψ *of* $\mathcal{S}'$ *onto* $\mathcal{S}$ *such that* $\phi' = \phi \circ \psi$.

Proof. Let $\mathcal{S}$ be the Stone representation space for the Boolean algebra $\mathfrak{D}(X)$ defined in Lemma 3.1. We now define the map ϕ. A point p of $\mathcal{S}$ is a maximal collection of elements of $\mathfrak{D}(X)$ closed under finite meets but not containing the null set. Maximality implies that if $D_1 \notin p$, we can choose $D_2 \in p$ so that $D_1 \wedge D_2 = 0$. From the formula for meets it follows that p is a family of closed subsets of X having the finite intersection property. Since X is compact, $\bigcap p$ is not void; we shall prove that this set contains only one point. Suppose $\bigcap p$ contained as many as two points, say x and y. Let G be an open set in X such that $x \in G$ but $y \notin \bar{G}$. Now $\bar{G} \in \mathfrak{D}(X)$ but $\bar{G} \notin p$ because $y \notin \bar{G}$ and $y \in \bigcap p$. Therefore, we can find $D \in \mathfrak{D}(X)$ such that $D \in p$ and $\bar{G} \wedge D = 0$. Since $x \in \bigcap p \subset D = \overline{\text{Int } D}$ we know that $G \cap \text{Int } D$ is a nonempty open set and therefore $\bar{G} \wedge D = \overline{\text{Int } (\bar{G} \cap D)} \supset \overline{\text{Int } (G \cap \text{Int } D)} \neq 0$, a contradiction. Thus $\bigcap p$ contains only one point, and we may define a map ϕ from S to X by the relation $\phi(p) \in \bigcap p$.

We remark for future reference that, if G is any open set containing $\phi(p)$, then $\bar{G} \in p$. This follows from arguments similar to the preceding.

Next we shall prove that ϕ is continuous. Let p be any point of $\mathcal{S}$, and let N be any neighborhood of $\phi(p)$. Since a compact space is regular, there is an open set G such that $\phi(p) \in G$ and $\bar{G} \subset N$. The set $\bar{G}$ defines an open set in $\mathcal{S}$, namely $U = \xi(\bar{G}) = \{q \mid q \in \mathcal{S}, \bar{G} \in q\}$, ξ being the isomorphism of the Stone representation theory. By the remark of the preceding paragraph, $p \in U$. If $q \in U$, we have $\phi(q) \in \bigcap q \subset \bar{G} \subset N$; this establishes the continuity of ϕ.

To show that ϕ maps $\mathcal{S}$ onto X, we choose any point $x \in X$ and consider $\mathcal{A} = \{D \mid x \in \text{Int } D, D \in \mathfrak{D}(X)\}$. This set has the finite meet property but does not contain the null set, therefore it can be extended to be a maximal subset of $\mathfrak{D}(X)$ having these properties; in other words, there is a point p of $\mathcal{S}$ with $\mathcal{A} \subset p$. Since x has arbitrarily small closed neighborhoods, we see that $\{x\} = \bigcap \mathcal{A}$. Therefore $\phi(p) \in \bigcap p \subset \bigcap \mathcal{A} = \{x\}$ or $\phi(p) = x$.

Suppose now that $\mathcal{S}_0$ is a proper closed subset of $\mathcal{S}$. There is a nonvoid open and closed subset U of $\mathcal{S}$ such that $\mathcal{S}_0 \cap U = 0$. By the Stone theorem, there is a nonvoid closed domain D in X such that

$$U = \xi(D) = \{q \mid q \in \mathcal{S}, D \in q\}.$$

Then $\phi(\mathcal{S}_0)$ contains no point of Int D. For if $s \in \mathcal{S}_0$ and $\phi(s) \in \text{Int } D$, then as remarked in the second paragraph, $D = \overline{\text{Int } D} \in s$ or $s \in U$, which implies the absurdity $s \in \mathcal{S}_0 \cap U$. This proves that $\mathcal{S}$ satisfies condition (i).

Finally, we must prove the unicity statement concerning pairs $\langle \mathcal{S}, \phi \rangle$ satisfying condition (i). Suppose $\langle \mathcal{S}', \phi' \rangle$ is another such pair. Since $\mathcal{S}'$ is projective, there is a map ψ of S' into S such that $\phi' = \phi \circ \psi$. Since $\psi(\mathcal{S}')$ is a closed subset of $\mathcal{S}$ and $\phi(\psi(\mathcal{S}')) = X$, condition (i) for $\mathcal{S}$ implies $\psi(\mathcal{S}') = \mathcal{S}$. On the other hand, if $\mathcal{S}_0'$ is a proper closed subset of $\mathcal{S}'$, then $\psi(\mathcal{S}_0')$ cannot be all of $\mathcal{S}$ because then $\phi'(\mathcal{S}_0') = \phi(\psi(\mathcal{S}_0')) = X$, contrary to condition (i) for $\mathcal{S}'$. Now by Lemma 2.3, ψ is a homeomorphism. This completes the proof of Theorem 3.2.

3.3 COROLLARY. *In the category of compact spaces and continuous maps, every space is the admissible image of a projective space.*

4. Locally compact spaces

We consider now the category of locally compact spaces and proper maps. (A map is said to be proper if and only if it is continuous and the inverse image of every compact set is compact.) With minor modifications the theorems and proofs of Sections 2 and 3 are valid in this category, but the quickest way to obtain the results is to pass to the Stone-Čech compactification of all the spaces involved and apply the theorems already developed. Since the details are all straightforward, we shall give no proofs of the following theorems.

4.1 THEOREM. *A completely regular space is extremally disconnected if and only if its Stone-Čech compactification is extremally disconnected.*

4.2 THEOREM. *In the category of locally compact spaces and proper maps, the projective spaces are precisely the extremally disconnected spaces.*

4.3 THEOREM. *In the category of locally compact spaces and proper maps, every space is the admissible image of a projective space. Moreover, there is a natural choice of this space and map which is unique within isomorphism as in Theorem* 3.2.

5. Duality

In homology theory the term injective is applied to a module which has the property dual to projectivity. Specifically, a member X of a category is injective if, whenever f is an admissible map of Y into X and i is a one-to-one admissible map of Y into Z, then there exists an admissible map g of Z into X such that $f = g \circ i$. Generally speaking the dual of an injective object is projective and vice versa. Since it is known that the category of compact spaces and continuous maps is dual to the category of commutative C^* algebras, we deduce immediately

5.1 THEOREM. *In the category of commutative C^* algebras and $*$-homomorphisms, the injective algebras are precisely the algebras of continuous functions on extremally disconnected compact spaces.*

If we restrict ourselves to begin with to the category of totally disconnected compact spaces, then the appropriate dual category is that of Boolean algebras and homomorphisms. We obtain immediately, then, the following theorem of Sikorsky [1].

5.2 THEOREM. *In the category of Boolean algebras and homomorphisms, the injective algebras are precisely the complete algebras.*

REFERENCES

1. R. SIKORSKI, *A theorem on extensions of homomorphisms*, Ann. Soc. Polon. Math., vol. 21 (1948), pp. 332–335.
2. M. H. STONE, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., vol. 41 (1937), pp. 375–481.

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