CLOSURE AND DISPERSION OF FINITE GROUPS

In commemoration of G. A. Miller

BY

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If $\mathfrak{r}$ is a set of primes, then we term $\mathfrak{r}$-group [$\mathfrak{r}$-element] every finite group [every group element] whose order is divisible by primes in $\mathfrak{r}$ only. A group is termed $\mathfrak{r}$-closed, if its set of $\mathfrak{r}$-elements is a characteristic $\mathfrak{r}$-subgroup; and this is equivalent to requiring that products of $\mathfrak{r}$-elements are again $\mathfrak{r}$-elements. Several well known theorems in finite group theory may be interpreted as criteria for $\mathfrak{r}$-closure; and our principal concern in this investigation will be with such criteria.

If $\mathfrak{r}$ is a set of primes, then we denote by $P\mathfrak{r}$ the complementary set of primes (= set of primes prime to $\mathfrak{r}$); and we say that a group is $P\mathfrak{r}$-homogeneous if its elements induce $P\mathfrak{r}$-automorphisms in its $P\mathfrak{r}$-subgroups. It is easy to see that $\mathfrak{r}$-closed groups are $P\mathfrak{r}$-homogeneous; but there exist $P\mathfrak{r}$-homogeneous groups which are not $\mathfrak{r}$-closed. The clarification of this relation is our main problem. The most comprehensive criterion obtained in this direction is Theorem 5.3: The finite group G is $\mathfrak{r}$-closed if, and only if, it is $P\mathfrak{r}$-homogeneous and $\{R, P\}$ is an $\mathfrak{r}$-p-group whenever R is a maximal $\mathfrak{r}$-subgroup of G, P a p-Sylow subgroup of G, and p a prime, not in $\mathfrak{r}$.

On our way we have to focus attention on Pp-closure (and dually on p-closure); and the analysis of Pp-closure is closely related to an investigation of groups with the property that all epimorphic images of subgroups of index prime to p are p-normal. The auxiliary results obtained here appear to be of independent interest [§4].

By its very definition dispersion is a concatenation of an involved array of closure requirements. We shall, however, show in §1 that dispersion may be reduced essentially to p-closure and Pp-closure. Combining this reduction theorem with the closure criteria obtained in §§2 to 5 we obtain a number of interesting dispersion criteria in §6.

Notations

$o(G)$ = order of group G.

$o(g)$ = order of group element g.

G' = commutator subgroup of G.

$G^{(i)}$ = i^{th} derivative of group G (inductively defined by $G = G^{(0)}, G^{(i+1)} = [G^{(i)}]'$).

ZG = center of G.

$Z_i G$ = i^{th} term in ascending central series of G (inductively defined by $Z_0 G = 1, Z_{i+1} G/Z_i G = Z[G/Z_i G]$).

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ΦG = Frattini subgroup of G = intersection of all maximal subgroups of G.
NS = normalizer of subgroup S of G in G.
CS = centralizer of subgroup S of G in G.
$\mathfrak{r}$-group = group whose order is divisible by primes in the set $\mathfrak{r}$ only.
$\mathfrak{r}$-element = group element whose order is divisible by primes in $\mathfrak{r}$ only.
$G_{\mathfrak{r}}$ = set of $\mathfrak{r}$-elements in group G.
$P\mathfrak{r}$ = set of primes, not in $\mathfrak{r}$.

All groups considered are *finite*.

1. The reduction theorems

We begin by explaining some of the relevant terms. The group G is $\mathfrak{S}$-*closed* if products of $\mathfrak{S}$-elements in G are $\mathfrak{S}$-elements. This latter property is equivalent to the requirement that the set $G_{\mathfrak{S}}$ of all $\mathfrak{S}$-elements in G is a characteristic $\mathfrak{S}$-subgroup of G. If the set $\mathfrak{S}$ consists of one prime p only, then we speak of p-closure; and this property amounts to requiring the existence of one and only one p-Sylow subgroup.

Next we consider a partial ordering σ of the set $\mathfrak{S}$ of primes. Then $p \sigma p$ is false for every prime p in $\mathfrak{S}$; and $a \sigma b$, $b \sigma c$ implies $a \sigma c$. A σ-*segment* is a subset $\mathfrak{a}$ of $\mathfrak{S}$ with the following property: if p belongs to $\mathfrak{a}$ and $q \sigma p$, then q too belongs to $\mathfrak{a}$.

Definition. *The group G is σ-dispersed if G is $\mathfrak{a}$-closed for every σ-segment $\mathfrak{a}$ of $\mathfrak{S}$.*

If $\mathfrak{S}(G)$ is the totality of prime divisors of $o(G)$ belonging to $\mathfrak{S}$, then σ defines a partial ordering of $\mathfrak{S}(G)$; and G is clearly σ-dispersed for the partial ordering σ of $\mathfrak{S}$ if, and only if, G is σ-dispersed for the partial ordering σ of $\mathfrak{S}(G)$. This shows that all relevant sets of primes will be finite. For a more detailed discussion of closure and dispersion see Baer [1; §4 and §9].

Theorem 1.1. *The group G is σ-dispersed if, and only if, every subgroup S of G is p-closed for every σ-minimal prime p in $\mathfrak{S}(S)$.*

Proof. The necessity of our condition is an immediate consequence of the fact that subgroups of σ-dispersed groups are σ-dispersed and that σ-minimal prime divisors of $o(S)$ form σ-segments of $\mathfrak{S}(S)$.

If conversely the condition of our theorem is satisfied by G, then we are going to show that every subgroup S of G is $\mathfrak{a}$-closed for every σ-segment $\mathfrak{a}$ of $\mathfrak{S}(S)$. This we are going to do by complete induction with respect to the number of primes in $\mathfrak{a}$. It is clear that S is $\mathfrak{a}$-closed whenever $\mathfrak{a}$ is the empty set; and thus we may assume that $\mathfrak{a}$ is not vacuous and that a subgroup T is $\mathfrak{b}$-closed whenever the σ-segment $\mathfrak{b}$ of $\mathfrak{S}(T)$ contains fewer primes than $\mathfrak{a}$. Since $\mathfrak{a}$ is not vacuous, there exists a σ-minimal prime p in $\mathfrak{a}$; and since $\mathfrak{a}$ is a σ-segment of $\mathfrak{S}(S)$, p is a σ-minimal prime divisor of $o(S)$. Application of our condition shows that S is p-closed. Hence there exists a characteristic p-subgroup $P = S_p$ of S whose index $[S:P]$ is prime to p. From Schur's Theorem we deduce the existence of a complement T of P in S; see Zassenhaus [1; p. 125, Satz 25]. Noting that $S = PT$, $1 = P \cap T$ and $T \simeq S/P$ we see that $\mathfrak{s}(T)$ arises from $\mathfrak{s}(S)$ by omission of p. By omitting p from $\mathfrak{a}$, a σ-segment $\mathfrak{b}$ of $\mathfrak{s}(T)$ is obtained which contains fewer primes than $\mathfrak{a}$. Application of the inductive hypothesis shows that T is $\mathfrak{b}$-closed. The totality B of $\mathfrak{b}$-elements in T is consequently a characteristic $\mathfrak{b}$-subgroup of T. Now one verifies without difficulty that PB is the totality of $\mathfrak{a}$-elements in S. Hence S is $\mathfrak{a}$-closed; and this completes the inductive argument and the proof of our theorem.

THEOREM 1.2. *Assume that every prime divisor of $o(G)$ belongs to $\mathfrak{s}$. Then G is σ-dispersed if, and only if, every subgroup S of G is Pp-closed for every σ-maximal prime divisor p of $o(S)$.*

Remark 1.1. The hypothesis that every prime divisor of $o(G)$ be in $\mathfrak{s}$ is clearly indispensable for the validity of our theorem. Assume, for instance, that $\mathfrak{s}(G)$ consists of one and only one prime p. Then p is certainly a σ-maximal prime divisor of $o(G)$. Furthermore σ-dispersion of G is clearly equivalent to p-closure of G. But p-closure and Pp-closure of G are independent properties. Thus without our general hypothesis our condition is neither necessary nor sufficient for σ-dispersion.

Remark 1.2. It is fairly easy to derive our present result from a former result; see Baer [1; p. 165–166, §9, Theorem 1, (xiii) to (xvi)]. We prefer to give a direct derivation which is quite analogous to the proof of Theorem 1.1.

Proof. The necessity of our condition is an immediate consequence of the facts that subgroups of σ-dispersed groups are σ-dispersed and that for every σ-maximal prime divisor p of $o(S)$ the totality of primes, not p, dividing $o(S)$ is a σ-segment of $\mathfrak{s}(S)$.

If conversely our condition is satisfied by G, then we are going to show by complete induction with respect to the number of different prime divisors of $o(S)$ that the subgroup S of G is σ-dispersed. This is clearly true for $S = 1$; and thus we may assume that $S \neq 1$ and that every subgroup T of G is σ-dispersed whose order is divisible by fewer primes than $o(S)$. Consider a σ-segment $\mathfrak{a}$ of $o(S)$. If $\mathfrak{a}$ is the set of all prime divisors of $o(S)$, then S is certainly $\mathfrak{a}$-closed. Thus we may assume that $\mathfrak{a}$ does not contain every prime divisor of $o(S)$. Among the prime divisors of $o(S)$ which do not belong to $\mathfrak{a}$ there exists a σ-maximal one, say p; and p is a σ-maximal prime divisor of $o(S)$, since every prime divisor of $o(S)$ is in $\mathfrak{s}$, and since $\mathfrak{a}$ is a σ-segment of the set $\mathfrak{s}(S)$ of all prime divisors of $o(S)$. Our condition shows that S is Pp-closed. Consequently there exists a characteristic Pp-subgroup T of S whose index $[S:T]$ is a power of p. It follows that T contains every $\mathfrak{a}$-element in S and that $o(T)$ is divisible by fewer primes than $o(S)$. Application of the inductive hypothesis shows that T is $\mathfrak{a}$-closed; and this implies the

$\mathfrak{a}$-closure of S. Hence S is σ-dispersed; and this completes the inductive argument and the proof of our theorem.

2. Homogeneity and closure

In order to apply the theorems of §1, characterizations of p-closed and of Pp-closed groups are needed. It is the objective of §3 and §5 to supply such characterizations. For a convenient enunciation of these criteria a concept is needed which may be of independent interest.

DEFINITION 2.1. *The group G is $\mathfrak{r}$-homogeneous, for $\mathfrak{r}$ a set of primes, if elements in G induce $\mathfrak{r}$-automorphisms in $\mathfrak{r}$-subgroups of G.*

More elaborately stated: Whenever g belongs to the normalizer NS of the $\mathfrak{r}$-subgroup S of G, then g induces an $\mathfrak{r}$-automorphism in S. This is equivalent to the assertion that NS/CS is an $\mathfrak{r}$-group whenever S is an $\mathfrak{r}$-subgroup of G, since NS/CS is essentially the same as the group of automorphisms induced in S by elements in NS.

LEMMA 2.1. *$\mathfrak{r}$-closed groups are $P\mathfrak{r}$-homogeneous.*

Proof. If $\mathfrak{r}$ is a set of primes, and if the group G is $\mathfrak{r}$-closed, then the totality R of $\mathfrak{r}$-elements in G is a characteristic $\mathfrak{r}$-subgroup of G, and $[G:R]$ is prime to every prime in $\mathfrak{r}$. Suppose now that S is a $P\mathfrak{r}$-subgroup of G and that the element g belongs to $R \cap NS$. Then every commutator $g^{-1}s^{-1}gs$ with s in S belongs to $R \cap S = 1$, so that $R \cap NS \leqq CS$. Since $[NS:R \cap NS] = [R \cdot NS:R]$, it follows that $[NS:R \cap NS]$ and its divisor $[NS:CS]$ are prime to every prime in $\mathfrak{r}$. Consequently NS/CS is a $P\mathfrak{r}$-group, proving the $P\mathfrak{r}$-homogeneity of G.

LEMMA 2.2. *If G is not $\mathfrak{r}$-homogeneous, though every proper subgroup of G is $\mathfrak{r}$-homogeneous, then there exist a prime p in $\mathfrak{r}$ and a prime q, not in $\mathfrak{r}$, such that G is an extension of a p-group by a cyclic q-group and G is not q-closed.*

Proof. Since G is not $\mathfrak{r}$-homogeneous, there exists an $\mathfrak{r}$-subgroup R of G such that elements in G induce automorphisms in R which are not $\mathfrak{r}$-automorphisms. Among these automorphisms there is necessarily one whose order is a prime q, not in $\mathfrak{r}$. It is easily verified that such an automorphism of order q may be induced by a suitable q-element g in G. It is clear that the subgroup $\{R, g\}$ of G is not $\mathfrak{r}$-homogeneous. Since every proper subgroup of G is $\mathfrak{r}$-homogeneous, we have $G = \{R, g\}$. Since g belongs to the normalizer of R, the $\mathfrak{r}$-subgroup R of G is a normal subgroup of G; and $G/R \simeq \{g\}$ is a cyclic q-group. Since g induces an automorphism of order q in the Pq-group R, the group G is not q-closed.

$R \neq 1$, since an automorphism of order q is induced in R. Thus there exists at least one prime divisor p of $o(R)$. Assume now by way of contradiction that R is not a p-group. Every prime divisor x of $o(R)$ belongs to $\mathfrak{r}$. Consider an x-Sylow subgroup X of R. Since R is a normal subgroup of G, and since any two x-Sylow subgroups of R are conjugate in R, the Frattini argument shows that $G = RY$ where Y is the normalizer of X in G; see, for instance, Baer [1; p. 117, Lemma 1]. Since G/R is a cyclic q-group, and since q is not in $\mathfrak{r}$ and consequently not a divisor of $o(R)$, it follows that Y contains some q-Sylow subgroup Q of G. Since $\{g\}$ is also a q-Sylow subgroup of G, there exists an element t in G such that $t^{-1}Qt = \{g\}$. Thus g belongs to the normalizer of the x-Sylow subgroup $t^{-1}Xt = X_0$ of R. Since R is not primary, $R \neq X_0$ and $X_0\{g\}$ is a proper subgroup of G. Hence $X_0\{g\}$ is $\mathfrak{r}$-homogeneous. Since the x-group X_0 is an $\mathfrak{r}$-group, and since g is not an $\mathfrak{r}$-element, but a q-element, g induces the identity automorphism in X_0, i.e., g commutes with every element in the x-Sylow subgroup X_0 of R. The centralizer of g contains therefore g and, for every prime divisor x of $o(R)$, an x-Sylow subgroup of R. Hence g belongs to the center of G. But g induces an automorphism of order q in R, an impossibility. Hence R is a p-group, completing the proof.

LEMMA 2.3. *Subgroups, direct products, and epimorphic images of $\mathfrak{r}$-homogeneous groups are $\mathfrak{r}$-homogeneous.*

Proof. It is obvious that subgroups of $\mathfrak{r}$-homogeneous groups are $\mathfrak{r}$-homogeneous. Consider next the direct product $G = A \otimes B$ of the $\mathfrak{r}$-homogeneous groups A and B. Suppose that S is an $\mathfrak{r}$-subgroup of G. Then $S(A) = BS \cap A$, and $S(B) = AS \cap B$ are $\mathfrak{r}$-subgroups of A and B respectively; and $S \leqq S(A) \otimes S(B)$. If the element $g = ab$ for a in A and b in B belongs to NS, then a belongs to $NS(A)$ and b belongs to $NS(B)$. The $\mathfrak{r}$-homogeneity of A and B implies that a induces an $\mathfrak{r}$-automorphism in $S(A)$ and that b induces an $\mathfrak{r}$-automorphism in $S(B)$. Consequently g induces an $\mathfrak{r}$-automorphism in $S(A) \otimes S(B)$ and in its subgroup S. Thus G is $\mathfrak{r}$-homogeneous too.

Assume next the $\mathfrak{r}$-homogeneity of the group G, and consider a normal subgroup K such that G/K is not $\mathfrak{r}$-homogeneous. Then there exists among the subgroups of G/K which are not $\mathfrak{r}$-homogeneous a minimal one, say H/K. Then every proper subgroup of H/K is $\mathfrak{r}$-homogeneous. Thus we may apply Lemma 2.2. Consequently there exist a prime p in $\mathfrak{r}$ and a prime q not in $\mathfrak{r}$ such that H/K is an extension of a p-group by a cyclic q-group without being q-closed. Accordingly there exists a characteristic p-subgroup P/K of H/K such that H/P is a cyclic q-group. Denote by S a p-Sylow subgroup of the normal subgroup P of H. If T is the normalizer of S in H, then we deduce $H = PT$ from the well known Frattini argument; see, for instance, Baer [1; p. 117, Lemma 1]. The isomorphy $H/P \simeq T/(T \cap P)$ shows that $T/(T \cap P)$ is a cyclic q-group. Consequently there exists a q-element t in T such that $T = (T \cap P)\{t\}$. Since G is $\mathfrak{r}$-homogeneous, since S is a p-subgroup of G and p is in $\mathfrak{r}$, and since t is a q-element in NS and q is not in $\mathfrak{r}$, the element t belongs to CS. Consequently Kt commutes with every element in $KS/K = P/K$. But $H = PT = P(T \cap P)\{t\} = P\{t\} = KS\{t\}$;

and thus we see that H/K is the direct product of the p-group P/K and the q-group $\{Kt\}$. Hence H/K is in particular q-closed, a contradiction showing the $\mathfrak{r}$-homogeneity of G/K.

Clearly every $\mathfrak{r}$-group is $\mathfrak{r}$-homogeneous. This simple remark will provide a counterexample to many a conjecture. We shall, however, be mainly interested in two special cases of $\mathfrak{r}$-homogeneity, namely p-homogeneity and Pp-homogeneity for p a prime.

LEMMA 2.4. *If K is a normal subgroup of the $P\mathfrak{r}$-homogeneous group G, and if K and G/K are $\mathfrak{r}$-closed, then G is $\mathfrak{r}$-closed.*

Proof. Since K is $\mathfrak{r}$-closed, there exists a characteristic $\mathfrak{r}$-subgroup R of K such that K/R is a $P\mathfrak{r}$-group. A characteristic subgroup of a normal subgroup is normal. Hence R is a normal subgroup of G; and we may form $G^* = G/R$ and $K^* = K/R$. Then K^* is a normal $P\mathfrak{r}$-subgroup of G^*, and $G^*/K^* \simeq G/K$ is $\mathfrak{r}$-closed. Since G is $P\mathfrak{r}$-homogeneous, so is G^* by Lemma 2.3. The $\mathfrak{r}$-closure of G^*/K^* implies the existence of a normal subgroup T^* of G^* which contains K^* such that T^*/K^* is an $\mathfrak{r}$-group and G^*/T^* is a $P\mathfrak{r}$-group. Since K^* is a $P\mathfrak{r}$-group and T^*/K^* is an $\mathfrak{r}$-group, there exists by Schur's Theorem a complement S^* of K^* in T^*, so that $T^* = K^*S^*$, $1 = K^* \cap S^*$, $T^*/K^* \simeq S^*$; see Zassenhaus [1; p. 125, Satz 25]. Since K^* is a $P\mathfrak{r}$-group and G^* is $P\mathfrak{r}$-homogeneous, only $P\mathfrak{r}$-automorphisms are induced in K^* by elements in G^*. Since S^* is an $\mathfrak{r}$-group, it follows that every element in S^* commutes with every element in K^*. Hence $T^* = K^* \otimes S^*$ is the direct product of the $P\mathfrak{r}$-group K^* and the $\mathfrak{r}$-group S^*. This implies in particular that S^* is a characteristic $\mathfrak{r}$-subgroup of the normal subgroup T^*; and so S^* is a normal subgroup of G^*. Since $T^*/S^* \simeq K^*$ and G^*/T^* are both $P\mathfrak{r}$-groups, so is G^*/S^*. Hence G^* is $\mathfrak{r}$-closed; and G is consequently an extension of the $\mathfrak{r}$-group R by the $\mathfrak{r}$-closed group $G^* = G/R$. This implies the $\mathfrak{r}$-closure of G.

The group G shall be termed $\mathfrak{r}$-*separated*, if its composition factors are either $\mathfrak{r}$-groups or $P\mathfrak{r}$-groups. Thus $\mathfrak{r}$-separation and $P\mathfrak{r}$-separation are equivalent properties. This concept has also been named $\mathfrak{r}$-solubility; see Baer [1; p. 145].

THEOREM 2.5. *The group G is $\mathfrak{r}$-closed if, and only if, G is $\mathfrak{r}$-separated and $P\mathfrak{r}$-homogeneous.*

Proof. If G is $\mathfrak{r}$-closed, then G is $P\mathfrak{r}$-homogeneous by Lemma 2.1; and its composition factors are simple $\mathfrak{r}$-closed groups. But simple $\mathfrak{r}$-closed groups are either $\mathfrak{r}$-groups or $P\mathfrak{r}$-groups. If conversely G is $\mathfrak{r}$-separated and $P\mathfrak{r}$-homogeneous, then there exist subgroups $S(i)$ of G such that $S(0) = 1$, $S(i)$ is a normal subgroup of $S(i + 1)$ and $S(i + 1)/S(i)$ is an $\mathfrak{r}$-group or a $P\mathfrak{r}$-group, $S(n) = G$. Since every $S(i + 1)/S(i)$ is in particular $\mathfrak{r}$-closed, the $\mathfrak{r}$-closure of $S(i)$ and the $P\mathfrak{r}$-homogeneity of $S(i + 1)$ imply, by Lemma 2.4, the $\mathfrak{r}$-closure of $S(i+1)$. Hence it follows by complete induction that every $S(i)$, and in particular G, is $\mathfrak{r}$-closed.

Remark 2.6. There exist many examples of simple groups showing the indispensability of the separation requirement in Theorem 2.5. The alternating group of degree 5, for instance, is not 5-closed, but $P5$-homogeneous. On the other hand, soluble groups are $\mathfrak{r}$-separated; and thus for soluble groups $\mathfrak{r}$-closure and $P\mathfrak{r}$-homogeneity are equivalent properties.

3. p-closure

Every p-closed group is Pp-homogeneous [Lemma 2.1], and there exist Pp-homogeneous groups which are not p-closed [Remark 2.6]. Consequently we are interested in groups which are Pp-homogeneous without being p-closed; and we propose in this section to investigate those members of this class of groups which are, in a sense, minimal. More precisely we are going to investigate groups G with the following properties:

($\mathfrak{C}.p$) *G is not p-closed; every proper subgroup and every proper epimorphic image of G is p-closed; G is Pp-homogeneous.*

Throughout this section we shall assume that the group G under investigation has property ($\mathfrak{C}.p$), and we shall refrain from explicit restatement of this hypothesis.

(3.1) *$o(G)$ is a multiple of p, but not a power of p; and G is simple.*

Proof. p-groups and Pp-groups are p-closed which G is not. This proves our first claim. Assume by way of contradiction the existence of a normal subgroup K of G such that $1 < K < G$. Then K is a proper subgroup and G/K a proper epimorphic image of G. Hence K and G/K are both p-closed. Since G is Pp-homogeneous, application of Lemma 2.4 shows the p-closure of G. This is impossible; and hence G is simple.

(3.2) *The subgroup S of G is the normalizer of a (necessarily uniquely determined) p-Sylow subgroup of G if, and only if, S is a maximal subgroup of G and $o(S)$ is a multiple of p.*

Proof. Assume first that $S = NP$ is the normalizer of the p-Sylow subgroup P of G. Since $P \neq 1$ (by (3.1)) and $P \leqq NP = S$, $o(S)$ is a multiple of p; and P is not a normal subgroup of G (by (3.1)) so that $NP \neq G$. Consequently there exists a maximal subgroup T of G which contains $S = NP$. Since $T \neq G$ is p-closed, its p-Sylow subgroup P is a normal subgroup of T. Hence $T \leqq NP \leqq T$, so that $S = T$ is a maximal subgroup of G.

Assume conversely that S is a maximal subgroup of G and that $o(S)$ is a multiple of p. Since $S \neq G$ is p-closed, its p-Sylow subgroup P is a characteristic subgroup of S. Hence $S \leqq NP$. If $S \neq NP$, then we would deduce $NP = G$ from the maximality of S so that P would be a proper normal subgroup of G, contradicting (3.1). Hence $S = NP$. If P were not a p-Sylow subgroup of G, then there would exist a p-Sylow subgroup Q of G such that $P < Q$. Clearly $P = S \cap Q < NP \cap Q$ by the fundamental properties of p-groups; and this would imply $S < NP$, an impossibility. Hence P is a p-Sylow subgroup of G, completing the proof of (3.2).

(3.3) *There exists a pair of different p-Sylow subgroups A, B of G such that $NA \cap NB \neq 1$.*

Proof. Since the p-Sylow subgroups P of G form a complete class of conjugate subgroups of G, the same is true of their normalizers NP. They are maximal subgroups of G by (3.2), but not normal ones by (3.1). Hence $NP = NNP$ for every p-Sylow subgroup P of G. If $NA \cap NB = 1$ for every pair of different p-Sylow subgroups A, B, then we could apply a celebrated Theorem of Frobenius asserting the existence of a normal subgroup W of G complementary to the subgroups NP. This would contradict the simplicity of G (see (3.1)); and consequently there exists a least one pair of p-Sylow subgroups A, B of G such that $NA \cap NB \neq 1$.

(3.4) *If A and B are two different p-Sylow subgroups of G, then $CZA \cap CZB = 1$.*

Proof. Assume first by way of contradiction the existence of a pair of different p-Sylow subgroups with nontrivial intersection. Among these pairs there would exist one A, B with maximal intersection $J = A \cap B \neq 1$. It is clear that $J < A$ and $J < B$, since otherwise $A = B$. By using the well known properties of p-groups it follows that $J < A \cap NJ$ and $J < B \cap NJ$. Since $1 < J < G$, it is not a normal subgroup of G (by (3.1)). Hence $NJ \neq G$, implying the existence of a maximal subgroup S of G which contains NJ. From $J \leqq S$ we deduce that $o(S)$ is a multiple of p. Application of (3.2) shows the existence of a uniquely determined p-Sylow subgroup R of G such that $S = NR$. Clearly $\{A \cap NJ, B \cap NJ\} \leqq R$. Hence $J < A \cap NJ \leqq A \cap R$; and we deduce $A = R$ from the maximality of J. Likewise we see that $R = B$, contradicting $A \neq B$. This contradiction shows that $A \cap B = 1$ for any two different p-Sylow subgroups A, B of G.

Consider again a pair of two different p-Sylow subgroups A, B of G, and let $J = CZA \cap CZB$. Then ZA and ZB are both contained in CJ and a fortiori in NJ. If NJ were different from G, then NJ would be p-closed, so that $\{ZA, ZB\}$ would be a p-subgroup of NJ and consequently part of a p-Sylow subgroup R of G. Hence $1 < ZA \leqq A \cap R$, proving $A = R$, since we have shown already that different p-Sylow subgroups of G have trivial intersection. Likewise we see that $R = B$, so that $A = B$, a contradiction. Hence $NJ = G$; and this implies $J = 1$, since G is, by (3.1), simple.

(3.5) $P \leqq (NP)'$ *for every p-Sylow subgroup P of G.*

Proof. Different p-Sylow subgroups of G have by (3.4) trivial intersection. This implies (much more than) p-normality of G. Application of Grün's Second Theorem shows that the p-component of G/G' is isomorphic to the p-component of J/J' where $J = NZP$ for some p-Sylow subgroup P of G; see Zassenhaus [1; p. 135, Satz 6]. The simplicity of G implies therefore that $[J:J']$ is prime to p. Since ZP is a characteristic subgroup of P, we have $NP \leqq NZP$. Since NP is a maximal subgroup of G (by (3.2)), and since $ZP \neq 1$ is not a normal subgroup of G (by (3.1)), we find that $J = NP$. Since $[J:J']$ is prime to p, and since P is the p-Sylow subgroup of $J = NP$, it follows that $P \leqq J' = (NP)'$.

(3.6) *If U is a proper subgroup of G, and if A, B are p-Sylow subgroups of G such that $A \cap NU \neq 1 \neq B \cap NU$, then $A = B$.*

Proof. Since U is a proper subgroup of G, we deduce $NU \neq G$ from (3.1); and this implies p-closure of NU. Consequently there exists a p-Sylow subgroup P of G containing the p-subgroup $\{A \cap NU, B \cap NU\}$ of the p-closed group NU. It follows that $1 < A \cap NU \leqq P$. Hence $A \cap P \neq 1$; and this implies $A = P$ by (3.4). Likewise we see that $P = B$. Hence $A = B$.

(3.7) *The Pp-subgroup U of G is part of NP for P a p-Sylow subgroup of G if, and only if, $UP = PU$ and U is not a maximal subgroup of G.*

Proof. If the Pp-subgroup U of G is part of NP for P a p-Sylow subgroup of G, then we recall that NP is a maximal subgroup of G by (3.2). But $U < NP$, since $P \cap U = 1$; and so U is not a maximal subgroup of G. That $U < NP$ implies $UP = PU$, is obvious.

Assume conversely that the Pp-subgroup U of G is not a maximal subgroup of G and that $UP = PU$ for P a p-Sylow subgroup of G. Then UP is a subgroup of G; and we have $o(UP) = o(U)o(P)$, since $U \cap P = 1$. It is clear that $U < G$; and since U is not maximal, there exists a maximal subgroup V of G such that $U < V$. If V is not a Pp-group, then $V = NQ$ for some p-Sylow subgroup Q of G (by (3.2)) so that $o(U)o(Q) = o(U)o(P) = o(UP)$ is a divisor of $o(V)$. If V is a Pp-group, then $V \cap P = 1$ so that $o(V)o(P)$ is a divisor of $o(G)$; and thus we see in either case that $o(UP)$ is a proper divisor of $o(G)$. Hence $UP < G$; and as a proper subgroup of G the subgroup UP is p-closed. But P is a p-Sylow subgroup of (G and) UP. Hence $U \leqq UP \leqq NP$, as we wanted to show.

(3.8) *If P is a p-Sylow subgroup of G, then there exists a common prime divisor of the sequence $[Z_i P:Z_{i-1} P] - 1$ for $0 < i$.*

Proof. It is a consequence of (3.3) that there exists a p-Sylow subgroup $Q \neq P$ of G satisfying $NP \cap NQ \neq 1$. If q is a prime divisor of $o(NP \cap NQ)$, then $q \neq p$ by (3.4); and there exists an element w of order q in $NP \cap NQ$. If i were a positive integer such that q is not a divisor of $[Z_i P:Z_{i-1} P] - 1$, then we deduce from $p \neq q$ and the fact that $o(Z_{i-1} P)$ is a power of p that q is not a divisor of

$$o(Z_i P) - o(Z_{i-1} P) = o(Z_{i-1} P)([Z_i P : Z_{i-1} P] - 1)$$

either. This number is just the number of elements in the set $Z_i P - Z_{i-1} P$ of elements in $Z_i P$ which do not belong to $Z_{i-1} P$. Since w belongs to NP, w belongs likewise to the normalizer of every $Z_j P$. The inner automorphism induced by w effects therefore a permutation of the elements in $Z_i P - Z_{i-1} P$. Since w is of order q, we find for every element x in G that the set $x, x^w, \cdots, x^{w^{q-1}}$ which is invariant under w is either a one-element set or a set consisting of exactly q elements. Since q is prime to $o(Z_i P) - o(Z_{i-1} P)$, the permutation effected by w cannot divide the set $Z_i P - Z_{i-1} P$ into cycles of q elements each. Hence there exists an element a in $Z_i P - Z_{i-1} P$ such that $a = a^w$; and this is equivalent to saying that $aw = wa$. It follows in particular that $Cw \cap P \neq 1$. Since P and Q are isomorphic groups, we have $o(Z_j P) = o(Z_j Q)$; and it follows from the preceding discussion that $Cw \cap Q \neq 1$. Consequently $P \cap N\{w\} \neq 1 \neq Q \cap N\{w\}$; and an immediate application of (3.6) gives $P = Q$, contradicting our choice of Q. Thus we have shown that every prime divisor q of $o(NP \cap NQ)$ is a common divisor of the sequence $[Z_i P : Z_{i-1} P] - 1$ for $0 < i$.

4. Complete p-normality

We recall that the group G is termed *p-normal* if $ZP = ZQ$ for every pair of p-Sylow subgroups P, Q of G such that $ZP \leqq Q$. We need in the sequel a considerably stronger concept which may be characterized by a number of equivalent properties.

LEMMA 4.1. *The following properties of the group G (and the prime p) are equivalent:*

(i) *If S is a subgroup of G whose index $[G:S]$ is prime to p, and if K is a normal subgroup of S, then S/K is p-normal.*

(ii) *If S is a subgroup of G whose index $[G:S]$ is prime to p, if K is a normal p-subgroup of S, if P and Q are p-Sylow subgroups of S/K, and if t is an element of order p in $Q \cap ZP$, then t belongs to ZQ.*

(iii) *If P and Q are p-Sylow subgroups of G, and if the normal subgroup J of P is contained in Q, then J is a normal subgroup of Q.*

(iv) *$J \cap P^x = J^x \cap P$ for every p-Sylow subgroup P of G, every element x in G, and every normal subgroup J of NP with $J \leqq P$.*

(v) *If TX is, for every p-group X, a characteristic subgroup of X such that $T(X^\sigma) = (TX)^\sigma$ for every isomorphism σ of X, then*

$$TP \cap Q = TQ \cap P \quad \textit{for every pair of p-Sylow subgroups } P, Q \textit{ of } G.$$

(vi) *$Z_i P = Z_i Q$ for every pair of p-Sylow subgroups P, Q such that $Z_i P \leqq Q$ (for every positive i).*

Proof. Assume the validity of (i), and consider a subgroup S of G whose index $[G:S]$ is prime to p, a normal p-subgroup K of S, a pair of p-Sylow subgroups P, Q of S/K, and an element t of order p in $Q \cap ZP$. Denote by C the centralizer of t in S/K. Since t belongs to $Q \cap ZP$, we find that $\{ZQ, P\} \leqq C$; and we note that t belongs to ZC. Clearly ZQ is part of a p-Sylow subgroup R of C. Since $P \leqq C$, the index $[S/K:C]$ is prime to p, and R is a p-Sylow subgroup of S/K. Since S/K is p-normal (by (i)), $ZQ \leqq R$ implies $ZQ = ZR$. Since t belongs to ZC, since $o(t) = p$, and since R is a p-Sylow subgroup of C, t belongs to $ZR = ZQ$. Thus (ii)—and more—is a consequence of (i).

Assume next the validity of (ii), and consider a pair of p-Sylow subgroups P, Q of G and a normal subgroup J of P which is part of Q. Let $S = \{P, Q\}$ and note that $[G:S]$ is prime to p. Since J is a normal subgroup of the p-group P, there exist normal subgroups $J(i)$ of P such that $J(0) = 1$, $J(i) < J(i+1)$, and $[J(i+1):J(i)] = p$, $J(n) = J$. We are going to prove by complete induction with respect to i that every $J(i)$ is normal in Q. This is certainly true for $i = 0$; and thus we may assume that $0 < i$ and that $J(i-1)$ is a normal subgroup of Q. Then $K = J(i-1)$ is a normal p-subgroup of S, P/K and Q/K are p-Sylow subgroups of S/K, and $J(i)/K$ has order p and is a normal subgroup of P/K and also part of Q/K. Normal subgroups of order p of p-groups are contained in the center. Hence $J(i)/K \leqq Z(P/K)$. We may apply condition (ii) to show that $J(i)/K \leqq Z(Q/K)$. In particular therefore $J(i)/K$ is a normal subgroup of Q/K so that $J(i)$ is a normal subgroup of Q. This completes the inductive argument proving that $J = J(n)$ is a normal subgroup of Q and that (iii) is a consequence of (ii).

Next we are going to deduce (i) from (iii). Consider a subgroup S of G whose index $[G:S]$ is prime to p and a normal subgroup K of S. Consider furthermore a pair P, Q of p-Sylow subgroups of S/K such that $ZP \leqq Q$. Let D be the uniquely determined subgroup which contains K and satisfies $D/K = P \cap Q$. If E is a p-Sylow subgroup of D, then $D = KE$, since D/K is a p-group. Denote by P^* and Q^* the uniquely determined subgroups which contain K and satisfy $P = P^*/K$ and $Q = Q^*/K$ respectively. Then $D \leqq P^* \cap Q^*$, and the p-subgroup E of D is contained in a p-Sylow subgroup P^{**} of P^* and a p-Sylow subgroup Q^{**} of Q^*. Since P is a p-Sylow subgroup of S/K, the index $[S:P^*]$ is prime to p. Hence $[G:P^*]$ is likewise prime to p, so that the p-Sylow subgroup P^{**} of P^* is a p-Sylow subgroup of G. Likewise Q^{**} is a p-Sylow subgroup of G. Next denote by F the uniquely determined subgroup which contains K and satisfies $F/K = ZP$. From $ZP \leqq P \cap Q$ we deduce $F \leqq D = KE$, so that $F = K(E \cap F)$ by Dedekind's Law. From $F/K = ZP$ and $P = P^*/K$ we deduce that F is a normal subgroup of P^*. Hence $F \cap P^{**}$ is a normal subgroup of P^{**}. Since $K \cap P^{**}$ and $K \cap E$ are both p-Sylow subgroups of K, we deduce $P^{**} \cap K = K \cap E$

from $E \leqq P^{**}$. Application of Dedekind's Law shows now that

$$F \cap P^{**} = K(E \cap F) \cap P^{**} = (E \cap F)(K \cap P^{**}) \leqq E \leqq Q^{**},$$

since the p-Sylow subgroup E of D/K is part of P^{**} and Q^{**}. We apply condition (iii) to see that $F \cap P^{**}$ is a normal subgroup of Q^{**}. Hence

$$ZP = P \cap ZP = (P^* \cap F)/K = (KP^{**} \cap F)/K = K(F \cap P^{**})/K$$

is a normal subgroup of $KQ^{**}/K = Q^*/K = Q$. Thus ZP is a normal subgroup of $T = \{P, Q\}$; and the centralizer C of ZP in T is a normal subgroup of T which contains P. Thus T/C is a group of order prime to p. Hence $CQ/C = 1$ so that $Q \leqq C$. Consequently $ZP \leqq ZQ$. But all centers of p-Sylow subgroups of S/K have the same order. Hence $ZP = ZQ$. Thus S/K is p-normal; and we have verified that (i) is a consequence of (iii).

If (iv) were not a consequence of the equivalent properties (i) to (iii), then there would exist a group G of minimal order, satisfying (i) to (iii) without satisfying (iv). There would then exist a p-Sylow subgroup P of G, an element x in G, and a normal subgroup J of NP with $J \leqq P$ such that $J \cap P^x \neq J^x \cap P$. It is impossible that at the same time $J \cap P^x \leqq J^x$ and $J^x \cap P \leqq J$, since this would imply

$$J \cap P^x \leqq J \cap J^x \leqq P \cap J^x \leqq J \cap J^x \leqq J \cap P^x,$$

so that

$$J \cap P^x = J \cap J^x = J^x \cap P.$$

Thus we may assume without loss in generality that $P \cap J^x \not\leqq J$. It will be convenient to term P, J, x a *critical triplet*, if P is a p-Sylow subgroup of G, x an element in G, J a normal subgroup of NP with $J \leqq P$, and $P \cap J^x \not\leqq J$. If P, J, x is a critical triplet, then we term P, x a *critical pair*; and we note that we have shown the existence of critical triplets and critical pairs.

Consider a critical triplet P, J, x. Then P and P^x are both p-Sylow subgroups of $T = \{P, P^x\}$. Consequently there exists an element t in T such that $P^x = P^t$. Since xt^{-1} belongs to NP, and since the member J of a critical triplet is a normal subgroup of NP, we have $J = J^{xt^{-1}}$, and therefore $J^x = J^t$. Since T contains a p-Sylow subgroup of G, it meets requirement (i). Since P is a p-Sylow subgroup of T, since J is a normal subgroup of the normalizer of P in T, and since t is an element in T such that $J^t \cap P = J^x \cap P$ is not part of J, we deduce $G = T = \{P, P^x\}$ from the minimality of G. Thus we have shown that

(1) $\qquad G = \{P, P^x\}$ for every critical pair P, x.

Consider a critical triplet P, J, x, and let $D = P \cap J^x$. The normalizer $N = ND$ of D in G naturally contains the centers ZP and ZP^x of P and P^x respectively. Then ZP is part of a p-Sylow subgroup A of N, and ZP^x is part of a p-Sylow subgroup B of N. There exists an element t in N such that $A^t = B$. Furthermore B is contained in a p-Sylow subgroup R of G.

Then

$$Z(P^t) = (ZP)^t \leqq A^t = B \leqq R \quad \text{and} \quad Z(P^x) \leqq B \leqq R.$$

Since G is p-normal by (i),

$$Z(P^t) = ZR = Z(P^x), \qquad Z(P^{xt^{-1}}) = Z(P^x)^{t^{-1}} = Z(P^t)^{t^{-1}} = ZP.$$

Since t belongs to the normalizer of D, so does t^{-1}. Consequently

$$P \cap J^x = D = D^{t^{-1}} = P^{t^{-1}} \cap J^{xt^{-1}} \leqq P \cap J^{xt^{-1}}.$$

Since P, J, x is a critical triplet, $P \cap J^x$ is not part of J; and this implies a fortiori that $P \cap J^{xt^{-1}}$ is not part of J. Hence P, J, xt^{-1} is likewise a critical triplet. Consequently $G = \{P, P^{xt^{-1}}\}$ by (1). Since $ZP = ZP^{xt^{-1}}$, it follows that $ZP = ZP^{xt^{-1}} \leqq ZG$. But the centers of p-Sylow subgroups of G form a complete class of conjugate subgroups of G; and inner automorphisms leave invariant every center element. Thus we have shown that

(2) $\qquad ZP = ZP^x \leqq ZG$ for every critical pair P, x.

Consider again a critical triplet P, J, x. Since $P \cap J^x$ is not part of J, we conclude that J^x, and hence J, is different from 1. Since J is a normal subgroup of the p-Sylow subgroup P, we have

$$1 \neq J \cap ZP \leqq ZG$$

by (2). We let $W = J \cap ZP$, and deduce from (2) that

$$W = W^x = J^x \cap Z(P)^x = J^x \cap Z(P^x) = J^x \cap ZP.$$

Thus W is a normal subgroup, not 1, of G which is part of $J \cap J^x$. Since G satisfies (i), so does its quotient group G/W. Since $W \neq 1$, the order of G/W is smaller than the order of G. Because of the minimality of G, condition (iv) holds in G/W. Clearly P/W is a p-Sylow subgroup of G/W, $N(P/W) = NP/W$, J/W is a normal subgroup of $N(P/W)$ since J is a normal subgroup of NP and $N(P/W) = (NP)/W$. Hence

$$\begin{aligned}(P \cap J^x)/W = (P/W) \cap (J^x/W) &= (P/W) \cap (J/W)^{Wx} \\ &= (J/W) \cap (P/W)^{Wx} \leqq J/W;\end{aligned}$$

and this implies $P \cap J^x \leqq J$. This is impossible, since P, J, x is a critical triplet; and this contradiction shows that (iv) is a consequence of (i) to (iii).

Assume next the validity of (iv), and consider a "characteristic functor" T as described in (v). If P is a p-Sylow subgroup of G, then TP is a characteristic subgroup of P; and this implies that TP is a normal subgroup of NP. Application of (iv) shows that

$$TP \cap P^x = (TP)^x \cap P = T(P^x) \cap P \quad \text{for every element } x \text{ in } G;$$

and this shows the validity of (v), since p-Sylow subgroups are conjugate.

Assume next the validity of (v), and consider a characteristic functor T

(in the sense of (v)) and a pair of p-Sylow subgroups P, Q of G such that $TP \leqq Q$. Application of (v) shows that

$$TP = TP \cap Q = TQ \cap P \leqq TQ.$$

But the isomorphy of P and Q implies the isomorphy of TP and TQ. Hence $TP = TQ$; and we have shown that

(v*) *$TP = TQ$, if T is a characteristic functor (in the sense of (v)) and if P, Q are p-Sylow subgroups of G such that $TP \leqq Q$.*

Since Z_i is a characteristic functor, as used in (v), condition (vi) is a special case of (v*).

Next we note that the conditions (i) to (iii) whose equivalence has already been verified are equivalent to the following condition:

(i*) *If S is a subgroup of G whose index $[G:S]$ is prime to p, and if K is a normal p-subgroup of S, then S/K is p-normal.*

Assume now by way of contradiction that (i*) is not a consequence of (vi). Then there would exist a group G of minimal order which satisfies (vi) without satisfying (i*). Consequently there exist a subgroup S of G whose index $[G:S]$ is prime to p and a normal p-subgroup K of S such that S/K is not p-normal. Since S/K is not p-normal, there exists a pair of p-Sylow subgroups P^*, Q^* of S/K such that $ZQ^* \neq ZP^* \leqq Q^*$. Since K is a normal p-subgroup of S, every p-Sylow subgroup of S contains K, and there exist uniquely determined p-Sylow subgroups P, Q of S such that $P^* = P/K$, $Q^* = Q/K$. Since $[G:S]$ is prime to p, the p-Sylow subgroups of S are p-Sylow subgroups of G. Thus (vi) is satisfied by $\{P, Q\}$, since (vi) is satisfied by G. But (i*) is patently not satisfied by $\{P, Q\}$. Hence $G = \{P, Q\}$ is a consequence of the minimality of G. Next denote by U the uniquely determined subgroup of G which contains K and satisfies $U/K = ZP^*$. From $ZP^* \leqq P^* \cap Q^*$ we deduce $U \leqq P \cap Q$. It is clear that $K \cdot ZP/K \leqq ZP^* = U/K$. Hence $ZP \leqq U \leqq Q$; and application of (vi) shows $ZP = ZQ$. Since $G = \{P, Q\}$, it follows even that

$$W = ZP = ZQ \leqq ZG.$$

Thus W is a normal p-subgroup of G; and $W \neq 1$ is a consequence of $P \neq 1$ which in turn is a consequence of $ZQ^* \neq ZP^*$. Since the centers of p-Sylow subgroups form a complete class of conjugate subgroups of G, we deduce

$$ZP = ZX \quad \text{for every } p\text{-Sylow subgroup } X \text{ of } G$$

from $ZP \leqq ZG$. It follows that G/W likewise meets requirement (vi). But the order of G/W is smaller than the order of G. Hence (i*) is satisfied by G/W; and this implies that (i*) and all its consequences are satisfied by

G/W. Since U/KW is a normal subgroup of the p-Sylow subgroup P/KW of G/KW, and since U/KW is part of the p-Sylow subgroup Q/KW of G/KW, application of the consequence (iii) of (i*) to G/KW shows that U/KW is a normal subgroup of Q/KW. Hence U is a normal subgroup of P and Q. Since $G = \{P, Q\}$, we see that U is a normal subgroup of G. Consequently $U/K = ZP^*$ is a normal subgroup of G/K. Since the inner automorphism transforming P into Q also transforms P^* into Q^* and ZP^* into ZQ^*, and since it leaves invariant the normal subgroup ZP^* of G/K, we find that $ZP^* = ZQ^*$, a contradiction which shows that (i*) is a consequence of (vi), and that therefore the conditions (i) to (vi) are equivalent.

DEFINITION 4.1. *The group G is completely p-normal, if it meets the equivalent requirements* (i) *to* (vi) *of Lemma* 4.1.

By using the defining property (iii) of complete p-normality it is readily seen that the following result is just a restatement of

BURNSIDE'S THEOREM. *p-homogeneous groups are completely p-normal.*

See Burnside [1; p. 156] or Zassenhaus [1; p. 103, Satz 8].

It is quite easy to see that a group is completely p-normal, if it is p-closed or Pp-closed or if its p-Sylow subgroups are abelian or hamiltonian.

Remark. If the p-Sylow subgroup P of G is normal and abelian, then G is certainly completely p-normal, and every subgroup J of P is a normal subgroup of P. But in general subgroups of P are not going to be normal subgroups of $G = NP$. Thus it is impossible to prove (iv) in the stricter form where J is required only to be a normal subgroup of P.

5. Pp-closure

We begin with a short discussion of the commutator subgroups of p-Sylow subgroups.

LEMMA 5.1. *Assume that P is a p-Sylow subgroup of the group G.*

(a) *$P' = P \cap G'$ if, and only if, $P' = P \cap (NP)'$ and $P \cap Q' = Q \cap P'$ for every p-Sylow subgroup Q of G.*

(b) If *$P' = P \cap G'$, then p-automorphisms are induced in P by elements in NP.*

Proof. Assume first the validity of $P' = P \cap G'$. If the element s in NP induces in P an automorphism σ of order prime to p, then $x^{1-\sigma} = xs^{-1}x^{-1}s$ belongs to $P \cap G' = P'$ for every x in P. Hence σ induces the identity automorphism in P/P'; and this implies that σ induces the identity automorphism in $P/\Phi P$, since $P' \leqq \Phi P$ as P is a p-group. Application of a result due to Ph. Hall [1, p. 38] proves $\sigma = 1$, showing (b). Since Sylow subgroups are conjugate, we deduce $Q' = Q \cap G'$ for every p-Sylow subgroup Q of G from

$P' = P \cap G'$. It follows that

$$P \cap Q' = P \cap G' \cap Q = P' \cap Q$$

and

$$P \cap (NP)' = P \cap G' \cap (NP)' = P' \cap (NP)' = P'.$$

Thus the conditions stated in (a) are necessary.

Assume conversely the validity of the conditions of (a). Denote by t the transfer of G into P/P', and let K be the kernel of this homomorphism t. As a consequence of our conditions we find that

$$P' = \{P \cap (NP)', P \cap Q'\},$$

where Q ranges over all the p-Sylow subgroups of G. This implies $P/P' \simeq G/K$ by Grün's First Theorem; see Zassenhaus [1; p. 134, Satz 5]. Hence G/K is an abelian p-group so that $G' \leqq K$ and $G = KP$. Consequently

$$P' \leqq P \cap G' \leqq P \cap K,$$

$$P/P' \simeq G/K = KP/K \simeq P/(P \cap K) \simeq [P/P']/[(P \cap K)/P'].$$

Hence $P' = P \cap K = P \cap G'$, as we wanted to show.

THEOREM 5.1. *The following properties of the group G (and of the prime p) are equivalent*:

(i) *G is Pp-closed.*

(ii) *G is p-homogeneous.*

(iii) *G is completely p-normal; and if P is a p-Sylow subgroup of G, $0 \leqq i$, then only p-automorphisms are induced in $P^{(i)}$ by elements in G.*

(iv) *If P, Q are p-Sylow subgroups of G, $0 \leqq i$, then $P \cap Q^{(i)} = Q \cap P^{(i)}$, and only p-automorphisms are induced in $P^{(i)}$ by elements in G.*

(v) *$P^{(i)} = G^{(i)} \cap P$ for every positive i and every p-Sylow subgroup P of G.*

(vi) *$P' = S' \cap P$ for every subgroup S of G and every p-Sylow subgroup P of S.*

Proof. It is a consequence of Lemma 2.1 that (i) implies (ii). That every p-homogeneous group is completely p-normal is the content of Burnside's Theorem (as stated in §4). The second condition (iii) is an immediate special case of p-homogeneity. Hence (iii) is a consequence of (ii). Formation of the i^{th} derivative is a characteristic functor in the sense of Lemma 4.1 (v). Complete p-normality of G implies therefore $P \cap Q^{(i)} = Q \cap P^{(i)}$ for every pair of p-Sylow subgroups P, Q. Hence (iv) is a consequence of (iii).

Assume next the validity of (iv). We are going to prove by complete induction with respect to j the validity of

(v.j) $P^{(j)} = G^{(j)} \cap P$ for every p-Sylow subgroup P of G.

It is clear that (v.0) is true; and thus we may assume that $0 < j$, and that the validity of (v.$j-1$) is already verified. Let $H = G^{(j-1)}$. If A is a p-Sylow subgroup of H, then A is part of a p-Sylow subgroup B of G. Application of (v.$j-1$) shows that

$$B^{(j-1)} = G^{(j-1)} \cap B = H \cap B = A.$$

It follows that every p-Sylow subgroup of H has the form $P^{(j-1)}$ for P a suitable p-Sylow subgroup of G, and that every $P^{(j-1)}$ is a p-Sylow subgroup of H.

Consider now a pair of p-Sylow subgroups A, B of H. Then there exist p-Sylow subgroups P, Q of G such that $A = P^{(j-1)}$ and $B = Q^{(j-1)}$. Application of (iv) shows that

$$B \cap A' = Q \cap H \cap P^{(j)} = P \cap H \cap Q^{(j)} = A \cap B'.$$

If furthermore A is any p-Sylow subgroup of H, then A is the $(j-1)^{\text{st}}$ derivative of a p-Sylow subgroup of G. By (iv), p-automorphisms only are induced in A by elements in H. If E is the normalizer of A in H, then the normal p-subgroup A of E is likewise a p-Sylow subgroup of E. Application of Schur's Theorem shows the existence of a complement D of A in E so that $E = AD$ and $1 = A \cap D$; see Zassenhaus [1; p. 125, Satz 25]. Then $o(D) = [E:A]$ is prime to p, so that elements in D induce the identity automorphism in A. Hence elements in A and in D commute, so that E is the direct product of A and D. Then E' is the direct product of A' and D'; and this implies $A' = A \cap E'$.

Thus we have verified that the conditions of Lemma 5.1 (a), are satisfied by H. Hence $A' = A \cap H'$ for every p-Sylow subgroup A of H. If P is a p-Sylow subgroup of G, then $P^{(j-1)}$ is a p-Sylow subgroup of $H = G^{(j-1)}$. Consequently

$$P^{(j)} = [P^{(j-1)}]' = P^{(j-1)} \cap H' = P \cap H' = P \cap G^{(j)};$$

and this completes the inductive proof of (v.j). Accordingly (v) is a consequence of (iv).

If it were not true that (i) is a consequence of (v), then there would exist a group G of minimal order which satisfies (v) without being Pp-closed. Hence G is in particular not a Pp-group nor a p-group. If P is a p-Sylow subgroup of G, then $P \neq 1$. Hence $P' < P$. But $P' = P \cap G'$ by (v), so that P' is p-Sylow subgroup of G'. It follows in particular that the order of G' is smaller than the order of G. Noting that P' is a p-Sylow subgroup of G' and that $(X')^{(i)} = X^{(i+1)}$, one sees that condition (v) is satisfied by G' too. Because of the minimality of G it follows that G' is Pp-closed. Hence there exists a characteristic Pp-subgroup W of G' such that G'/W is a p-group. As a characteristic subgroup of a characteristic subgroup, W is a characteristic subgroup of G.

Let $H = G/W$. Then $H' = G'/W$ is a p-group. Since subgroups containing the commutator subgroup are normal, and since every p-Sylow subgroup of H contains the characteristic p-subgroup H' of H, the p-Sylow subgroup K of H is normal and hence characteristic (H is p-closed). If P is a p-Sylow subgroup of G, then $K = WP/W$; and now we deduce from $W \leqq G'$, condition (v), and Dedekind's Law that

$$K' = WP'/W = W(P \cap G')/W = (WP \cap G')/W = K \cap H'.$$

Application of Lemma 5.1 (b) shows that elements in H induce p-automorphisms in K.

Since K is the p-Sylow subgroup of H, we deduce from Schur's Theorem the existence of a complement D of K in H so that $H = KD$, $1 = K \cap D$, $D \simeq H/K$; see Zassenhaus [1; p. 125, Satz 25]. Since every element in D has order prime to p and induces a p-automorphism in K, elements in D and in K commute. Hence H is the direct product of K and D. Thus D is a characteristic Pp-subgroup with p-quotient group $H/D \simeq K$. Consequently H is Pp-closed. Hence G is an extension of the Pp-group W by the Pp-closed group $H = G/W$. Such a group is likewise Pp-closed. Thus we have arrived at a contradiction by assuming that (i) is not a consequence of (v); and this completes the proof of the equivalence of conditions (i) to (v).

If G is Pp-closed, then every subgroup S of G is Pp-closed. If P is a p-Sylow subgroup of S, then Pp-closure of S and (v) imply $P' = S' \cap P$, showing that (vi) is a consequence of the equivalent conditions (i) to (v).

Assume now that (vi) is satisfied by G, that P is a p-Sylow subgroup of G, and that $P^{(i)} = P \cap G^{(i)}$ is verified for some i. Since $P^{(i)}$ is a p-Sylow subgroup of $G^{(i)}$, application of (vi) shows that

$$P^{(i+1)} = [P^{(i)}]' = P^{(i)} \cap [G^{(i)}]' = P \cap G^{(i)} \cap G^{(i+1)} = P \cap G^{(i+1)}.$$

Hence (v) follows from (vi) by complete induction; and this completes the proof of our theorem.

Remark 5.1. The equivalence of conditions (i) and (ii) is a theorem due to Frobenius [1, p. 1324, I]. For another proof of this equivalence, compare a forthcoming book by Marshall Hall, Jr.

Remark 5.2. Assume that the p-Sylow subgroups of G are abelian. Then the first half of condition (iv) is trivially satisfied; and its second half holds if, and only if, normalizers and centralizers of p-Sylow subgroups coincide. This shows that the equivalence of conditions (i) and (iv) contains as a special case a well known Theorem of Burnside; see for instance, Zassenhaus [1; p. 133, Satz 4]. One notes that Burnside's Theorem is likewise a special case of the equivalence of conditions (i) and (iii). It is a consequence of these remarks that the second half of conditions (iii) and (iv) cannot be omitted without invalidating the theorem.

Remark 5.3. If the group G is Pp-closed, so are all its subgroups and their quotient groups. Accordingly all the conditions of Theorem 5.1 are satisfied by these too. For instance, if G is Pp-closed and Q is a quotient group of a subgroup of G, then Q is p-normal. This is considerably stronger than the requirement of complete p-normality appearing in (iii).

COROLLARY 5.2. *The group G is the direct product of a p-group and a Pp-group if, and only if, G is both p-homogeneous and Pp-homogeneous.*

Proof. G is clearly a direct product of a p-group and a Pp-group if, and only if, G is both p-closed and Pp-closed. Thus the necessity of our conditions is an immediate consequence of Lemma 2.1.

If conversely G is both p-homogeneous and Pp-homogeneous, then G is Pp-closed by Theorem 5.1. Accordingly there exists a characteristic Pp-subgroup F of G whose index $[G:F]$ is a power of p. If P is a p-Sylow subgroup of G, then $G = FP$ and $1 = F \cap P$. Because of Pp-homogeneity, elements in P commute with elements in the characteristic Pp-subgroup F of G, so that $G = F \otimes P$.

As another application of Theorem 5.1 we offer the following characterization of $\mathfrak{r}$-closure. It will contain the equivalence of Pp-closure and p-homogeneity as a special case.

THEOREM 5.3. *The group G is $\mathfrak{r}$-closed if, and only if, G is $P\mathfrak{r}$-homogeneous and $\{R, P\}$ is an $\mathfrak{r}$-p-group whenever R is a (maximal) $\mathfrak{r}$-subgroup of G and P a p-Sylow subgroup of G for p a prime, not in $\mathfrak{r}$.*

Proof. If G is $\mathfrak{r}$-closed, then G is $P\mathfrak{r}$-homogeneous by Lemma 2.1; and there exists one and only one maximal $\mathfrak{r}$-subgroup R of G which naturally is a characteristic subgroup of G. This shows the necessity of our conditions.

Assume conversely the validity of our conditions. Suppose that R is some maximal $\mathfrak{r}$-subgroup of G. Consider a prime divisor p of $o(G)$ which does not belong to $\mathfrak{r}$. If P is any p-Sylow subgroup of G, then $P \neq 1$, and $Q = \{R, P\}$ is by hypothesis an $\mathfrak{r}$-p-subgroup of G. Thus p is the one and only one prime divisor of $o(Q)$ which does not belong to $\mathfrak{r}$; and the $P\mathfrak{r}$-homogeneity of G implies consequently the p-homogeneity of Q. Application of Theorem 5.1 shows the Pp-closure of Q which—as has been noted—is equivalent to the $\mathfrak{r}$-closure of Q. The totality $Q_{\mathfrak{r}}$ of $\mathfrak{r}$-elements in Q is consequently a characteristic $\mathfrak{r}$-subgroup of Q with index $[Q:Q_{\mathfrak{r}}]$ a power of p. In particular $R \leqq Q_{\mathfrak{r}}$; and the maximality of R shows that $R = Q_{\mathfrak{r}}$ is a characteristic subgroup of Q. In particular $P \leqq NR$.

Denote next by H the subgroup of G which is generated by all the $P\mathfrak{r}$-elements in G. It is clear that H is a characteristic subgroup of G and that G/H is an $\mathfrak{r}$-group. From the result verified in the preceding paragraph of the proof, we deduce $H \leqq NR$ for the maximal $\mathfrak{r}$-subgroup R of G, since H is generated by all the p-Sylow subgroups of G with p not in $\mathfrak{r}$. In particular R is a normal subgroup of RH. If U is an $\mathfrak{r}$-subgroup of RH, then RU is an $\mathfrak{r}$-subgroup, since the $\mathfrak{r}$-group R is normalized by the $\mathfrak{r}$-subgroup U of RH. But R is a maximal $\mathfrak{r}$-subgroup of G. Hence $R = RU$ so that $U \leqq R$, proving that R is the totality of $\mathfrak{r}$-elements in RH. Consequently

$K = R \cap H$ is the totality of $\mathfrak{r}$-elements in H. Since R is a subgroup, K is a characteristic subgroup of the characteristic subgroup H of G. Thus we see that K is a characteristic $\mathfrak{r}$-subgroup of G and that H/K is a $P\mathfrak{r}$-group. Since G/H is an $\mathfrak{r}$-group, we have shown that G is $\mathfrak{r}$-separated. Since G is $P\mathfrak{r}$-homogeneous by hypothesis, we deduce the $\mathfrak{r}$-closure of G from Theorem 2.5, Q.E.D.

Remark 5.4. Every $\mathfrak{r}$-closed group G has the following property:

(*) *If R is a maximal $\mathfrak{r}$-subgroup of G, if the prime p is not in $\mathfrak{r}$, and if P is a p-Sylow subgroup of G, then $RP = PR$.*

It is furthermore clear that the last condition of Theorem 5.3 is a consequence of (*) and that we may consequently substitute (*) for the last condition of Theorem 5.3. If on the other hand p is some fixed prime and $\mathfrak{r} = Pp$, then every group satisfies trivially the last condition of Theorem 5.3, though it need not satisfy (*).

6. Dispersion

In the present section we shall obtain dispersion criteria by combination of the results obtained so far. Accordingly throughout this section (as in §1) we shall denote by $\mathfrak{S}$ a set of primes and by σ a partial ordering of $\mathfrak{S}$.

THEOREM 6.1. *The following properties of the group G are equivalent:*

(i) *G is σ-dispersed.*

(ii) *G is $\mathfrak{a}$-separated and $P\mathfrak{a}$-homogeneous for every σ-segment $\mathfrak{a}$ of $\mathfrak{S}$.*

(iii) *Every subgroup S of G is p-separated and Pp-homogeneous for every σ-minimal prime p in $\mathfrak{S}(S)$.*

The equivalence of (i) and (ii) is an immediate consequence of Theorem 2.5; and the equivalence of (i) and (iii) may be deduced from Theorem 1.1 and Theorem 2.5.

THEOREM 6.2. *If every prime divisor of $o(G)$ belongs to $\mathfrak{S}$, then the following properties of the group G are equivalent:*

(i) *G is σ-dispersed.*

(ii) *Every subgroup S of G is p-homogeneous for every σ-maximal prime divisor p of $o(S)$.*

(iii) *If S is a subgroup of G and p a σ-maximal prime divisor of $o(S)$, then S is completely p-normal and $NP^{(i)}/CP^{(i)}$ is, for P a p-Sylow subgroup of G and every i, a p-group.*

This is easily deduced from Theorems 1.2 and 5.1.

Remark. These results make it possible to obtain simplified proofs of some of the theorems of Baer [2] and [3]. In particular, Baer [3, p. 243, Ergänzungssatz] is an immediate consequence of Theorem 6.2.

LEMMA 6.3. *Assume that every prime divisor of $o(G)$ belongs to $\mathfrak{S}$, or else that G is p-separated for every prime p in $\mathfrak{S}$. If G is not σ-dispersed, though* *every proper subgroup of G is σ-dispersed, then there exists a σ-minimal prime p in $\mathfrak{S}(G)$ such that G is not p-closed and such that G is an extension of a q-group with $q \neq p$ by a cyclic p-group.*

Proof. Assume first that every prime divisor of $o(G)$ belongs to $\mathfrak{S}$. Since G is not σ-dispersed, we deduce from Theorem 1.2 the existence of a subgroup H of G which is not Pp-closed, though p is a σ-maximal prime in $\mathfrak{S}(H)$. Application of Theorem 1.2 shows that H is not σ-dispersed. Since every proper subgroup of G is supposed to be σ-dispersed, we find that $G = H$. If J is a proper subgroup of G, then either p is no divisor of $o(J)$, in which case J is certainly Pp-closed; or else p is σ-maximal in $\mathfrak{S}(J)$, in which case Pp-closure of J is a consequence of Theorem 1.2. Noting that Pp-closure is equivalent to p-homogeneity by Theorem 5.1, we have verified the following fact:

There exists a σ-maximal prime divisor p of $o(G)$ such that G is not p-homogeneous, though every proper subgroup of G is p-homogeneous.

We apply Lemma 2.2 and find that G is an extension of a p-group by a cyclic q-group ($p \neq q$) and that G is not q-closed. Since every prime divisor of $o(G)$ belongs to $\mathfrak{S}$, and since q is certainly a prime divisor of $o(G)$, q is in $\mathfrak{S}$. Since $\mathfrak{S}(G)$ consists of p and q only, and since p is σ-maximal, q is, of necessity, σ-minimal; and thus we have shown in the present case that G has all the properties claimed.

Assume next that G is p-separated for every prime p in $\mathfrak{S}$. Since G is not σ-dispersed, we deduce from Theorem 1.1 the existence of a subgroup H of G which is not p-closed, though p is a σ-minimal prime in $\mathfrak{S}(H)$. Application of Theorem 1.1 shows that H is certainly not σ-dispersed. Since every proper subgroup of G is σ-dispersed, we find $G = H$. If J is a proper subgroup of G, then either p is no divisor of $o(J)$, in which case J is certainly p-closed; or else p is σ-minimal in $\mathfrak{S}(J)$. In the latter case we recall that J as a proper subgroup of G is σ-dispersed, and Theorem 1.1 implies p-closure of J. Since by hypothesis G is p-separated, every subgroup of G is p-separated too. It is a consequence of Theorem 2.5 that for p-separated groups p-closure and Pp-homogeneity are equivalent properties. Thus we have verified the following fact:

There exists a σ-minimal prime p in $\mathfrak{S}(G)$ such that G is not Pp-homogeneous, though every proper subgroup of G is Pp-homogeneous.

We apply Lemma 2.2 and find that G is an extension of a q-group with $q \neq p$ by a cyclic p-group and that G is not p-closed; and thus we have shown in the present case too that G has all the properties claimed.

Remark. It is noteworthy that all the groups appearing in Lemma 6.3 are soluble. Considering that a σ-dispersed group G is certainly soluble, if every prime divisor of $o(G)$ belongs to $\mathfrak{S}$, we obtain the generalized Theorem of Iwasawa-Schmidt which asserts that the group G is soluble if all its proper subgroups are σ-dispersed and if every prime divisor of $o(G)$ belongs to $\mathfrak{S}$; see Baer [1; p. 172, Corollary 1].

COROLLARY 6.4. *If every prime divisor of $o(G)$ belongs to $\mathfrak{S}$ or else G is p-separated for every prime p in $\mathfrak{S}$, and if every soluble subgroup of G is σ-dispersed, then G is σ-dispersed.*

Proof. If G were not σ-dispersed, then there would exist a subgroup W of G which is not σ-dispersed, though every proper subgroup of W is σ-dispersed. Since p-separation is inherited by subgroups, W meets all the requirements of Lemma 6.3. Hence W is soluble. But soluble subgroups of G are supposed to be σ-dispersed. This is a contradiction proving the σ-dispersion of G.

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