ALGEBRAIC CLOSURE OF FIELDS AND RINGS OF FUNCTIONS

Dedicated to L. J. Mordell in gratitude and friendship on his seventieth birthday, January 28, 1958

BY

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The class of rings of functions that is going to be the object of our discussion may be described as follows: There are given firstly a [commutative] field F, the field of values of the ring of functions; secondly a set D of elements [called points], the domain of the ring of functions; and thirdly and mainly a ring R of single-valued functions, defined on D with values in F. [Addition and multiplication of functions in R are defined in the natural fashion:

$$(f+g)(x) = f(x) + g(x), \qquad (fg)(x) = f(x)g(x)$$

for x in D and f, g in R.] These rings will always be subject to the following requirements:

R contains all the constants;

if x and y are different points in D, then there exists a function f in R such that $f(x) \neq f(y)$.

All these rings are commutative and contain a ring unit 1, namely the constant 1. The requirement that all constants are present in R is not quite as harmless as it appears. The field of constants which is naturally isomorphic with the field F of values shall be denoted by C. The requirement on the other hand that there exists to any pair of different points in D a function in R which takes different values on these points does not constitute a loss of generality, since we would form otherwise the classes of points in D on which all functions in R take the same value, and since we could consider these classes as the "points".

With such a configuration $[F, D, R]$ we connect two topological spaces.

The space of maximal ideals

We denote by $T = T(R)$ the totality of maximal ideals in R. If p is a point in T and S is a subset of T, then p is said to belong to the closure $\bar{S}$ of S if, and only if,

$$S^* = \bigcap_{s \in S} s \leqq p.$$

It is well known that T with the topology just described is a compact T_1-space [so that in particular every point is a closed set and every covering of T with open sets contains a finite covering of T]; see Jacobson [1] or Samuel [1; pp.

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118–120, Chapter II, 7]. We note furthermore that an element r in R possesses an inverse in R if, and only if, r does not belong to any maximal ideal, since [because of the existence of the ring identity in R] the element r does not belong to any maximal ideal if, and only if, $Rr = R$. Note that T is often referred to as the structure space of R.

The zero set topology of D

A subset S of D shall be termed closed if, and only if, there exists a set Y of functions in R with the following property:

The element d in D belongs to S if, and only if, $f(d) = 0$ for every f in Y.

It is well known and easily seen that with this topology D is turned into a T_1-space. For the convenience of the reader we indicate the principal points of the proof of this fact. If p and y are different points, then there exists a function f in R such that $f(p) \neq f(y)$. If $f(p) = v$, then $g(x) = f(x) - v$ belongs to R too [since R contains the constants]; and we have $g(p) = 0 \neq g(y)$. It follows that points are closed sets. It is clear that the whole space, the empty set, and intersections of closed sets are closed sets. If finally A and B are closed sets, and if the point p does not belong to the join $A \vee B$ of A and B, then there exist functions v and w in R such that $v(A) = w(B) = 0$ whereas neither $v(p)$ nor $w(p)$ vanishes. It is clear then that $(vw)(A \vee B) = 0 \neq (vw)(p)$; and this shows the closure of the join of two closed sets.

Implicit in this proof is the following fact: If the point p does not belong to the closed subset A of D, then there exists a function f in R such that $f(A) = 0 \neq f(p)$; and this fact may be called the "complete R-regularity of D". The reader will verify without difficulty that our zero set topology is completely determined by the following two requirements:

The zero sets of functions in R are closed sets in D, and D is completely R-regular.

The canonical mapping of D into T

If p is a point in D, then we denote by p^σ the totality of functions f in R such that $f(p) = 0$. It is clear that p^σ is an ideal in R and that the field C of constants is a field of representatives of R/p^σ. Hence every p^σ is a maximal ideal in R. If y is a point, not p, in D, then there exists, as we noted before, a function f in R such that $f(p) = 0 \neq f(y)$. Since f belongs to p^σ, but not to y^σ, these maximal ideals are different; and we see that σ is a one to one mapping of D into T.

The point d in D belongs to the closure $\bar{S}$ of the subset S of D if, and only if, $f(d) = 0$ is, for every f in R, a consequence of $f(S) = 0$. This is equivalent to saying that

$$(S^\sigma)^* = \bigcap_{s \in S} s^\sigma \leqq d^\sigma.$$

Hence d belongs to the closure of S if, and only if, d^σ belongs to the closure of S^σ; and this shows that the canonical mapping σ is a topological mapping of D into T.

It is clear that $(D^\sigma)^* = \bigcap_{x \in D} x^\sigma = 0$; and this implies that D^σ is everywhere dense in T.

The necessary conditions for $D^\sigma = T$

Since T is compact, and since D and D^σ are topologically equivalent, $T = D^\sigma$ implies the compactness of D. Since elements in R, not belonging to any maximal ideal, possess inverses in R, and since elements in R, not belonging to any ideal in D^σ are just the functions which do not vanish anywhere, we see that $T = D^\sigma$ implies the existence in R of inverses to any function in R which does not vanish anywhere.

It will be convenient to say that R is a *full* ring of functions, if D is compact and if every nowhere vanishing function f in R possesses an inverse function $1/f$ in R. Thus we have seen that $T = D^\sigma$ implies the fullness of R. We note the following partial converse: If T happens to be a Hausdorff space, then the compact subspaces of T are closed in T; see, for instance, Alexandroff-Urysohn [1; p. 263, Satz V]. Dense compact subspaces of T would then be identical with T. Hence *$T = D^\sigma$, if T is a Hausdorff space and R is full.*

We are now ready to state and prove our principal result.

THEOREM. *$T = D^\sigma$ for every full ring of functions over F if, and only if, the field F is not algebraically closed.*

Note that the condition $T = D^\sigma$ signified the existence of a common zero for all the functions in any given maximal ideal of R. Thus the presence of a common zero for all the functions in any given maximal ideal of every given full ring of functions over F is equivalent to the existence of a zerofree polynomial [of positive degree] over R.

We precede the proof of our theorem by a proof of the following

LEMMA. *If R is a ring of functions over F, if the field F is not algebraically closed, if the finitely many functions $f_1, \cdots, f_k$ in R do not possess any common zero, then the ideal $\sum_{i=1}^{k} Rf_i$ contains a function which does not vanish anywhere in D.*

Proof. By hypothesis there exists a zerofree polynomial $\sum_{i=0}^{n} a_i x^i$ over F. We may assume without loss in generality that $a_0 \neq 0 \neq a_n$. We define

$$p_2(x_1, x_2) = \sum_{i=0}^{n} a_i x_1^i x_2^{n-i};$$

$$p_{j+1}(x_1, \cdots, x_{j+1}) = p_2[p_j(x_1, \cdots, x_j), x_{j+1}];$$

and we note that the polynomials p_k are well defined for every $k > 1$, that they are homogeneous, and that none of them possesses an absolute term. Since

$\sum_{i=0}^{n} a_i x^i$ is zerofree over F, $p_2(x_1, x_2) = 0$ if, and only if, $x_1 = x_2 = 0$ [provided, of course, that x_1 and x_2 are in F]; and now it follows by complete induction that

$$p_j(x_1, \cdots, x_j) = 0 \quad \text{if, and only if,} \quad x_1 = \cdots = x_j = 0.$$

Since the ring R contains the constants, and since the ideal $J = \sum_{i=1}^{k} Rf_i$ contains all the positive powers of the functions f_i, it follows that

$$f(x) = p_k[f_1(x), \cdots, f_k(x)]$$

belongs to J. If d were a point in D such that $f(d) = 0$, then the numbers $f_i(d)$ in F would satisfy $p_k[f_1(d), \cdots, f_k(d)] = 0$. But we noted before that this implies $f_1(d) = \cdots = f_k(d) = 0$. Hence d would be a common zero of the functions $f_1, \cdots, f_k$, contradicting our hypothesis. Thus the function f in J does not vanish anywhere on D.

Proof of the Theorem. We assume first that F is not algebraically closed and that R is a full ring of F-valued functions over the domain D. Assume that the ideal J in R is not contained in any of the ideals d^σ for d in D. Then there exists to every point d in D a function f_d in J satisfying $f_d(d) \neq 0$. Denote by $N(d)$ the set of all points x in D such that $f_d(x) \neq 0$. It is clear that d belongs to $N(d)$ and that $N(d)$ is just the complement of the set of zeros of the function f_d. Since the latter set is closed in the zero set topology, every $N(d)$ is open. These sets $N(d)$ form consequently a covering of D by open sets. Since R is full, the space D is compact. Consequently D may be covered by finitely many of the sets $N(d)$. Hence there exist finitely many points $d_1, \cdots, d_k$ in D such that D is covered by the sets $N(d_1), \cdots, N(d_k)$. The finitely many functions $f_{d_1}, \cdots, f_{d_k}$ in J do not possess any common zero in D. Application of the Lemma [which is applicable, since F is not algebraically closed] proves the existence of a function f in the ideal $\sum_{i=1}^{k} Rf_{d_i} \leqq J$ which does not vanish anywhere on D. Since R is full, the inverse function $1/f$ belongs to R. Hence $(1/f)f = 1$ belongs to J so that $J = R$.

Consider now some maximal ideal M in R. Since $M \neq R$, there exists at least one point d in D such that $M \leqq d^\sigma$. Since M is maximal and $d^\sigma \neq R$, we have $M = d^\sigma$. Hence $T = D^\sigma$.

Assume conversely that the field F is algebraically closed. Denote by E the set of all pairs $x = (x_1, x_2)$ of elements x_i in F, and by P the ring of all polynomials $f(x_1, x_2)$ of two variables x_1 and x_2 with coefficients in F. Every polynomial f in P defines a function on E. The function 0 is induced by the polynomial 0 only, since F is, as an algebraically closed field, infinite. Hence we may identify every polynomial in P with the induced function on E; and consequently P will be considered as a ring of functions on E. It is clear that this ring of functions on E contains the constants and that to every pair of different points in E there exists a function in P which vanishes on one of these two points, but not on the other one.

If S is a subset of E, then we denote by S^P the totality of functions f in P such that $f(S) = 0$. Clearly S^P is an ideal in P. The set S is closed [in the zero set topology] if, and only if, S is exactly the set of all the common zeros of all the functions in S^P. It follows that for closed subsets A and B of E the statements $A \leqq B$ and $B^P \leqq A^P$ are equivalent. Noting the well known fact that the maximum condition is satisfied by the ideals in P, we conclude that every descending chain of closed subsets of E terminates after a finite number of steps. This is, of course, a property considerably sharper than compactness.

Assume now that the polynomial f in P is not a constant. Then $f(x_1, x_2) = \sum_{i=0}^{n} f_i(x_1)x_2^i$, where each of the $f_i(x_1)$ is a polynomial in x_1 with coefficients in F, and where in particular the polynomial f_n is not the zero polynomial. Since f is not a constant, $0 < n$. Consequently there exists only a finite number of elements v in F such that $f_n(v) = 0$. Since the algebraically closed field F is infinite, there exists an infinity of numbers w in F such that $f_n(w) \neq 0$; and for each of these infinitely many numbers w the polynomial $f(w, x)$ has degree n. Since $0 < n$ and F is algebraically closed, the equation $f(w, x) = 0$ has at least one solution x in F; and this shows that f possesses an infinity of zeros in E.

Assume again that the polynomial f in P is not a constant. If c is any number in F, then $f - c$ is likewise a polynomial in P which is not a constant. Hence $f - c$ possesses an infinity of zeros in E so that the equation $f(x_1, x_2) = c$ possesses an infinity of solutions in E.

Denote by D the subset of E arising by the removal of one point, say $(0, 0)$—we could equally well remove from E any finite number of points. Denote by R the ring of F-valued functions on D which are induced by functions in P.

If the function f in P vanishes everywhere on D, then there exists at least one number c in F—since F contains an infinity of numbers—such that the equation $f(x_1, x_2) = c$ has no solutions in E. It follows that f is a constant; and this implies that f is the constant 0. Every function in R is consequently induced by one and only one function in P so that R and P are essentially the same.

Assume next that the function f in P does not vanish anywhere in D. Then f has only a finite number of zeros in E. Consequently f is a constant which is of necessity different from 0. This implies in particular that f possesses an inverse in P. We conclude that a function in R possesses an inverse $1/f$ in R if, and only if, f does not vanish anywhere in D [and is a constant, not 0].

The zero set topology defined by R in D is clearly the same topology as is induced by the space E in its subspace D. Since descending chains of closed subsets of E terminate after a finite number of steps, the same holds true for D; and this implies in particular that D is compact. Thus we have verified that R is a full ring of F-valued functions on D.

It is clear that the totality of functions f in P satisfying $f(0, 0) = 0$ is a maximal ideal in P; and the functions in this maximal ideal induce a maximal ideal M in R. Since to every point (u, v) in D there exists a function in P vanishing in $(0, 0)$, but not in (u, v), M is different from all the ideals $(u, v)^{\sigma}$ with (u, v) in D. Hence M does not belong to $D^{\sigma} \neq T$; and this concludes the proof.

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