TWISTED RANKS AND EULER CHARACTERISTICS

BY ALEX HELLER[1]

In a previous paper [2] the author introduced the notion of the twisted Euler characteristic of a complex on which a group of prime order operates. The twisted Euler characteristic of the complex is equal to that of the fixed-point set; this generalizes part of a classical result of P. A. Smith.

In view of the theorem of Artin and Tate on periodicity in the homology of a finite group [1, XII, 11] the twisted Euler characteristic may be generalized to other groups of operators, and the theorem quoted above remains true if the most generous notion of fixed-point set is adopted. This generalization is made here.

The standpoint is that of the theory of abstract abelian categories (the "exact" categories of Buchsbaum [1, appendix]). Although no application, other than the one just mentioned, is considered here, it is clear that similar constructions may be made, for example, in categories of sheaves.

1. Ranks and Euler characteristics

If $\mathcal{K}$ is an abelian category, a *rank* on $\mathcal{K}$ is a function ρ on the objects of $\mathcal{K}$ with values in an additive group, such that if $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is exact, then $\rho A = \rho A' + \rho A''$. In particular, then, $\rho 0 = 0$.

For example, on the category of finite dimensional vector spaces over a field, the dimension is a rank. On the category of finite abelian groups $\mathfrak{o}G$, the logarithm of the order of G is a rank.

LEMMA 1. *If ρ is a rank on $\mathcal{K}$ and the diagram*

$$\begin{tikzcd} A_{2n-1} \arrow[r] & B_{2n-1} \arrow[r] & C_{2n-1} \arrow[r] & \cdots \arrow[r] & B_0 \arrow[r] & C_0 \arrow[lllll, bend left] \end{tikzcd}$$

is exact in $\mathcal{K}$, then

$$\sum_{i=0}^{2n-1} (-1)^i \rho B_i = \sum_{i=0}^{2n-1} (-1)^i \rho A_i + \sum_{i=0}^{2n-1} (-1)^i \rho C_i .$$

For if $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ are the kernels in A_i, B_i, C_i, then, writing indices modulo $2n$:

$$\rho A_j = \rho \bar{A}_j + \rho \bar{B}_j$$

$$\rho B_j = \rho \bar{B}_j + \rho \bar{C}_j$$

$$\rho C_j = \rho \bar{C}_j + \rho \bar{A}_{j-1}$$

from which the lemma follows immediately.

The notation $\mathcal{K}'$ will be used for the category of finitely graded objects

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[1] Fellow of the Alfred P. Sloan Foundation.

of $\mathcal{K}$, that is, the category whose objects are sequences

$$A = \{ \cdots , A_{-1} , A_0 , A_1 , \cdots \}$$

of objects of $\mathcal{K}$ such that $A_i = 0$ except for a finite set of i. If A, B are objects of $\mathcal{K}'$, then $\text{Hom}(A, B)$ is the graded group $\sum_k \text{Hom}_k(A, B)$ where $\text{Hom}_k(A, B) = \sum_i \text{Hom}(A_i , B_{i+k})$ is the subgroup of maps homogeneous of degree k. Clearly $\mathcal{K}$ is again an abelian category.

If ρ is a rank on $\mathcal{K}$, then the function χ_ρ on the objects of $\mathcal{K}'$ is defined by $\chi_\rho A = \sum_i (-1)^i \rho A_i$. Then if $0 \to A' \xrightarrow{\varphi'} A \xrightarrow{\varphi''} A'' \to 0$ is exact in $\mathcal{K}'$ and φ', φ'' are homogeneous of degrees k', k'',

$$\chi_\rho A = (-1)^{k'} \chi_\rho A' + (-1)^{k''} \chi_\rho A''.$$

With respect to the maps of even degree in $\mathcal{K}'$, which form a subcategory, χ_ρ is again a rank.

The category $d\mathcal{K}$ of finite complexes in $\mathcal{K}$ has as objects the pairs (A, d) where A is an object of $\mathcal{K}'$ and $d:A \to A$ is a map of degree -1 such that $d^2 = 0$. Hom $((A, d), (A', d'))$ is the subgroup of Hom_0 (A, A') consisting of maps $f:A \to A'$ such that $d'f = fd$. Again, $d\mathcal{K}$ is an abelian category.

In this situation, of course, Z, B, $H:d\mathcal{K} \to \mathcal{K}'$: the cycle, boundary, and homology functors are defined. These are related by natural exact sequences

$$0 \to Z(A, d) \to A \to B(A, d) \to 0$$

$$0 \to B(A, d) \to Z(A, d) \to H(A, d) \to 0$$

where all the maps are of degree 0 except $A \to B(A, d)$ which is of degree -1. Thus if ρ is a rank on $\mathcal{K}$, then $\chi_\rho A = \chi_\rho Z(A, d) - \chi_\rho B(A, d)$, and $\chi_\rho Z(A, d) = \chi_\rho H(A, d) + \chi_\rho B(A, d)$.

Proposition 1. $\chi_\rho A = \chi_\rho H(A, d)$.

Corollary 1. *If* $0 \to (A', d') \to (A, d) \to (A'', d'') \to 0$ *is exact in dK, then* $\chi_\rho H(A, d) = \chi_\rho H(A', d') + \chi_\rho H(A'', d'')$.

Of especial interest will be *twisted* ranks on $\mathcal{K}$, that is, ranks ρ such that $\rho X = 0$ whenever X is projective in $\mathcal{K}$. Then also $\chi_\rho X = 0$ if X is a projective of $\mathcal{K}'$, for then each X_i is projective in $\mathcal{K}$.

If (X, d) is an object of $d\mathcal{K}$ and X is projective in $\mathcal{K}'$, then (X, d) is a *projective complex* over $\mathcal{K}$. (It is not, however, necessarily projective in $d\mathcal{K}$.) If ρ is a twisted rank on $\mathcal{K}$, then $\chi_\rho H(X, d) = \chi_\rho X = 0$.

Corollary 2. *If* $0 \to (A', d') \to (A, d) \to (A'', d'') \to 0$ *is exact in* $d\mathcal{K}$, ρ *is a twisted rank on* $\mathcal{K}$, *and* A'' *is projective in* $\mathcal{K}'$, *then* $\chi_\rho H(A', d') = \chi_\rho H(A, d)$.

2. Existence of twisted ranks

If $\mathcal{K}$ and $\mathcal{L}$ are abelian categories and every object of $\mathcal{K}$ is the epimorphic image of a projective, a covariant additive functor $T:\mathcal{K} \to \mathcal{L}$ is periodic of period n if, for all $k > 0$, $S_{k+n}\,T = S_kT$, where S_kT is the k'^{th} left satellite of T [1].

For example, suppose Q is a finite group and G a finite right Q-module, and that Q is an Artin-Tate group with respect to G (an ATG-group), i.e. that for every prime p dividing the order of G, the p-Sylow subgroup of Q is either cyclic or a generalized quaternion group. Then the tensor product $G \otimes_Q A$ is periodic of some even period on the category of left Q-modules.

Proposition 2. *If* $T: \mathcal{K} \to \mathcal{L}$ *is half-exact and periodic of period* $2n$, *and if* ρ *is a rank on* $\mathcal{L}$, *then*

$$\sigma A = \sum_{k}^{k+2n-1} (-1)^i \rho S_i TA$$

defines a twisted rank on $\mathcal{K}$.

In any case it is clear that σ vanishes on projectives. But if $0 \to A' \to A \to A'' \to 0$ is exact in $\mathcal{K}$, then

$$\begin{tikzcd} S_{k+2n-1}TA' \arrow[r] & S_{k+2n-1}TA \arrow[r] & S_{k+2n-1}TA'' \arrow[r] & \cdots \arrow[r] & S_k TA'' \arrow[llll, bend left] \end{tikzcd}$$

is exact. The proposition follows from Lemma 1.

In the example cited above, the functor, $G \otimes_Q$, is half-exact. If it is restricted to the category of finitely generated left Q-modules, the values of the satellites, $\operatorname{Tor}_k^Q(G, A)$, lie in the category of finite abelian groups. Thus if $2n$ is a period,

$$\sigma A = \sum_{k}^{k+2n-1} (-1)^i \rho \operatorname{Tor}_i^Q(G, A)$$

is a twisted rank on the category of finitely generated Q-modules.

All the above observations may, of course, be dualized in several ways.

3. A topological application

Suppose the ATG-group Q operates cellularly on the cell-complex $\mathcal{X}$. Then the points of $\mathcal{X}$ fixed under some nontrivial element of Q form a subcomplex $\mathcal{A}$. The integral chain complex of $\mathcal{X}$ is a finite complex in the finitely generated left Q-modules, and the chain complex of $\mathcal{A}$ a subcomplex. The quotient is free, thus *a fortiori* a projective complex.

Theorem. *If* $2n$ *is a period of* $G \otimes_Q$, *then*

$$\sum_i \sum_{j=k}^{k+2n-1} (-1)^{i+j} \rho \operatorname{Tor}_j^Q(G, H_i(\mathcal{A})) = \sum_i \sum_{j=k}^{k+2n-1} (-1)^{i+j} \rho \operatorname{Tor}_j^Q(G, H_i(\mathcal{X})).$$

Bibliography

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University of Illinois
Urbana, Illinois