CLASSES OF FINITE GROUPS AND THEIR PROPERTIES

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Of the various properties that a class Θ of finite groups may or may not have, those of interest to us in our present investigation can be described roughly as follows:

1. The formal or inheritance properties: Subgroups, homomorphic images, and direct products of groups in Θ may or may not belong to Θ; and somewhat less superficial is the question whether a product of normal Θ-subgroups is itself a Θ-group; see e.g. Specht [1; §1.4.4. etc.].

2. The material properties: These are concerned with the structure of the minimal normal subgroups of homomorphic images of Θ-groups and with the structure of the automorphism groups induced by homomorphic images of Θ-groups in their minimal normal subgroups. They are furthermore concerned with the situation of maximal subgroups in Θ-groups. In particular one wants to derive from such properties criteria for a group to be a Θ-group, criteria that will lead to theorems asserting that a group G is a Θ-group if, and only if, $G/\Phi(G)$ is a Θ-group.

3. Somewhat in between **1** and **2** are questions of the following type: Is a group G a Θ-group if, and only if, every n-tuplet of elements in G, for n a fixed integer, generates a Θ-group? Is furthermore G a Θ-group, if there exists a normal subgroup N of G such that G/N and every $\{N, x_1, \cdots, x_n\}$ for x_i in G and n a fixed integer is a Θ-group?

One might try to undertake such an investigation completely in abstracto, attempting to derive relations between such general properties of a class Θ of finite groups; and some few results of such generality will be found in the present investigation. But we have been concerned here mainly with more concrete questions; and the starting point of our investigation was the observation that such properties as supersolubility, nilpotence, dispersion, existence of Sylow towers, nilpotence of the commutator subgroup are highly complex and may be reduced to more elementary properties in the sense that they are equivalent to certain concatenations of these elementary properties—that different sets of such elementary properties may characterize one and the same complex property, leads to particularly intriguing problems. We have considered here just two types of elementary properties. The first one is

Σ-*closure*: If Σ is a set of primes and if the set of elements in the group G whose orders are divisible by primes in Σ only is actually a subgroup of G, then G is termed Σ-closed.

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The other type is constituted by the

(Θ, $\mathfrak{K}$)-*groups*: If Θ is some class of finite groups, and if $\mathfrak{K}$ is a simple group, then the group G is termed a (Θ, $\mathfrak{K}$)-group, if every homomorphic image H of G induces Θ-groups of automorphisms in those minimal normal subgroups M of H which are direct products of groups isomorphic to $\mathfrak{K}$.

To illustrate how complex properties may be reduced to such elementary ones, we mention that a group is nilpotent if, and only if, it is p-closed for every prime p and likewise if, and only if, every homomorphic image induces in each of its minimal normal subgroups the identity automorphism only.

We have here investigated these elementary classes from the point of view of the general properties of classes of finite groups outlined in the beginning of this introduction. We have then considered specially interesting concatenations of these elementary properties, specializing the results so obtained to nilpotency and supersolubility. These discussions and successive specializations led among other things to an elementary proof of Huppert's Theorem that a group is supersoluble if, and only if, its maximal subgroups have index a prime, to new criteria for nilpotency and solubility, and to the discovery of a great number of classes Θ of finite groups with the property that G is a Θ-group if, and only if, $G/\Phi(G)$ is a Θ-group; actually this seems to be quite the rule.

Some of the auxiliary results, in particular those in §2, may be of independent interest.

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Notations

$S^x = x^{-1}Sx$.

t^G = set of all t^g for g in G.

$[A, B]$ = subgroup generated by all commutators $[a, b] = a^{-1}b^{-1}ab$ for a in A and b in B.

$G' = [G, G]$ = commutator subgroup of G.

$Z(G)$ = center of G.

G^n = subgroup generated by the n^{th} powers of elements in G.

$F(G)$ = Fitting subgroup of G = product of all normal nilpotent subgroups of G.

$D(G)$ = hypercommutator of G = intersection of all normal subgroups X with nilpotent quotient group G/X.

Maximal subgroup = proper subgroup, not contained in any greater proper subgroup.

$\Phi(G)$ = Frattini subgroup = intersection of all maximal subgroups of G.

Minimal normal subgroup of G = normal subgroup $M \neq 1$ of G which does not contain normal subgroups of G except 1 and M.

Complement of normal subgroup N of G = subgroup S of G such that $G = NS$, $1 = N \cap S$.

Σ-element, for Σ a set of primes, is an element whose order is divisible by primes in Σ only.
Σ-group = group all of whose elements are Σ-elements.
Σ-Sylow subgroup S of G = Σ-subgroup of G whose index in G is prime to every prime in Σ.
$P\Sigma$ = set of primes not in Σ (so that a $P\Sigma$-element is an element whose order is prime to every prime in Σ etc.).
Core S_G of subgroup S of G = intersection of all subgroups conjugate to S in G.

Only *finite* groups will be considered.

1. The Frattini lemmas

The arguments used in the proofs of results of this section are due to Frattini. The results and their proofs, probably well known, are here reproduced for the convenience of the reader.

LEMMA 1. *If N is a normal subgroup of the group G, and if T is the normalizer in G of a Sylow subgroup of N, then $G = NT$.*

Proof. Assume that T is the normalizer of the p-Sylow subgroup S of N. If g is an element in G, then S and $g^{-1}Sg$ are both p-Sylow subgroups of N' since g induces an automorphism in the normal subgroup N of G. Consequently there exists an element n in N such that $n^{-1}Sn = g^{-1}Sg$. Hence $S = (gn^{-1})^{-1}Sgn^{-1}$ so that gn^{-1} belongs to the normalizer T of S in G. The element g belongs therefore to the subset Tn of TN, proving that $G = TN$.

LEMMA 2. *If K is a normal subgroup of the normal subgroup N of G such that the orders of K and N/K are relatively prime, then there exists a complement of K in N. If furthermore K or N/K is soluble and T is normalizer in G of some complement of K in N, then $G = KT$.*

Here as always we term *complement of K in N* a subgroup S of N such that $N = KS$ and $1 = K \cap S$ (so that in particular $S \simeq N/K$).

Proof. The first part of Lemma 2 is nothing but a restatement of Schur's Theorem; see, for instance, Zassenhaus [1; p. 125, Satz 25]. Assume next that S is a complement of K in N, that T is the normalizer of S and that K or N/K is soluble. Because of the last hypothesis and the fact that the orders of K and N/K are relatively prime, any two complements of K in N are conjugate in N; see Zassenhaus [1; p. 126, Satz 27]. Consider now some element g in G. Then g induces an automorphism in the normal subgroup N of G so that S and $g^{-1}Sg$ are both complements of K in N. Consequently there exists an element n in N such that $n^{-1}Sn = g^{-1}Sg$. Hence gn^{-1} belongs to the normalizer T of S in G so that g belongs to TN, proving $G = TN$. Since S is part of its normalizer T, it follows now that

$$G = TN = TSK = TK.$$

LEMMA 3. *If the group G possesses one and only one minimal normal subgroup M, and if G/M possesses a normal subgroup, not* 1, *whose order is prime to the order of M, then $\Phi(G) = 1$.*

Proof. If firstly M is not soluble, then M is not part of the nilpotent group $\Phi(G)$. Since M is on the other hand contained in every normal subgroup, not 1, of G, this implies $\Phi(G) = 1$. Assume therefore next the solubility of M. By hypothesis there exists a normal subgroup $N/M \neq 1$ of G/M such that the orders of M and N/M are relatively prime. By Lemma 2 there exists a complement S of M in N; and the normalizer T of S in G satisfies $G = MT$. Since $S \neq 1$ does not contain the one and only one minimal normal subgroup M of G, S is not a normal subgroup of G so that $T \neq G$. Consequently there exists a maximal subgroup R of G which contains S. Clearly $G = MT = MR \neq R$ so that M is not part of R. Consequently M is not part of $\Phi(G)$ either; and this implies as before $\Phi(G) = 1$.

2. The core of a maximal subgroup

Maximal subgroups and minimal normal subgroups are dual concepts and their relations dominate our investigation.

LEMMA 1. *If M is a soluble minimal normal subgroup of G, then M is abelian and $M^p = 1$ for some prime p. If a maximal subgroup S of G does not contain M, then $G = MS$ and $1 = M \cap S$.*

The simple proof of these well known facts may be indicated for the reader's convenience. Since M is free of proper characteristic subgroups (as these would be normal subgroups of G), and since the commutator subgroup M' of the soluble group M is different from M, $M' = 1$ so that M is abelian. If p is a prime divisor of the order of the abelian group M, then M^p is a characteristic subgroup of M and $M^p < M$ so that $M^p = 1$. If the maximal subgroup S of G does not contain M, then clearly $G = MS$. If $t \neq 1$ is an element in M, then the minimal normal subgroup M of G is generated by the set t^G of the elements in G conjugate to t in G. Since M is abelian, $t^G = t^{MS} = t^S$. Hence $G = MS = \{t^S\}S = \{t, S\}$ so that $M \cap S = 1$.

If S is any subgroup of the group G, then *the core S_G of S in G* is the intersection of all the subgroups conjugate to S in G, i.e.

$$S_G = \bigcap_{g \in G} S^g.$$

It is clear that S_G is a normal subgroup of G which is part of S; and that every normal subgroup of G which is contained in S is likewise part of S_G. Thus S_G is the product of all the normal subgroups of G which are contained in S. The only normal subgroup of G/S_G which is contained in S/S_G is consequently 1; and this will make it possible in many situations to assume that $S_G = 1$. In this latter case a true representation of G is obtained by mapping every element in G upon the permutation it induces in the set S^G of subgroups conjugate to S in G.

LEMMA 2. *If S is a maximal subgroup of the group G, if $S_G = 1$, if $N \neq 1$ is a normal subgroup of G and C the centralizer of N in G, then $C \cap S = 1$ and C is either* 1 *or a minimal normal subgroup of G.*

Remark. The reader will observe that Lemma 2 and its proof remain valid without the hypothesis that G be finite.

Proof. Since 1 is the only normal subgroup of G which is contained in the maximal subgroup S of G, N is not contained in S, and $G = NS$. The centralizer C of N is a normal subgroup of G, since N is a normal subgroup of G. Consequently $C \cap S$ is a normal subgroup of S so that S is part of the normalizer of $C \cap S$ in G. Since N is part of the centralizer of C, and hence of $C \cap S$, $NS = G$ is the normalizer of $C \cap S$. Thus $C \cap S$ is a normal subgroup of G which is contained in S; and this implies $C \cap S = 1$ because of $S_G = 1$.

Suppose now that the normal subgroup $X \neq 1$ of G is contained in C. As before we see that X is not part of the maximal subgroup S of G and that therefore $G = XS$. Hence $X \leqq C \leqq XS$; and application of Dedekind's Law shows that

$$C = X(C \cap S) = X.$$

Hence either $C = 1$ or else C is a minimal normal subgroup of G.

COROLLARY 1. *If S is a maximal subgroup of the group G and if $S_G = 1$, then*

(a) *there exists at most one abelian normal subgroup, not* 1, *of G; and*

(b) *there exist at most two different minimal normal subgroups of G.*

Remark. These results and their proofs are valid without the hypothesis that G be finite.

Proof. If $X \neq 1$ is an abelian normal subgroup of G, then X is part of its centralizer C in G. Hence $1 < X \leqq C$; and, by Lemma 2, C is a minimal normal subgroup of G. Consequently $X = C$ is a minimal normal subgroup of G. Assume now by way of contradiction the existence of abelian normal subgroups U and V of G such that $1 \neq U \neq V \neq 1$. By the preceding result U and V are both minimal normal subgroups of G so that in particular $U \cap V = 1$. Consequently U is part of the centralizer of V. But V has been shown to be its own centralizer so that $U \leqq V$, a contradiction which proves (a).

Assume next by way of contradiction the existence of three different minimal normal subgroups P, Q, and R of G. Then $P \cap R = P \cap Q = 1$ so that R and Q are both contained in the centralizer of P. Since $R \cap Q = 1$, RQ is not a minimal normal subgroup of G so that the centralizer of P in G is neither 1 nor a minimal normal subgroup of G. This contradicts Lemma 2; and this contradiction proves (b).

COROLLARY 2. *If S is a maximal subgroup of the group G, if $S_G = 1$, and if A and B are two different minimal normal subgroups of G, then*

(a) $G = AS = BS, 1 = A \cap S = B \cap S$;

(b) *A is the centralizer of B in G [and B the centralizer of A]*;

(c) *A, B, and $AB \cap S$ are isomorphic non-abelian groups.*

Remark. These results and their proofs remain valid without the hypothesis that G be finite.

Proof. Since A and B are two different minimal normal subgroups of G, $A \cap B = 1$ so that B is part of the centralizer of A. We apply Lemma 2 to see that B is the centralizer of A in G; and likewise we see that A is the centralizer of B. This proves (b).

Because $S_G = 1$, neither A nor B is contained in the maximal subgroup S of G. Hence $G = AS = BS$. Since A is the centralizer of B in G, $A \cap S = 1$ is a consequence of Lemma 2; and likewise we see that $B \cap S = 1$. This proves (a).

That neither A nor B is abelian, may be deduced from Corollary 1, (a) or from the present property (b). From $A \leqq AB \leqq AS = G$ and Dedekind's Law we deduce that

$$A(AB \cap S) = AB = B(AB \cap S).$$

Since $1 = S \cap A = A \cap B = B \cap S$, this implies the isomorphy of the groups A, AB/B, $AB \cap S$, AB/A, B; and this completes the proof.

The structure of groups G possessing two minimal normal subgroups and a maximal subgroup S satisfying $S_G = 1$

If M is one of the two minimal normal subgroups of G, then the centralizer C of M in G is the second minimal normal subgroup of G [Corollary 2, (b)]. We note that M and C are isomorphic non-abelian groups [Corollary 2, (c)]. We recall from the proof of Corollary 2 that $MC = C(MC \cap S) = (MC \cap S)M$; and now it is easy to see that the elements in $MC \cap S$ induce in M exactly the full group of inner automorphisms of M. By Corollary 2, (a), we have $G = MS = SC$ and $1 = M \cap S = S \cap C$. It follows that G is a splitting extension of M, that S is essentially the same as a group of automorphisms of M which contains the group of inner automorphisms of M; and every automorphism of M which is induced in M by an element in G is already induced by an element in S.

If T happens to be a second maximal subgroup of G satisfying $T_G = 1$, then T too induces in M the group of all automorphisms induced by elements in G; and different elements in T induce different automorphisms in M. Consequently there exists one and only one isomorphism σ of S upon T such that the element x in S and its image x^σ in T induce the same automorphism in M. Since S and T are both complements of M in G, there exists one and only one automorphism σ^* of G which leaves invariant every element in M and which maps x in S upon x^σ in T.

The preceding remarks make it fairly clear how to construct the most general example of such a situation: Denote by N some non-abelian group and by Σ a group of automorphisms of N with the following two properties: Σ contains every inner automorphism of N, and 1 and N are the only Σ-invariant subgroups of N (i.e. N is Σ-simple). Denote by G the splitting extension of N by Σ so that the elements of a complement S of N in G induce in N exactly the automorphisms in Σ. Then $S_G = 1$, S is a maximal subgroup of G; and N and its centralizer C in G are two different minimal normal subgroups of G.

LEMMA 3. *If the group G possesses a maximal subgroup with core* 1, *then the following properties of G are equivalent*:

(i) *The indices in G of all the maximal subgroups with core* 1 *are powers of one and the same prime p.*

(ii) *There exists one and only one minimal normal subgroup of G; and there exists a common prime divisor of all the indices in G of all the maximal subgroups with core* 1.

(iii) *There exists a soluble normal subgroup, not* 1, *in G.*

Proof. If condition (i) is satisfied by G, then it is clear that the second part of condition (ii) is also satisfied by G. Assume by way of contradiction the existence of two minimal normal subgroups A and B of G. Because of the existence of maximal subgroups with core 1 we may apply Corollary 2. It follows that A is non-abelian and that $G = AX$, $1 = A \cap X$ for every maximal subgroup X of G whose core is 1. Since the order of A consequently equals the index $[G:X]$, and since the latter index is, by (i), a prime power, A is a minimal normal subgroup of prime power order. Since such groups are soluble, Lemma 1 implies the commutativity of A; and we have arrived at the contradiction which proves that (ii) is a consequence of (i).

Assume next the validity of (ii). Then there exists one and only one minimal normal subgroup M of G; and there exists a prime p such that $[G:X]$ is a multiple of p for every maximal subgroup X of G whose core $X_G = 1$. There exists, by hypothesis, a maximal subgroup S of core 1. It is clear that $M \not\leqq S$ and that therefore $G = MS$. It follows that $[G:S] = [M:M \cap S]$ is a multiple of p so that in particular p is a divisor of the order of M. Denote by P some p-Sylow subgroup of M; and assume by way of contradiction that $P < M$. Since $P \neq 1$ and M is a minimal normal subgroup, P is not a normal subgroup of G so that the normalizer Q of P in G is different from G. Application of §1, Lemma 1 shows $G = MQ$. Since $Q \neq G$, there exists a maximal subgroup R of G which contains Q; and it is clear that $G = MQ = MR$. But then $M \not\leqq R$; and this implies $R_G = 1$, since M is the one and only one minimal normal subgroup of G. By (ii), $[G:R]$ is a multiple of p. Since $G = MR$, $[G:R] = [M:M \cap R]$. Since $P \leqq M \cap Q \leqq M \cap R$,

$[M:P]$ is a multiple of $[G:R]$. Since R is a maximal subgroup of G and $R_G = 1$, $[G:R]$ is a multiple of p and hence $[M:P]$ is a multiple of p. But P is a p-Sylow subgroup of M so that $[M:P]$ is prime to p. We have arrived at a contradiction which proves that $M = P$ is a p-group, not 1. Since p-groups are soluble, we have shown that (iii) is a consequence of (ii).

Assume finally the validity of (iii). Then there exists a soluble minimal normal subgroup N of G; and N is, by Lemma 1, an elementary abelian p-group. If X is a maximal subgroup of G and the core $X_G = 1$, then $N \not\leqq X$; and $G = NX$, $1 = N \cap X$ by Lemma 1. It follows that $[G:X]$ is exactly the order of N; and the latter number is a power of p. Hence (i) is a consequence of (iii), completing the proof.

Remark 1. Note that we have proved condition (i) in the following stricter form:

(i′) *If X and Y are maximal subgroups of G and $1 = X_G = Y_G$, then $[G:X] = [G:Y]$ is a power of a prime.*

Remark 2. Assume that the group G possesses maximal subgroups of core 1 and that G possesses two minimal normal subgroups A and B. Then A and B are non-abelian isomorphic groups; and $G = AX$, $1 = A \cap X$ for every maximal subgroup X of core 1 [Corollary 2]. Thus the second half of condition (ii) is always satisfied when the first part of condition (ii) is not satisfied. Since the first half of condition (ii) is not a consequence of the existence of maximal subgroups with core 1, it is impossible to omit the first half of condition (ii). Likewise it does not suffice to assume that all the maximal subgroups of core 1 have the same index in G (or even that they are conjugate in G).

Application to the product of all soluble normal subgroups

It is well known and easily verified that *the product $A(G)$ of all the soluble normal subgroups of the group G* is itself a soluble characteristic subgroup of G. If the normal subgroup N of G does not contain $A(G)$, then $NA(G)/N$ is a soluble normal subgroup, not 1, of G/N. Consequently $A(G) \leqq N$ whenever N is a normal subgroup of G such that G/N is free of soluble normal subgroups except 1. If we denote by $A^*(G)$ the intersection of all the normal subgroups N of G such that G/N is free of soluble normal subgroups except 1, then we have seen that $A(G) \leqq A^*(G)$.

Next we denote by $A^{**}(G)$ the intersection of all the cores C of maximal subgroups of G such that G/C is free of soluble normal subgroups except 1. It is clear that $A^*(G) \leqq A^{**}(G)$.

Assume now by way of contradiction that $A(G) < A^{**}(G)$. Then there exists a minimal normal subgroup M of $H = G/A(G)$ which is contained in the normal subgroup $A^{**}(G)/A(G) \neq 1$ of H. If $M = W/A(G)$ were soluble, then W would be soluble as an extension of the soluble group $A(G)$ by the soluble group M. But this would imply $W \leqq A(G) < W$, an impossibility. Hence M is not soluble. Among the normal subgroups of H which do not contain M there exists a maximal one, say K. Since M is a minimal normal subgroup of H, we have $M \cap K = 1$. If U is a normal subgroup of H and $K < U$, then $M \leqq U$ because of the maximality of K. Hence KM/K is the one and only one minimal normal subgroup of H/K. Since $M \simeq KM/K$, the latter group is not soluble either; and this implies in particular that $KM/K \not\leqq \Phi(H/K)$. Consequently there exists a maximal subgroup S/K of H/K which does not contain the one and only one minimal normal subgroup KM/K of H/K. It follows that 1 is the core of S/K and that therefore K is the core of the maximal subgroup S of H. Since the non-soluble group KM/K is the one and only one minimal normal subgroup of H/K, this latter group does not possess soluble normal subgroups, except 1. Consequently $A^{**}(H) \leqq K$. Since $A^{**}(H) = A^{**}(G)/A(G)$, we have furthermore $M \leqq A^{**}(H) \leqq K$. But K has been chosen in such a way as not to contain M; and we have arrived at a contradiction which proves that

$$A(G) = A^*(G) = A^{**}(G).$$

Recalling that $A^{**}(G)$ is the intersection of all the cores C of maximal subgroups of G whose quotient groups G/C are free of soluble normal subgroups, not 1, and applying Lemma 3, we find that

the product $A(G)$ of all the soluble normal subgroups of G is the intersection of all the cores C of maximal subgroups of G with the following property:

There exist maximal subgroups X and Y of core C in G whose indices $[G:X]$ and $[G:Y]$ are not powers of the same prime.

LEMMA 4. *Assume that G possesses a soluble normal subgroup, not* 1, *and that the core of the maximal subgroup S of G is* 1.

(a) *The existence of a soluble normal subgroup, not* 1, *of S implies the existence of a normal subgroup, not* 1, *of S whose order is relatively prime to* $[G:S]$.

(b) *If there exists a normal subgroup, not* 1, *of S whose order is relatively prime to* $[G:S]$, *then S is conjugate to every maximal subgroup T of G whose core* $T_G = 1$.

Remark. The general situation will be reduced to the one considered in Lemma 4 by considering the maximal subgroup S/S_G of the group G/S_G. Clearly our lemma provides a criterion for conjugacy of equicore maximal subgroups which generalizes the result of Ore [1] that equicore maximal subgroups of soluble groups are conjugate.

Proof. From our hypothesis we deduce first the existence of a soluble minimal normal subgroup M of G. By Lemma 1, M is abelian and $M^p = 1$ for some prime p. If furthermore X is a maximal subgroup of G whose core is 1, then $M \not\leqq X$; and Lemma 1 implies $G = MX$, $1 = M \cap X$. In particular $G/M \simeq X$, and $[G:X]$ equals the order of M which happens to be a power of p. Note that this may be applied to $X = S$ too.

If there exists a soluble normal subgroup, not 1, of S, then the same is true of the isomorphic group G/M. Consequently there exists a soluble minimal normal subgroup N/M of G/M. By Lemma 1, N/M is abelian and $(N/M)^q = 1$ for some prime q. Assume by way of contradiction that $p = q$. Then N is a p-group. Since M is a normal subgroup, not 1, of the p-group N, M contains center elements, not 1, of N. But the center of N is a characteristic subgroup of a normal subgroup of G; and as such $Z(N)$ is a normal subgroup of G. The minimality of M and $1 \neq M \cap Z(N)$ imply that M is part of $Z(N)$ and that therefore N is part of the centralizer C of M. Since G possesses maximal subgroups with core 1, and since $1 < M \leqq C$ (as M is abelian), C is a minimal normal subgroup of G [Corollary 2]. Hence $C = M < N \leqq C$, a contradiction proving $p \neq q$. The isomorphic groups G/M and S contain therefore a normal subgroup, not 1, of order a power of q, whereas $[G:S]$ equals the order of M which is a power of the prime $p \neq q$. This completes the proof of (a).

Assume next the existence of a normal subgroup, not 1, of S whose order is prime to $[G:S]$. Then the group $G/M \simeq S$ contains a normal subgroup $P/M \neq 1$ whose order is prime to $[G:S]$. Since the order of M equals $[G:S]$, we see that the orders of M and P/M are relatively prime. Consider now some maximal subgroup X of G whose core $X_G = 1$. Then $G = MX$, $1 = M \cap X$. Because of $M \leqq P \leqq MX$ and Dedekind's Law, we have $P = M(P \cap X)$ so that $P \cap X$ is a complement of M in P. Since $P \cap X$ is a normal subgroup of X, X is part of the normalizer of $P \cap X$. Since G possesses maximal subgroups with core 1 as well as the abelian minimal normal subgroup M, M is, by Corollaries 1 and 2, the one and only one minimal normal subgroup of G. Hence $P \cap X$ is not a normal subgroup of G so that the normalizer of $P \cap X$ is exactly the maximal subgroup X of G.

Suppose now that the core of the maximal subgroup T of G is 1. Application of the results of the preceding paragraph of our proof shows that $P \cap S$ and $P \cap T$ are both complements of M in P, that S is the normalizer of $P \cap S$ and T the normalizer of $P \cap T$ in G. Since the orders of M and P/M are relatively prime, and since M is abelian, any two complements of M in P are conjugate in P; see Zassenhaus [1; p. 126, Satz 27]. Consequently there exists an element t in P transforming $P \cap S$ into $P \cap T$. But this element t naturally transforms the normalizer S of $P \cap S$ into the normalizer T of $P \cap T$, Q.E.D.

LEMMA 5. *The group G is soluble, if it possesses a maximal subgroup of core* 1, *and if every maximal subgroup of core* 1 *is nilpotent.*

Remark. This result constitutes, essentially, a generalization of the theorem of O. Schmidt [1] and Iwasawa [1] which asserts the solubility of every group G all of whose proper subgroups are nilpotent. We shall, however, make use of this theorem in the proof of our result.

Proof. If the group G happens to be simple, then every maximal subgroup of G has core 1. In this case, therefore, every proper subgroup of G is nilpotent so that G is soluble (and of order a prime) by the Theorem of O. Schmidt and Iwasawa referred to above.

Assume next that G is not simple. Consider a maximal subgroup S of core $S_G = 1$; and suppose by way of contradiction that the prime p divides both $[G:S]$ and the order of S. Since S is, by hypothesis, nilpotent, the p-Sylow subgroup P of S is a direct factor of S so that S is certainly part of the normalizer of P. Since the order of S is a multiple of p, $P \neq 1$; and since $S_G = 1$, P is not a normal subgroup of G. Since $[G:S]$ is a multiple of p, P is not a p-Sylow subgroup of G; and there exists therefore a p-subgroup Q of G such that $P < Q$. By the well known properties of p-groups, the normalizer P^* of P in Q satisfies $P < P^* \leqq Q$. The normalizer R of P in G contains consequently S and P^*. Since P is a p-Sylow subgroup of S, and since P^* is a p-group greater than P, P^* is not part of S. Since S is a maximal subgroup of G, $\{S, P^*\} = G$. Hence G is the normalizer R of P in G so that P is a normal subgroup of G, a contradiction. Thus we have shown that the order of S and the index $[G:S]$ are relatively prime for every maximal subgroup S of G whose core $S_G = 1$.

Since G is not simple, G possesses a minimal normal subgroup M, and $M \neq G$. If X and Y are maximal subgroups of G whose cores $X_G = Y_G = 1$, then neither X nor Y contains M so that $G = MX = MY$. It follows that $[G:M] = [X:M \cap X] = [Y:M \cap Y]$. Since $M \neq G$, there exists a prime p which divides $[G:M]$ and which consequently divides the orders of X and Y. Since X and Y are, by hypothesis, nilpotent, the p-Sylow subgroup X_p of X is a direct factor of X and the p-Sylow subgroup Y_p of Y is a direct factor of Y. We have shown in the preceding paragraph of our proof that the order of X is relatively prime to the index $[G:X]$; and this implies that X_p is a p-Sylow subgroup of G, since $X_p \neq 1$. Likewise Y_p is a p-Sylow subgroup of G. Consequently there exists an element t in G such that $t^{-1}X_p t = Y_p$. Since X is part of the normalizer of X_p, $t^{-1}Xt$ is part of the normalizer of Y_p. But Y is part of the normalizer of Y_p so that $\{Y, t^{-1}Xt\}$ is part of the normalizer of Y_p. Since $Y_p \neq 1 = Y_G$, Y_p is not a normal subgroup of G. Hence the normalizer of Y_p in G is different from G; and this implies $G \neq \{Y, t^{-1}Xt\}$. Since X, $t^{-1}Xt$, and Y are maximal subgroups of G, this implies $Y = t^{-1}Xt$; and we have shown that any two maximal subgroups of core 1 are conjugate in G.

Assume now by way of contradiction the existence of a second minimal normal subgroup W of G. Since G possesses by hypothesis maximal subgroups of core 1, then, by Corollary 2,

$G = MS = WS$, $1 = M \cap S = W \cap S$ for every maximal subgroup S of core $S_G = 1$; and

M and W are isomorphic non-abelian groups.

Since maximal subgroups S of core 1 are nilpotent, G/M is nilpotent. Hence $M \simeq W \simeq MW/M$ is nilpotent; and this implies, by Lemma 1, that the non-abelian minimal normal subgroup M is abelian, a contradiction. Thus we see that M is the one and only one minimal normal subgroup of G.

Consequently we have verified the validity of condition (ii) of Lemma 3; and this implies the existence of a soluble normal subgroup $N \neq 1$ of G. Since M is the one and only one minimal normal subgroup of G, $M \leqq N$. Hence M is soluble (and, by Lemma 1, abelian). But $G/M = MS/M$ for every maximal subgroup S of core 1; and since these maximal subgroups S are nilpotent, G/M is nilpotent. Hence G is soluble; and this completes the proof.

3. The Δ-commutator subgroup of a group

A group theoretical property Δ defines a class of groups. Thus every group G either has the property Δ or else it does not have this property. It will be convenient to term *Δ-group* every group with the property Δ.

Throughout we shall assume of such a property Δ that the identity group is a Δ-group. In this section and elsewhere we shall usually require that Δ *be homomorphism-invariant*, i.e. that homomorphic images of Δ-groups are Δ-groups. It should be noted, however, that there exist interesting classes of groups which are not homomorphism-invariant, for instance, the direct products of cyclic groups of equal order.

As all groups considered in this investigation are supposed to be finite, Δ is likewise supposed to be a property of finite groups only.

DEFINITION. *The Δ-commutator subgroup $[G, \Delta]$ of the group G is the intersection of all the normal subgroups X of G with Δ-quotient group G/X.*

Since G/G is always a Δ-group, $[G, \Delta]$ is a well determined characteristic subgroup of G. In general, $G/[G, \Delta]$ will not be a Δ-group, as may be seen from easily constructed examples; if, for instance, Δ is the class of cyclic groups, then $[G, \Delta]$ is the commutator subgroup G' of G and G/G' is cyclic in exceptional cases only.

It is easy to verify the equivalence of the following two properties of Δ:

(a) *Δ is homomorphism-invariant and $G/[G, \Delta]$ is a Δ-group for every group G.*

(b) *The homomorphic image X of the group Y is a Δ-group if, and only if, X is a homomorphic image of $Y/[Y, \Delta]$.*

Group theoretical properties Δ, meeting these two equivalent requirements (a) and (b), shall be termed *strictly homomorphism-invariant*. A useful characterization of these properties is contained in our next result.

PROPOSITION 1. *The following properties of the homomorphism-invariant group theoretical property Δ are equivalent*:

(i) *$G/[G, \Delta]$ is a Δ-group for every group G.*

(ii) *If the group G is the product of Δ-subgroups A and B such that $ab = ba$ for every a in A and b in B, and if S is a subgroup of G such that $G = AS = SB = BA$, then S is a Δ-group.*

(iii) *If U and V are normal subgroups of the group G such that G/U and G/V are Δ-groups, then $G/(U \cap V)$ is a Δ-group.*

Proof. Assume the validity of (i); and consider a product G of (necessarily normal) Δ-subgroups A and B such that $ab = ba$ for every a in A and b in B. Assume that S is a subgroup of G satisfying $G = AS = SB = BA$. We form the direct product D of the groups A and B. The elements in D are then the pairs (a, b) for a in A and b in B. Denote by σ the mapping of D onto G defined by $(a, b)^\sigma = ab$. This mapping is a homomorphism of D onto G, since

$$[(a, b)(a', b')]^\sigma = (aa', bb')^\sigma = aa'bb' = aba'b' = (a, b)^\sigma(a', b')^\sigma.$$

Denote by T the inverse image of S under σ. Then T is the totality of elements in D which are mapped by σ upon elements in S. It is clear that T is a subgroup of D and that $D = (A, 1)T = T(1, B)$, since $G = AS = SB$. The direct factors $(A, 1)$ and $(1, B)$ of D are Δ-groups, since A and B are Δ-groups. Since D is the direct product of $(A, 1)$ and $(1, B)$, we obtain now the following isomorphies:

$$(A, 1) \simeq D/(1, B) = (1, B)T/(1, B) \simeq T/[T \cap (1, B)].$$

Hence $T/[T \cap (1, B)]$ is a Δ-group; and that $T/[T \cap (A, 1)]$ is a Δ-group, is seen likewise. Consequently

$$[T, \Delta] \leqq (1, B) \cap T \cap (A, 1) = 1 \quad \text{or} \quad 1 = [T, \Delta].$$

Application of (i) shows now that $T/[T, \Delta] = T$ is a Δ-group. Since Δ is homomorphism-invariant, the homomorphic image $S = T^\sigma$ of T is a Δ-group; and thus we have shown that (ii) is a consequence of (i).

Assume next the validity of (ii); and consider normal subgroups U and V of a group G such that G/U and G/V are Δ-groups. Form the direct product E of G/U and G/V; and map the element g in G upon the element $g^\sigma = (Ug, Vg)$ in E. It is easily seen that σ is a homomorphism of G onto a subgroup S of E; and that $U \cap V$ is the kernel of this homomorphism σ. Since E is the direct product of its Δ-subgroups $(G/U, 1)$ and $(1, G/V)$, and since $E = (G/U, 1)S = S(1, G/V)$, we may apply condition (ii) to see that S is a Δ-group. Since S and $G/(U \cap V)$ are isomorphic, the latter group is a Δ-group too; and thus we have seen that (iii) is a consequence of (ii).

If finally (iii) is satisfied by Δ, then one verifies by an obvious inductive argument that $G/[N_1 \cap \cdots \cap N_k]$ is a Δ-group whenever every G/N_i is a Δ-group; and this shows that (i) is a consequence of (iii), Q.E.D.

COROLLARY 1. *If subgroups and homomorphic images of Δ-groups are Δ-groups, then the following two properties of Δ are equivalent:*

(a) *$G/[G, \Delta]$ is a Δ-group for every group G.*

(b) *Direct products of Δ-groups are Δ-groups.*

Proof. If G is the direct product of Δ-groups, then $[G, \Delta] = 1$ so that (a) implies (b). If conversely (b) is satisfied by Δ, and if U and V are normal subgroups of the group G with Δ-quotient groups G/U and G/V respectively, then $G/(U \cap V)$ is isomorphic to a subgroup of the direct product D of G/U and G/V. By (b), the direct product D of Δ-groups is a Δ-group; and, by hypothesis, every subgroup of a Δ-group is a Δ-group. Hence $G/(U \cap V)$ is a Δ-group; and we have verified the validity of condition (iii) of Proposition 1. Hence (a) is a consequence of (b), Q.E.D.

PROPOSITION 2. *If Δ is a strictly homomorphism-invariant property and if the group W is not a Δ-group, though every proper homomorphic image of W is a Δ-group, then W possesses one and only one minimal normal subgroup.*

Proof. $W \neq 1$, since W is not a Δ-group. Consequently there exist minimal normal subgroups of W. Assume by way of contradiction the existence of two different minimal normal subgroups A and B of W. Then W/A and W/B are both Δ-groups as proper homomorphic images of W; and $A \cap B = 1$. Consequently $[W, \Delta] \leqq A \cap B = 1$ so that $W/[W, \Delta] = W$ would, by hypothesis, be a Δ-group, an impossibility. Consequently there exists one and only one minimal normal subgroup of W.

LEMMA 1. *If Δ is strictly homomorphism-invariant, and if σ is a homomorphism of the group G upon the group H, then $[G, \Delta]^\sigma = [H, \Delta]$.*

Proof. It is clear that σ induces a homomorphism of the Δ-group $G/[G, \Delta]$ onto $H/[G, \Delta]^\sigma$ so that the latter group is a Δ-group too. Consequently $[H, \Delta] \leqq [G, \Delta]^\sigma$. Denote now by L the inverse image of $[H, \Delta]$ under σ. Since $G^\sigma = H$, the groups G/L and $H/[H, \Delta]$ are isomorphic groups. Since the latter group is a Δ-group, so is the former. Consequently $[G, \Delta] \leqq L$; and this implies $[G, \Delta]^\sigma \leqq L^\sigma = [H, \Delta]$ since $G^\sigma = H$. Hence $[G, \Delta]^\sigma = [H, \Delta]$, as we wanted to show.

4. Σ-closed groups

Throughout this section Σ is going to designate a set of primes which may, in extreme cases, be vacuous or the set of all primes. If Σ happens to consist of one prime p only, then we shall usually say p instead of Σ.

DEFINITION. *The group G is Σ-closed, if products of Σ-elements in G are Σ-elements.*

It is clear that the group G is Σ-closed if, and only if, the set of Σ-elements in G is a subgroup of G, a subgroup which is necessarily a characteristic Σ-subgroup of G. Likewise it is easy to see the equivalence of the following three properties of a group G: the group G is Σ-closed; the group G possesses one and only one maximal Σ-subgroup; the group G possesses a maximal Σ-subgroup which is at the same time a normal subgroup of G. It follows that a group G is Σ-closed for every set Σ of primes if, and only if, G is nilpotent.

Trivially the group G will be Σ-closed, if either none or all of the prime divisors of the order of G belong to Σ.

Subgroups and homomorphic images of Σ-closed groups are obviously Σ-closed groups. A simple group is Σ-closed if, and only if, it is either a Σ-group or of order prime to every prime in Σ. Since minimal normal subgroups are direct products of isomorphic simple groups, every minimal normal subgroup of a homomorphic image of a Σ-closed group is either a Σ-group or else of order prime to every prime in Σ; in other words: Σ-closed groups are Σ-soluble in the sense of Čunichin, though the converse is clearly false.

It is clear that direct products of Σ-closed groups are Σ-closed. By §3, Corollary 1, Σ-closure is a strictly homomorphism-invariant property.

If Θ is a set of Σ-closed normal subgroups of the group G, then we denote by X_Σ, for X in Θ, the set of all Σ-elements in X. It is clear that X_Σ is a normal Σ-subgroup of G. Next denote by P the product of all the X_Σ for X in Θ. It is clear that P is a normal Σ-subgroup of G. Finally denote by Q the product of all the subgroups X in Θ. It is clear that Q is a normal subgroup of G, and that Q/P is the product of all the normal subgroups

$$PX/P \simeq X/(X \cap P) \simeq [X/X_\Sigma]/[(X \cap P)/X_\Sigma]$$

of G/P. Since every $[X:X_\Sigma]$ is prime to every prime in Σ, $[Q:P]$ is likewise prime to every prime in Σ. Hence Q is Σ-closed; and we have shown that products of Σ-closed normal subgroups are Σ-closed.

It is easy to construct examples showing that the extension G of the Σ-closed group N by the Σ-closed group G/N need not be Σ-closed. It is, however, obvious that this extension G will be Σ-closed, if either N is a Σ-group or $[G:N]$ is prime to every prime in Σ. Somewhat deeper and more interesting is the following criterion.

PROPOSITION 1. *If N is a normal subgroup of the group G, and if G/N and $\{N, g\}$ for every Σ-element g of prime power order in G are Σ-closed, then G is Σ-closed.*

Proof. By hypothesis N and G/N are Σ-closed. The totality K of Σ-elements in N is consequently a characteristic Σ-subgroup of N; and the totality T/N of Σ-elements in G/N is likewise a characteristic Σ-subgroup of G/N. We note that T is a normal subgroup of G, that $[G:T]$ is prime to every prime in Σ, and that K is a normal subgroup of G.

Let $H = T/K$ and $M = N/K$. Then M is a normal subgroup of H and $H/M \simeq T/N$ is a Σ-group, whereas the order of M is prime to every prime in Σ. Thus order and index of M in H are relatively prime. We may therefore apply Schur's Theorem to assure the existence of a complement R

of M in H; see Zassenhaus [1; p. 125, Satz 25]. Suppose that r is an element of prime power order in R. Then $r = Kx$ where x is of prime power order too; and since r is a Σ-element, so is x. By hypothesis $\{N, x\}$ is Σ-closed; and this implies that $\{M, r\}$ is Σ-closed too. Thus the totality r^* of Σ-elements in $\{M, r\}$ is a characteristic Σ-subgroup of $\{M, r\}$ which contains r. Since the order of M is prime to every prime in Σ, it follows that $\{M, r\}$ is the direct product of M and r^*. Thus r commutes with every element in M. Since R is a complement of M in H, it follows that H is the direct product of M and R. Since $R \simeq H/M$ is a Σ-group whereas M is of order prime to every prime in Σ, it follows that R is the totality of Σ-elements in H and that therefore H is Σ-closed.

Since T is an extension of the Σ-group K by the Σ-closed group $H = T/K$, T is Σ-closed; and since G is an extension of the Σ-closed group T by the group G/T whose order is prime to every prime in Σ, G too is Σ-closed, Q.E.D.

If every pair of Σ-elements in G generates a Σ-closed subgroup of G, then G is clearly Σ-closed. A somewhat deeper question arises when requiring only that pairs of elements of prime power order generate Σ-closed sub groups.

PROPOSITION 2. *The following two conditions are necessary and sufficient for the group G to be Σ-closed:*

(a) *If x is a p-element and y a q-element in G, if p is in Σ and q is not in Σ, then $\{x, y\}$ is a Σ-closed subgroup of G.*

(b) *If the simple group S is a homomorphic image of a subgroup of G, and if A and B are two different maximal Σ-subgroups of S, then the normalizer of $A \cap B$ in A is different from $A \cap B$.*

Remark 1. If Σ happens to consist of one prime p only, then condition (b) is satisfied by every group, as every proper subgroup of a p-group is different from its normalizer. But it seems to be an open question whether (b) can be omitted in general.

Proof. If G is Σ-closed, then every subgroup of G is Σ-closed, proving the necessity of (a). If the simple group S is a homomorphic image of a subgroup of the Σ-closed group G, then S is Σ-closed so that either S itself is a Σ-group or else 1 is the only Σ-subgroup of S. In either case S possesses one and only one maximal Σ-subgroup so that (b) is satisfied by default. This shows the necessity of our conditions.

If the conditions (a) and (b) were not sufficient for Σ-closure, then there would exist a group G of minimal order with the following two properties:

(1) G is not Σ-closed.

(2) G meets requirements (a) and (b).

Every subgroup and every homomorphic image of G meets requirements (a) and (b). Because of the minimality of G it follows that

(3) every proper subgroup and every proper homomorphic image of G is Σ-closed.

Assume now by way of contradiction that N is a normal subgroup of G and $1 < N < G$. Then N is Σ-closed by (3); and the set N^* of Σ-elements in N is a characteristic Σ-subgroup of N. Clearly N^* is a normal subgroup of G; and $N^* \neq 1$ would imply, by (3), that G/N^* is Σ-closed. But extensions of Σ-groups by Σ-closed groups are Σ-closed so that G would be Σ-closed contradicting (1). Hence $N^* = 1$ so that the order of N is prime to every prime in Σ. By (3), G/N is Σ-closed so that the set G^*/N of Σ-elements in G/N is a characteristic Σ-subgroup of G/N. Then G^* is a normal subgroup of G and $[G:G^*]$ is prime to every prime in Σ. If $G^* < G$, then G^* would be Σ-closed by (3); and this would imply the Σ-closure of G, contradicting (1). Hence $G = G^*$ so that G/N is a Σ-group.

Since G/N is a Σ-group and the order of N is prime to every prime in Σ, order and index of N in G are relatively prime. Thus we may apply Schur's Theorem; see Zassenhaus [1; p. 125, Satz 25]. Consequently there exists a complement R of N in G; and $R \simeq G/N$ is a Σ-group. Consider elements x in R and y in N which are both of prime power order. Then condition (a) is applicable so that $\{x, y\}$ is Σ-closed. The totality X of Σ-elements in $\{x, y\}$ is consequently a characteristic Σ-subgroup of $\{x, y\}$ which contains x. Furthermore $N \cap \{x, y\}$ is a normal subgroup of $\{x, y\}$ which contains y and whose order is prime to every prime in Σ and therefore to the order of the Σ-group X. Consequently $\{x, y\}$ is the direct product of X and $N \cap \{x, y\}$; and this implies in particular that $xy = yx$. It follows that every element in R commutes with every element in N and that therefore G is the direct product of the Σ-group R and the group N whose order is prime to every prime in Σ. But then R is the totality of Σ-elements in G so that G is Σ-closed, contradicting (1). Hence

(4) G is simple.

Assume next by way of contradiction the existence of different maximal Σ-subgroups whose intersection is different from 1. Then there exists a pair $A \neq B$ of maximal Σ-subgroups of G whose intersection has maximal order. Because of (4) we may apply condition (b) to show that the normalizer A^* of $A \cap B$ in A is different from $A \cap B$ and that likewise the normalizer B^* of $A \cap B$ in B is different from $A \cap B$. Thus

$$1 < A \cap B < A^* \leqq A < G \quad \text{and} \quad 1 < A \cap B < B^* \leqq B < G.$$

By (4), $A \cap B$ is not a normal subgroup of G so that the normalizer of $A \cap B$ is different from G. Hence $\{A^*, B^*\} < G$. By (3), therefore, $\{A^*, B^*\}$ is Σ-closed; and since A^* and B^* are Σ-groups, $\{A^*, B^*\}$ is itself a Σ-group.

But then there exists a maximal Σ-subgroup C of G which contains $\{A^*, B^*\}$. Since B^* is not part of A, $C \neq A$; and $A \cap B < A^* \leqq A \cap C$, contradicting the maximality of the intersection $A \cap B$. We have arrived at a contradiction which proves that

(5) $A \cap B = 1$ if A and B are different maximal Σ-subgroups of G.

Suppose that the prime q does not belong to Σ and that t is a q-element in G. Consider a maximal Σ-subgroup A of G; and consider in A an element $a \neq 1$ of prime power order. We may apply condition (a) to the pair a, t; and consequently the set a^* of Σ-elements in $\{a, t\}$ is a characteristic Σ-subgroup of $\{a, t\}$ which contains a. There exists a maximal Σ-subgroup B of G which contains a^*. Since a belongs to $A \cap B$, we deduce $A = B$ from (5); and this implies $a^* \leqq A$. Hence $t^{-1}at$ belongs to A; and now it follows that $A = t^{-1}At$.

Next we note that G is simple, by (4), not Σ-closed, by (1). Hence G is generated by the elements of prime power order prime to Σ. Consequently every maximal Σ-subgroup of G is a normal subgroup of G which is different from G. By (4) this implies that 1 is the only maximal Σ-subgroup of G so that the order of G is prime to every prime in Σ. But then G would be Σ-closed, contradicting (1), and this contradiction shows the sufficiency of our conditions.

PROPOSITION 3. *The following properties of the group G are equivalent:*

(i) *G is Σ-closed.*

(ii) (a) *If S is a maximal subgroup of G and if $[G:S_G]$ is a multiple of some prime in Σ, then every prime divisor of $[G:S]$ belongs to Σ.*
(b) *If M is a minimal normal subgroup of the homomorphic image H of G, then M possesses Σ-Sylow subgroups, and any two Σ-Sylow subgroups of M are conjugate in M.*

(iii) *If S is a maximal subgroup of G, if G/S_G possesses one and only one minimal normal subgroup M, and if $[G:S_G]$ is a multiple of some prime in Σ, then M is a Σ-group.*

(iv) *$G/\Phi(G)$ is Σ-closed.*

Remark 2. Here as always we term the subgroup X of Y a *Σ-Sylow subgroup of Y*, if X is Σ-group and $[Y:X]$ is prime to every prime in Σ. It is clear that every Σ-Sylow subgroup is a maximal Σ-subgroup, though the converse is, in general, false. As a matter of fact maximal Σ-subgroups always exist, whereas Σ-Sylow subgroups may or may not exist. See in this context a recent investigation by Ph. Hall [1].

Remark 3. If Σ happens to consist of one prime p only, then condition (ii.b) is satisfied by every group, showing the indispensability of (ii.a).

Whether it is possible to omit condition (ii.b) in general, appears to be an open question.

Proof. Assume first that G is Σ-closed. If S is a maximal subgroup of G, and if $[G:S_G]$ is a multiple of some prime in Σ, then the set T of Σ-elements in G/S_G is a characteristic Σ-subgroup, not 1, of G/S_G. But then T is not part of the maximal subgroup S/S_G of G/S_G so that $G/S_G = T(S/S_G)$. Hence $[G:S]$ is a factor of the order of the Σ-group T so that (ii.a) is satisfied by G. Every minimal normal subgroup of a homomorphic image of G is a direct product of isomorphic simple Σ-closed groups. But a simple Σ-closed group is either a Σ-group or of an order prime to every prime in Σ; and thus (ii.b) is satisfied by G too. Hence (ii) is a consequence of (i).

Assume next the validity of (ii); and consider a maximal subgroup S of G with the following two properties:

$H = G/S_G$ possesses one and only one minimal normal subgroup M;

the order of H is a multiple of some prime in Σ.

Then $V = S/S_G$ is a maximal subgroup of H and the core $V_H = 1$. If U is some maximal subgroup of H whose core $U_H = 1$, then, by (ii.a), $[H:U]$ is divisible by primes in Σ only.

It is clear that M is not contained in the maximal subgroup V of H. Hence $H = MV$ so that $[H:V] = [M:M \cap V]$; and this implies in particular that the order of M is a multiple of some prime in Σ.

By (ii.b) there exists a Σ-Sylow subgroup B of M. Since the order of M is divisible by primes in Σ, $B \neq 1$. If M were not a Σ-group, then $1 < B < M$ so that B would not be a normal subgroup of H. The normalizer C of B in H is consequently different from H. If h is an element in H, then $h^{-1}Bh$ is likewise a Σ-Sylow subgroup of M. By (ii.b), Σ-Sylow subgroups of M are conjugate in M. Hence there exists an element m in M such that $m^{-1}Bm = h^{-1}Bh$. It follows that hm^{-1} belongs to the normalizer C of B and that h belongs therefore to the subset Cm of CM. Hence $H = MC$. Since $C < H$, there exists a maximal subgroup D of H which contains C. Since $D < H = MC = MD$, $M \not\leqq D$; and since M is contained in every normal subgroup, not 1, of H, we have $D_H = 1$. Consequently $[H:D]$ is divisible by primes in Σ only. Hence $[M:M \cap D] = [MD:D] = [H:D]$ is divisible by primes in Σ only. On the other hand $B \leqq M \cap C \leqq M \cap D$ so that $[M:B]$ is a multiple of $[M:M \cap D]$. Since B is a Σ-Sylow subgroup of M, $[M:B]$ is prime to every prime in Σ so that $[M:M \cap D]$ is likewise prime to every prime in Σ. It follows that $[M:M \cap D] = 1$; and this implies $M = M \cap D \leqq D$, a contradiction. Thus we have been led to a contradiction by assuming that M is not a Σ-group; and we have shown that (iii) is a consequence of (ii).

If (i) were not a consequence of (iii), then there would exist a group G of minimal order with the following two properties:

(1) G is not Σ-closed.

(2) Condition (iii) is satisfied by G.

It is clear that every homomorphic image of G meets requirement (iii); and thus it follows from the minimality of G that

(3) every proper homomorphic image of G is Σ-closed.

Since Σ-closure is strictly homomorphism-invariant, it is a consequence of (3) and §3, Proposition 2 that

(4) there exists one and only one minimal normal subgroup M of G.

Assume by way of contradiction the existence of a maximal subgroup S of G which does not contain M. Then $M \nleqq S_G$; and this implies $S_G = 1$ by (4). By (1), the order of G is a multiple of some prime in Σ. Thus we may apply (iii). Hence M is a Σ-group. Since G/M is Σ-closed by (3), it follows that G itself is Σ-closed. This contradicts (1). Hence M is part of every maximal subgroup of G so that

(5) $M \leqq \Phi(G)$.

Since the Frattini subgroup is nilpotent, the minimal normal subgroup M of G is soluble. We apply §2, Lemma 1 to see that

(6) M is an elementary abelian p-group.

If p were in Σ, then M would be a Σ-group; and this would imply, by (3), that G were Σ-closed, contradicting (1). Hence

(7) p is not in Σ.

By (3), G/M is Σ-closed. The set T/M of Σ-elements in G/M is consequently a characteristic Σ-subgroup of G/M; and this implies that T is a characteristic subgroup of G. Since the orders of M and G/T are prime to every prime in Σ, and since G is not Σ-closed, $M < T$. Since the order of M is prime to $[T:M]$, and since M is abelian, we may apply §1, Lemma 2. Consequently there exists a complement A of M in T and $G = MB$ where B is the normalizer of A in G. Since M is contained in every normal subgroup, not 1, of G, since $1 < T/M \simeq A$, and since $M \nleqq A$, it follows that A is not a normal subgroup of G; and that therefore $B < G$. Consequently there exists a maximal subgroup C of G which contains B; and this implies $C < G = MB = MC$ so that $M \nleqq C$. This contradicts (5), proving that (i) is a consequence of (iii).

The equivalence of (i) and (iv) is an immediate consequence of the equivalence of (i) and (iii).

5. Σ-dissolved groups

If Σ is a set of primes, and if the totality of Σ-elements in the group G is a soluble subgroup of G, then we shall say that G *is a Σ-dissolved group*. It is clear that Σ-dissolved groups are Σ-closed, but not conversely. If Σ happens to consist of at most two primes, then, according to a celebrated Theorem of Burnside, every Σ-closed group is also Σ-dissolved. Furthermore one verifies without any difficulty that subgroups, homomorphic images, and direct products of Σ-dissolved groups are likewise Σ-dissolved.

If Θ is a set of Σ-dissolved normal subgroups of the group G, then we denote by X_Σ, for X in Θ, the set of all Σ-elements in X. It is clear that X_Σ is a soluble normal Σ-subgroup of G; and this implies that the product P of all the X_Σ, for X in Θ, is likewise a soluble normal Σ-subgroup of G. If Q is the product of all the normal subgroups X in Θ, then Q/P is the product of all the normal subgroups $PX/P \simeq X/(X \cap P) \simeq [X/X_\Sigma]/[(X \cap P)/X_\Sigma]$; and all these groups have order prime to every prime in Σ so that Q/P has order prime to every prime in Σ. Products of Σ-dissolved normal subgroups are consequently Σ-dissolved.

LEMMA 1. *If N is a normal subgroup of the Σ-closed group G, and if N and G/N are both Σ-dissolved, then G is Σ-dissolved.*

Proof. The totality T of Σ-elements in G is a subgroup of G, since G is Σ-closed. Since N and G/N are Σ-dissolved, their Σ-subgroups $N \cap T$ and $NT/N \simeq T/(T \cap N)$ are soluble. Thus T is soluble as an extension of the soluble group $N \cap T$ by the soluble group $T/(T \cap N)$.

COROLLARY 1. *If N is a normal subgroup of the group G, and if G/N and $\{N, g\}$ for every g in G are Σ-dissolved, then G is Σ-dissolved.*

This is an immediate consequence of Lemma 1 and §4, Proposition 1.

COROLLARY 2. *G is Σ-dissolved if, and only if, $G/\Phi(G)$ is Σ-dissolved.*

This is an immediate consequence of Lemma 1 and §4, Proposition 3.

PROPOSITION 1. *The following properties of the group G (and of the set Σ of primes) are equivalent:*

(i) *G is Σ-dissolved.*

(ii) *If X and Y are maximal subgroups of G, if $X_G = Y_G$, and if $[G:X_G]$ is divisible by some prime in Σ, then $[G:X]$ and $[G:Y]$ are powers of the same prime in Σ.*

(iii) *If X and Y are maximal subgroups of G, if $X_G = Y_G$, and if $[G:X_G]$ is divisible by some prime in Σ, then X and Y are conjugate in G, and $[G:X]$ is a multiple of some prime in Σ.*

(iv) *If N is the core of some maximal subgroup of G and $[G:N]$ is divisible by some prime in Σ, then there exists a maximal subgroup S of G with core N such that $[G:S]$ is divisible by some prime in Σ, and there exists a common prime divisor p of all the indices $[G:X]$ of maximal subgroups X with core N.*

(v) *If N is the core of some maximal subgroup of G and $[G:N]$ is divisible by some prime in Σ, and if G/N possesses one and only one minimal normal subgroup M, then M is a soluble Σ-group.*

Proof. Assume first the validity of (i) and consider maximal subgroups X and Y of G such that $X_G = Y_G$ and such that $[G:X_G]$ is a multiple of some prime in Σ. The homomorphic image $H = G/X_G$ is likewise Σ-dissolved, and the order of H is a multiple of some prime in Σ; $U = X/X_G$ and $V = Y/Y_G = Y/X_G$ are maximal subgroups of H whose cores in H are 1.

The totality T of Σ-elements in the Σ-dissolved group H is a soluble characteristic Σ-subgroup of H; and $T \neq 1$, since the order of H is a multiple of some prime in Σ. Consequently there exists a minimal normal subgroup M of H which is contained in T; and M is not contained in U or V, since their cores are 1. Since M is a soluble minimal normal Σ-subgroup of H, application of §2, Lemma 1 shows the following facts: M is abelian, $M^p = 1$ for some prime p in Σ, $H = UM = MV$, $1 = U \cap M = M \cap V$.

These facts imply in particular the isomorphy of U, V, and H/M as well as the equality of $[H:U]$, $[H:V]$ and of the order of the p-group M. Consequently $[G:X] = [H:U] = [H:V] = [G:Y]$ is a power of the prime p in Σ.

To prove that U and V are conjugate subgroups of H we distinguish two cases.

Case 1. $M = T$.

Since $H/M = H/T$ is free of Σ-elements, not 1, whereas $M = T$ is a Σ-group, the orders of the groups M and H/M are relatively prime. Since U and V are complements of M in H, and since M is abelian, we may apply a theorem of Zassenhaus [1; p. 126, Satz 27] to prove that U and V are conjugate in H.

Case 2. $M < T$.

We recall that the cores of U and V are 1, that H contains the abelian normal subgroup $M \neq 1$, and that the isomorphic groups U, V, and H/M possess a soluble normal subgroup, not 1, since $T/M \neq 1$ is a soluble characteristic subgroup of H/M. Consequently we may apply §2, Lemma 4 to prove that U and V are conjugate in G.

Since $X_G = Y_G$, and since $U = X/X_G$ and $V = Y/X_G$ are conjugate in $H = G/X_G$, X and Y are conjugate in G; and now it is clear that (ii) and (iii) are both consequences of (i).

It is obvious that (iv) is a consequence of either of the conditions (ii) and (iii). Assume next the validity of (iv); and consider the core N of some maximal subgroup of G with the following two properties:

G/N possesses one and only one minimal normal subgroup M; and

$[G:N]$ is divisible by some prime in Σ.

Clearly G/N possesses a maximal subgroup with core 1; and it is a consequence of (iv) that there exists a common prime divisor of all the indices

$[G/N:X]$ of maximal subgroups X of G/N with core 1. Thus we may apply condition (ii) of §2, Lemma 3; and consequently there exists a soluble normal subgroup, not 1, of G/N. Since M is the one and only one minimal normal subgroup of G/N, M is contained in this soluble normal subgroup of G/N so that M itself is soluble.

We apply (iv) again to see that there exists a maximal subgroup S of G/N whose core is 1 and whose index $[G/N:S]$ is a multiple of some prime p in Σ. It is clear then that M is not part of S. Application of §2, Lemma 1 shows now that M is abelian and $M^q = 1$ for some prime q; and that $G/N = MS$, $1 = M \cap S$. It follows in particular that the order of M equals the index $[G/N:S]$. Since the latter is a multiple of p, so is the former. But M is a q-group. Hence $p = q$; and thus we have shown that M is an abelian p-group. Since p belongs to Σ, M is in particular a Σ-group; and thus we have shown that (v) is a consequence of (iv).

Assume now by way of contradiction that (i) is not a consequence of (v). Then there exists a group G of minimal order with the following two properties:

(a) G is not Σ-dissolved.

(b) Condition (v) is satisfied by G.

Since (v) is satisfied by every homomorphic image of G, we deduce from the minimality of G that

(c) every proper homomorphic image of G is Σ-dissolved.

Since Σ-dissolution is a strictly homomorphism-invariant property, application of §3, Proposition 2 to (c) shows that

(d) there exists one and only one minimal normal subgroup M of G.

If $\Phi(G)$ were not 1, then $G/\Phi(G)$ would be Σ-dissolved by (b); and this would imply that G itself is Σ-dissolved [Corollary 2], an impossibility. Hence

(e) $\Phi(G) = 1$.

By (e), there exists a maximal subgroup S of G which does not contain $M \neq 1$; and it is a consequence of (d) that $S_G = 1$. By (a), the order of G which happens to be $[G:S_G]$ is divisible by some prime in Σ. Because of (d) we may apply condition (v). It follows that $G = G/S_G$ possesses a soluble normal Σ-subgroup, not 1. By (d), M is contained in this soluble Σ-group so that M itself is a soluble Σ-group. The totality T/M of Σ-elements in G/M is, by (c), a soluble characteristic Σ-subgroup of G/M. It is clear then that T is a soluble Σ-group. Since T by its construction contains every Σ-element in G, T is actually the totality of all Σ-elements in G. Thus G is Σ-dissolved, contradicting (a). We have arrived at the desired contradiction which proves that (i) is a consequence of (v); and this completes the proof.

Remark. An obvious combination of conditions (ii) and (iii) shows the validity of the following condition (ii′) which is stricter than either of the conditions (ii) and (iii) and satisfied by all Σ-dissolved groups:

(ii′) *If X and Y are maximal subgroups of G, if $X_G = Y_G$, and if $[G:X_G]$ is divisible by some prime in Σ, then $[G:X] = [G:Y]$ is a power of a prime in Σ.*

Likewise it is almost trivial to show that Σ-dissolved groups G meet the following requirement (v′) which is somewhat stricter than our condition (v):

(v′) *If N is the core of some maximal subgroup of G and $[G:N]$ is divisible by some prime in Σ, then there exists a soluble normal Σ-subgroup, not 1, of G/N.*

The group G is clearly soluble if, and only if, G is Σ-dissolved for Σ the set of all primes. Consequently the following solubility criterion is a special case of Proposition 1.

COROLLARY 3. *The following properties of the group G are equivalent:*

(i) *G is soluble.*

(ii) *If X and Y are maximal subgroups of G and $X_G = Y_G$, then $[G:X]$ and $[G:Y]$ are powers of the same prime.*

(iii) *If X and Y are maximal subgroups of G and $X_G = Y_G$, then X and Y are conjugate in G.*

(iv) *If N is the core of some maximal subgroup of G, then there exists a common prime divisor p of all the indices $[G:X]$ of maximal subgroups X with core N.*

(v) *If N is the core of some maximal subgroup of G, and if G/N possesses one and only one minimal normal subgroup M, then M is soluble.*

Remark. The implication of (iii) by (i) had already been shown by Ore [1].

6. The elementary structure properties

Before defining these properties we recall the fact that every minimal normal subgroup M of a group G is a direct product of isomorphic simple groups; see, for instance, Zassenhaus [1; p. 77, Satz 2]. If $\mathfrak{K}$ is a class of isomorphic simple groups, and if the minimal normal subgroup M of G is the direct product of simple groups belonging to this class $\mathfrak{K}$, then we term *$\mathfrak{K}$ the characteristic of M.* If in particular $\mathfrak{K}$ is the class of cyclic groups of order the prime p, then we shall shortly write p instead of $\mathfrak{K}$; and it is a consequence of §2, Lemma 1 that soluble minimal normal subgroups have prime number characteristic.

Denote next by Θ some group theoretical property. No restrictions are imposed upon Θ so that, in particular, Θ may be the class of all groups or may be vacuous. If $\mathfrak{K}$ is some class of isomorphic simple groups, then we define the property $(\Theta, \mathfrak{K})$ of a group G as follows:

(Θ, $\mathfrak{K}$) *If the minimal normal subgroup M of the homomorphic image H of G has characteristic $\mathfrak{K}$, then the group of automorphisms induced in M by elements in H has property Θ.*

Groups with this property shall be called (Θ, $\mathfrak{K}$)-groups, exactly as the groups with property Θ shall be termed Θ-groups. A useful restatement of the above property is obtained as follows. If M is a minimal normal subgroup of H, then the centralizer M^* of M in H is a normal subgroup of H; and H/M^* is essentially the same as the group of automorphisms, induced in M by elements in H. Thus property (Θ, $\mathfrak{K}$) requires H/M^* to be a Θ-group whenever the characteristic of the minimal normal subgroup M of the homomorphic image H of G is $\mathfrak{K}$.

Two extreme cases will illustrate the range of this definition. We shall term (Θ, $\mathfrak{K}$) *unconditional*, if every group is a (Θ, $\mathfrak{K}$)-group, otherwise *conditional*. If, for instance, Θ is the class of all groups, then (Θ, $\mathfrak{K}$) is unconditional. A group G is certainly a (Θ, $\mathfrak{K}$)-group, if every minimal normal subgroup of every homomorphic image of G has characteristic different from $\mathfrak{K}$. If these groups are the only (Θ, $\mathfrak{K}$)-groups, then we term (Θ, $\mathfrak{K}$) *exclusive*. Whenever $\mathfrak{K}$ is a class of non-abelian simple groups and Θ is a class of soluble groups, (Θ, $\mathfrak{K}$) is going to be exclusive. For if M is a minimal normal subgroup of H, if M^* is the centralizer of M in H, and if the characteristic $\mathfrak{K}$ of M is not a prime, then $M \cap M^* = 1$ so that H/M^* contains a subgroup isomorphic to M and so that therefore H/M^* is not soluble. Likewise (Θ, $\mathfrak{K}$) is going to be exclusive whenever Θ is vacuous.

We have pointed out already that there exist always (Θ, $\mathfrak{K}$)-groups; and it is fairly clear from our definition that homomorphic images of (Θ, $\mathfrak{K}$)-groups are (Θ, $\mathfrak{K}$)-groups. On the other hand it is easy to construct examples which show that subgroups of (Θ, $\mathfrak{K}$)-groups need not be (Θ, $\mathfrak{K}$)-groups. If, for instance, p is a prime, then every non-abelian simple group is a (Θ, p)-group, though its subgroups will, in general, not have this property.

Dr. B. Huppert has pointed out that the group G is a (Θ, $\mathfrak{K}$)-group, if (and only if) there exists a principal chain of normal subgroups N_i of G with the following property:

If the minimal normal subgroup N_{i+1}/N_i of G/N_i has characteristic $\mathfrak{K}$, then a Θ-group of automorphisms is induced in N_{i+1}/N_i by the elements in G.

A proof of the sufficiency of this condition is easily obtained by applying the Jordan-Hölder-Schreier Theorem to the operator group G where we choose as operators the inner automorphisms of G.

Using this criterion it would be possible—as has been pointed out to me by Dr. Huppert—to shorten the proofs of Proposition 1, Corollary 1 and Theorem 1.

If G is a group, then we may form *the intersection* $[G, \Theta, \mathfrak{K}]$ *of all normal subgroups X of G whose quotient group G/X is a* (Θ, $\mathfrak{K}$)*-group*. This is nothing but the (Θ, $\mathfrak{K}$)-commutator subgroup of G introduced in §3.

Proposition 1. *$G/[G, \Theta, \mathfrak{K}]$ is always a $(\Theta, \mathfrak{K})$-group.*

Proof. Since the property $(\Theta, \mathfrak{K})$ is homomorphism-invariant (and is satisfied by some groups), we may apply §3, Proposition 1; and thus it suffices to verify condition (iii) of §3, Proposition 1. Consequently we consider normal subgroups U and V of a group G such that G/U and G/V are $(\Theta, \mathfrak{K})$-groups. To prove that $G/(U \cap V)$ is likewise a $(\Theta, \mathfrak{K})$-group, consider normal subgroups M and N of G such that $U \cap V \leqq N < M$ and M/N is a minimal normal subgroup of G/N whose characteristic is $\mathfrak{K}$. Denote by M^*/N the centralizer of M/N in G/N. Then M^* is a normal subgroup of G; and G/M^* is essentially the same as the group of automorphisms of M/N which are induced in M/N by elements in G/N (or by elements in G). We have to show that G/M^* is a Θ-group. We distinguish two cases.

Case 1. $M \leqq NU$.

Then $U \cap V \leqq N < M \leqq NU$; and we deduce from Dedekind's Law that

$$M = N(M \cap U), \qquad V(N \cap U) \cap (M \cap U) = (N \cap U)(V \cap U) = N \cap U.$$

Consequently

$$M/N = N(M \cap U)/N \simeq (M \cap U)/(N \cap U) = (M \cap U)/[V(N \cap U) \cap (M \cap U)]$$
$$\simeq V(M \cap U)/V(N \cap U).$$

Denote by $D/V(N \cap U)$ the centralizer of $V(M \cap U)/V(N \cap U)$ in $G/V(N \cap U)$; and recall that M^*/N is the centralizer of M/N in G/N. Then D is a normal subgroup of G; and

$$[M, D] = [N(M \cap U), D] = [N, D][M \cap U, D]$$
$$\leqq [N, D][V(M \cap U) \cap U, D] \leqq N[V(N \cap U) \cap U] = N$$

so that $D \leqq M^*$. On the other hand

$$[V(M \cap U), M^*] = [V, M^*][M \cap U, M^*]$$
$$\leqq [V, M^*]([M, M^*] \cap [U, M^*]) \leqq V(N \cap U)$$

so that $M^* \leqq D$. Hence $D = M^*$; and this implies in particular that

$$V(N \cap U) \leqq D = M^*.$$

The group $G/M^* = G/D$ is consequently essentially the same as the groups of automorphisms induced in the isomorphic groups M/N and $V(M \cap U)/V(N \cap U)$ by elements in G. Since M/N is a minimal normal subgroup of characteristic $\mathfrak{K}$ of G/N, the group $V(M \cap U)/V(N \cap U)$ is likewise a minimal normal subgroup of characteristic $\mathfrak{K}$ of $G/V(N \cap U)$. Since the latter group is a homomorphic image of the $(\Theta, \mathfrak{K})$-group G/V, $G/D = G/M^*$ is a Θ-group.

Case 2. $M \nleqq NU$.

Then $N \leqq M \cap NU < M$; and this implies, by Dedekind's Law,

$$N(M \cap U) = M \cap NU = N \quad \text{or} \quad M \cap U \leqq N,$$

since M, N, U are normal subgroups of G and M/N is a minimal normal subgroup of G/N. Consequently

$$M/N = M/(M \cap NU) \simeq UM/UN.$$

Denote by E/UN the centralizer of UM/UN in G/UN; and recall that M^*/N is the centralizer of M/N in G/N. Then

$$[M, E] \leqq M \cap [UM, E] \leqq M \cap UN = N$$

so that $E \leqq M^*$. On the other hand,

$$[UM, M^*] = [U, M^*][M, M^*] \leqq UN$$

so that $M^* \leqq E$. Consequently $M^* = E$; and this implies in particular that $UN \leqq E = M^*$.

The group $G/M^* = G/E$ is consequently essentially the same as the groups of automorphisms induced in the isomorphic groups M/N and UM/UN by elements in G. Since M/N is a minimal normal subgroup of characteristic $\mathfrak{K}$ of G/N, the group UM/UN is likewise a minimal normal subgroup of characteristic $\mathfrak{K}$ of G/UN. Since the latter group is a homomorphic image of the $(\Theta, \mathfrak{K})$-group G/U, $G/E = G/M^*$ is a Θ-group.

Thus we have shown in both cases that G/M^* is a Θ-group; and this completes the proof of the fact that $G/(U \cap V)$ is a $(\Theta, \mathfrak{K})$-group whenever G/U and G/V are both $(\Theta, \mathfrak{K})$-groups, Q.E.D.

COROLLARY 1. *If the normal subgroup N of G is contained in $[G, \Theta, \mathfrak{K}]$, and if $[G, \Theta, \mathfrak{K}]/N$ is a minimal normal subgroup of G/N, then $\mathfrak{K}$ is the characteristic of $[G, \Theta, \mathfrak{K}]/N$ and the group of automorphisms induced in $[G, \Theta, \mathfrak{K}]/N$ by elements in G/N is not a Θ-group.*

Proof. Among the normal subgroups X of G satisfying $N = X \cap [G, \Theta, \mathfrak{K}]$, there exists a maximal one, say K. If the normal subgroup Y of G satisfies $K < Y$, then $[G, \Theta, \mathfrak{K}] \leqq Y$, since $[G, \Theta, \mathfrak{K}]/N$ is a minimal normal subgroup of G/N. Consequently $K[G, \Theta, \mathfrak{K}]/K$ is the one and only one minimal normal subgroup of G/K. Next we note that, by Proposition 1, $G/[G, \Theta, \mathfrak{K}]$ and consequently $G/K[G, \Theta, \mathfrak{K}]$ are $(\Theta, \mathfrak{K})$-groups, that $[G/K, \Theta, \mathfrak{K}] = K[G, \Theta, \mathfrak{K}]/K$ is the one and only one minimal normal subgroup of G/K, and that therefore G/K is not a $(\Theta, \mathfrak{K})$-group. It follows now easily enough that $\mathfrak{K}$ is the characteristic of $K[G, \Theta, \mathfrak{K}]/K$ and that the group of automorphisms induced in $K[G, \Theta, \mathfrak{K}]/K$ by elements in G/K is not a Θ-group. But

$$[G, \Theta, \mathfrak{K}]/N = [G, \Theta, \mathfrak{K}]/(K \cap [G, \Theta, \mathfrak{K}]) \simeq K[G, \Theta, \mathfrak{K}]/K.$$

Furthermore an element x in G satisfies $[[G, \Theta, \mathfrak{K}], x] \leqq N$ if, and only if, $[K[G, \Theta, \mathfrak{K}], x] \leqq K$, since $N = K \cap [G, \Theta, \mathfrak{K}]$. Consequently essentially the same group of automorphisms is induced by elements in G in the isomorphic groups $[G, \Theta, \mathfrak{K}]/N$ and $K[G, \Theta, \mathfrak{K}]/K$. Thus the minimal normal subgroup $[G, \Theta, \mathfrak{K}]/N$ of G/N has characteristic $\mathfrak{K}$; and the group of automorphisms induced in it by elements in G/N is not a Θ-group.

COROLLARY 1'. *If there exist* Θ*-groups, then* $[G, \Theta, \mathfrak{K}] = [G, [G, \Theta, \mathfrak{K}]]$.

This is an immediate consequence of Corollary 1.

From now on we shall always *assume the existence of* Θ*-groups*; and we shall not make explicit reference to this hypothesis again. It is then an immediate consequence of Corollary 1' that a group has property $(\Theta, \mathfrak{K})$ if, and only if, its central quotient group has this property; and now it follows by an obvious inductive argument that a group has property $(\Theta, \mathfrak{K})$ if, and only if, its hypercenter quotient group is a $(\Theta, \mathfrak{K})$-group.

If Θ is homomorphism-invariant and G a Θ-group, and if M is a minimal normal subgroup of the homomorphic image H of G, then the group of automorphisms, induced in M by elements in H, is a homomorphic image of G and consequently a Θ-group. Hence Θ-groups are $(\Theta, \mathfrak{K})$-groups for every characteristic $\mathfrak{K}$. This implies clearly that

$[G, \Theta, \mathfrak{K}] \leqq [G, \Theta]$ *for every group* G*, every homomorphism-invariant property* Θ*, and every characteristic* $\mathfrak{K}$.

We recall that Θ is strictly homomorphism-invariant, if Θ is homomorphism-invariant and $G/[G, \Theta]$ is a Θ-group for every group G; see §3.

THEOREM 1. *If* Θ *is strictly homomorphism-invariant, then the following condition is necessary and sufficient for the group* G *to be a* $(\Theta, \mathfrak{K})$*-group*:

(C) *If* M *is a minimal normal subgroup of characteristic* $\mathfrak{K}$ *of the homomorphic image* H *of* $[G, \Theta]$*, then* $M \leqq Z(H)$.

Remark. The property of being the identity group we denote by 1. Then condition (C) may be restated as follows:

(C') $[G, \Theta]$ *is a* $(1, \mathfrak{K})$*-group.*

The property $(1, \mathfrak{K})$ is strictly homomorphism-invariant by Proposition 1. But unless the characteristic $\mathfrak{K}$ is a prime, the property $(1, \mathfrak{K})$ is not going to be subgroup-inherited. Still, it is not difficult to see that normal subgroups of $(1, \mathfrak{K})$-groups are likewise $(1, \mathfrak{K})$-groups; and on the basis of this remark one sees without difficulty that conditions (C) and (C') are equivalent to

(C'') G *is an extension of a* $(1, \mathfrak{K})$*-group by a* Θ*-group.*

Furthermore one verifies now quite easily that

$$[G, \Theta, \mathfrak{K}] = [[G, \Theta], 1, \mathfrak{K}] \text{ for every group } G.$$

Proof. Assume first that G is a $(\Theta, \mathfrak{R})$-group. To verify the necessity of our condition (C), consider normal subgroups A and B of $C = [G, \Theta]$ such that $A < B$ and B/A is a minimal normal subgroup of characteristic $\mathfrak{R}$ of C/A. There exist normal subgroups N of G such that $N \cap B \leqq A$, for instance $N = 1$; and among these there exists a maximal one, say U. Since $U \cap B \leqq A < B$, $G/U \neq 1$; and consequently there exists a minimal normal subgroup V/U of G/U. Because of the maximality of U we have $V \cap B \nleqq A$ so that $A < A(V \cap B) \leqq B$. Since $V \cap B$ is a normal subgroup of C and B/A is a minimal normal subgroup of C/A, we have $B = A(V \cap B) = B \cap AV$ or $B \leqq AV$. Furthermore

$$V \cap A = V \cap A(U \cap B) = V \cap AU \cap B = B \cap U(V \cap A)$$
$$= (B \cap V) \cap U(V \cap A)$$

so that

$$B/A = A(V \cap B)/A \simeq (V \cap B)/(V \cap A) = (V \cap B)/[(V \cap B) \cap U(V \cap A)]$$
$$\simeq U(V \cap B)/U(V \cap A).$$

Next we note that C is a characteristic subgroup of G, that therefore $V \cap C$ is a normal subgroup of G and that

$$U \leqq U(V \cap A) < U(V \cap B) \leqq U(V \cap C) \leqq V.$$

Since V/U is a minimal normal subgroup of G/U, V/U is a direct product of isomorphic simple groups, and the normal subgroups $U(V \cap A)/U$ and $U(V \cap B)/U$ of the normal subgroup $U(V \cap C)/U$ of V/U are direct factors of V/U. Because of the isomorphy $B/A \simeq U(V \cap B)/U(V \cap A)$, it follows now that V/U and B/A have the same characteristic $\mathfrak{R}$. Denote by W/U the centralizer of V/U in G/U. Then W is a normal subgroup of G, and G/W is essentially the same as the group of automorphisms induced in V/U by elements in G/U (or in G). But V/U is a minimal normal subgroup of characteristic $\mathfrak{R}$ of the homomorphic image G/U of the $(\Theta, \mathfrak{R})$-group G. Consequently G/W is a Θ-group; and this implies $[G, \Theta] \leqq W$. Hence $[V, C] \leqq [V, W] \leqq U$. Since A and B are normal subgroups of C, and since $U \cap B \leqq A$, we find that

$$[B, C] = [A(V \cap B), C] = [A, C][V \cap B, C]$$
$$\leqq [A, C]([V, C] \cap [B, C]) \leqq A(U \cap B) = A$$

and that therefore $B/A \leqq Z(C/A)$; and this completes the proof of the necessity of condition (C).

Assume conversely the validity of (C); and consider a minimal normal subgroup M of the homomorphic image H of G such that $\mathfrak{R}$ is the characteristic of M. Since $[H, \Theta]$ is by §3, Lemma 1 a homomorphic image of $[G, \Theta]$, and since condition (C) is preserved by epimorphisms, (C) is satisfied by $[H, \Theta]$.

Since M is a minimal normal subgroup of H and since $[H, \Theta]$ is a characteristic subgroup of H, either $M \cap [H, \Theta] = 1$ or $M \leqq [H, \Theta]$. In the former case we have certainly $[M, [H, \Theta]] = 1$. In the latter case M is the direct product of minimal normal subgroups M_i of $[H, \Theta]$. Since M is the direct product of simple groups of type $\mathfrak{R}$, each of the M_i has characteristic $\mathfrak{R}$. Application of (C) shows that every M_i is part of the center of $[H, \Theta]$; and consequently M itself is contained in the center of $[H, \Theta]$. Thus we have shown $[M, [H, \Theta]] = 1$ in both cases; and this implies that $[H, \Theta]$ is part of the centralizer M^* of M in H. Consequently H/M^* is a Θ-group, since Θ is strictly homomorphism-invariant. Hence a Θ-group of automorphisms is induced in M by the elements in H; and thus we have shown that G is a $(\Theta, \mathfrak{R})$-group.

COROLLARY 2. *If Θ is strictly homomorphism-invariant, and if products of normal Θ-subgroups are Θ-groups, then products of normal $(\Theta, \mathfrak{R})$-subgroups are $(\Theta, \mathfrak{R})$-groups.*

Proof. Assume that the group G is the product of two normal $(\Theta, \mathfrak{R})$-subgroups; and consider a minimal normal subgroup M of characteristic $\mathfrak{R}$ of the homomorphic image H of G. By hypothesis H is the product of its normal $(\Theta, \mathfrak{R})$-subgroups A and B. The Θ-commutator subgroups $[A, \Theta]$ and $[B, \Theta]$ are normal subgroups of H, since they are characteristic subgroups of normal subgroups of H. If firstly $M \cap [A, \Theta] = 1$, then $[A, \Theta]$ is part of the centralizer of M, since M and $[A, \Theta]$ are both normal subgroups of H. If $M \cap [A, \Theta] \neq 1$, then $M \cap [A, \Theta] = M$, since M is a minimal normal subgroup of H. Hence $M \leqq [A, \Theta]$. The product of all the minimal normal subgroups of $[A, \Theta]$ which are contained in M is a normal subgroup of H, since M and $[A, \Theta]$ are normal subgroups of H. It follows that the minimal normal subgroup M of H is a product of minimal normal subgroups of $[A, \Theta]$. Since M is of characteristic $\mathfrak{R}$, each minimal normal subgroup X of $[A, \Theta]$ which is part of M is likewise of characteristic $\mathfrak{R}$. Application of Theorem 1, (C) shows that each of these minimal normal subgroups X of $[A, \Theta]$ is part of the center of $[A, \Theta]$. Hence $M \leqq Z([A, \Theta])$; and thus we have shown in either case that $[A, \Theta]$ is part of the centralizer of M. Likewise we see that $[B, \Theta]$ is part of the centralizer of M. Hence $[A, \Theta][B, \Theta]$ is part of the centralizer of M. But $H/[A, \Theta][B, \Theta]$ is the product of its normal subgroups $[B, \Theta]A/[B, \Theta][A, \Theta]$ and $[A, \Theta]B/[A, \Theta][B, \Theta]$ which are both Θ-groups. By hypothesis therefore $H/[A, \Theta][B, \Theta]$ is a Θ-group; and this implies that H induces a Θ-group of automorphisms in M. Thus we see that G is a $(\Theta, \mathfrak{R})$-group; and our corollary follows by an obvious inductive argument.

COROLLARY 3. *If the property Θ is strictly homomorphism-invariant and if the characteristic $\mathfrak{R}$ is not a prime, then the following properties of the group G are equivalent:*

(i) *G is a $(\Theta, \mathfrak{R})$-group.*

(ii) *No minimal normal subgroup of a homomorphic image of $[G, \Theta]$ has characteristic $\mathfrak{R}$.*

(iii) *If the homomorphic image H of G possesses one and only one minimal normal subgroup M, and if $\mathfrak{K}$ is the characteristic of M, then H is a Θ-group.*

(iv) *$G/F[G]$ is a $(\Theta, \mathfrak{K})$-group.*

Here as always we denote by $F(G)$ the *Fitting* subgroup of G. This is the product of all nilpotent normal subgroups of G; and as such it is a nilpotent characteristic subgroup of G.

Proof. Since the characteristic $\mathfrak{K}$ is not a prime, groups of characteristic $\mathfrak{K}$ are not abelian; and as such they cannot be subgroups of the center of any other group. Now one verifies without difficulty that in our case the present condition (ii) and the condition (C) of Theorem 1 are equivalent; and this shows the equivalence of conditions (i) and (ii).

(i) implies (iv), since $(\Theta, \mathfrak{K})$ is homomorphism-invariant.—If (iv) is satisfied by G, and if the one and only one minimal normal subgroup M of the homomorphic image H of G has characteristic $\mathfrak{K}$, then $F(H) = 1$, since M is not nilpotent as $\mathfrak{K}$ is not a prime. Hence H is a homomorphic image of the $(\Theta, \mathfrak{K})$-group $G/F(G)$; and as such H is a $(\Theta, \mathfrak{K})$-group. Since M is not abelian, M is not part of its centralizer; and since M is part of every normal subgroup, not 1, of H, the centralizer of M equals 1. Thus H is essentially the same as the group of automorphisms, induced in M by elements in H; and this group is a Θ-group, since H is a $(\Theta, \mathfrak{K})$-group and $\mathfrak{K}$ is the characteristic of M. Hence (iii) is a consequence of (iv).

Assume finally the validity of (iii); and consider a minimal normal subgroup U of characteristic $\mathfrak{K}$ of the homomorphic image V of G. Among the normal subgroups X of V, satisfying $X \cap U = 1$, there exists a maximal one, say W. Then U and WU/W are isomorphic groups; and one verifies without difficulty that WU/W is a minimal normal subgroup of V/W, that the characteristic of WU/W is $\mathfrak{K}$, and that the groups of automorphisms, induced in U and WU/W by elements in V, are essentially the same. But WU/W is, because of the maximality of W, the one and only one minimal normal subgroup of V/W. Thus, we may apply (iii) to see that V/W is a Θ-group. Hence a Θ-group of automorphisms is induced in WU/W and in U by elements in V. Consequently G is a $(\Theta, \mathfrak{K})$-group. Thus (i) is a consequence of (iii), completing the proof.

To obtain a better insight into the nature of (Θ, p)-groups for p a prime, we shall investigate a certain subclass of this class. With this in mind we introduce the following concept.

Definition. *The group G is p-separated, if every minimal normal subgroup of every homomorphic image of G is either a p-group or of order prime to p.*

This concept has been discussed first, under the name p-soluble, by Čunichin. The change of name is motivated by our having used the word n-soluble for designation of a different concept.

It is almost obvious that subgroups, homomorphic images, and direct products of p-separated groups are p-separated. This property is therefore strictly homomorphism-invariant [§3, Corollary 1].

PROPOSITION 2. *The following properties of the group G are equivalent:*

(i) *The set of elements of order prime to p in G is a subgroup.*

(ii) *G is p-separated; and if the minimal normal subgroup M of the homomorphic image H of G has characteristic p, then $M \leqq Z(H)$.*

(iii) *If the order of the minimal normal subgroup M of the homomorphic image H of G is a multiple of p, then $M \leqq Z(H)$.*

Remark 0. The class of groups, characterized by Proposition 2, has often been termed p-nilpotent. A result equivalent to ours may also be found in an as yet unpublished investigation of B. Huppert.

Remark 1. If Pp is the set of all primes $q \neq p$, then condition (i) just requires Pp-closure of G. Note that the set of elements of order prime to p, if it is a subgroup, is a characteristic subgroup of order prime to p.

Remark 2. The second part of (ii) is just a restatement of condition (C) of Theorem 1 in case $\mathfrak{K} = p$ and $\Theta = 1$.

Proof. Assume first the validity of (i); and suppose that the order of the minimal normal subgroup M of the homomorphic image H of G is a multiple of p. Since G meets requirement (i), so does its homomorphic image H. The totality T of elements of order prime to p in H is consequently a characteristic subgroup of order prime to p. Since M contains elements of order p, $M \not\leqq T$; and $M \cap T = 1$ is a consequence of the minimality of M. Thus M does not contain elements of order prime to p, except 1; and this implies that M is a p-group. Furthermore $[M, T] = 1$ so that T is part of the centralizer of M in H. It follows that a p-group of automorphisms is induced in M by the elements in H. But a p-group of automorphisms, operating on a p-group, not 1, possesses fixed elements, not 1. Because of the minimality of M this implies that every element in M is a fixed element of this group of automorphisms, i.e. $M \leqq Z(H)$. This shows that (ii) is a consequence of (i); and it is almost obvious that (iii) is a consequence of (ii).

Assume finally the validity of condition (iii). Assume that $N \neq 1$ is a normal subgroup of G and that the set T/N of elements of order prime to p in G/N is a characteristic subgroup of order prime to p. Because $N \neq 1$ there exists a normal subgroup K of G such that $K < N$ and such that $M = N/K$ is a minimal normal subgroup of $H = G/K$. If firstly the order of M is prime to p, then T/K is a group of order prime to p. If Kx is an element of order prime to p in H, then Nx is an element of order prime to p so that Nx belongs to T/N and Kx to T/K. Hence T/K is the totality of elements of order prime to p. If next the order of M is a multiple of p, then $M \leqq Z(H)$ by (iii). Application of §2, Lemma 1 shows that M is an elementary abelian p-group. Since $T/N = (T/K)/M$ is of order prime to p and M is a p-group, we may apply Schur's Theorem; see Zassenhaus [1; p. 125, Satz 25]. Consequently there exists a complement R of M in T/K. Since $M \leqq Z(H)$, T/K is the direct product of M and R so that R is the totality of elements of order prime to p in T/K. Hence R is the set of all elements of order prime to p in H. Thus we have shown in either case that (i) is satisfied by H; and an obvious inductive argument completes the proof of the fact that (i) is a consequence of (iii).

PROPOSITION 3. *The following properties of the group G (and the prime p) are equivalent:*

(i) *G is p-separated.*

(ii) *$G/F(G)$ is p-separated.*

(iii) *If the homomorphic image H of G possesses one and only one minimal normal subgroup M, and if the order of M is a multiple of p, then $[H:S]$ is a power of p for every maximal subgroup S of H whose core $S_H = 1$.*

Proof. The equivalence of (i) and (ii) is almost obvious. Likewise one sees easily, using §2, Lemma 1, that (i) implies (iii). If the group G satisfies (iii) without being p-separated, then there exists a homomorphic image H of minimal order of G which is not p-separated. Then H possesses a minimal normal subgroup M whose order is a multiple of p, though M is not a p-group. Because of the minimality of H it is impossible that a normal subgroup, not 1, of H does not contain M; and so M is the one and only one minimal normal subgroup of H. Since the minimal normal subgroup M is not a p-group, M is not part of $\Phi(H)$; and consequently there exist maximal subgroups of H which do not contain M. But these must all have core 1. Application of (iii) and §2, Lemma 3 leads to a contradiction which proves our result.

THEOREM 2. *If Θ is strictly homomorphism-invariant, then the following properties of the group G (and the prime p) are equivalent:*

(i) *G is a p-separated (Θ, p)-group.*

(ii) *$G/[G, \Theta]$ is p-separated and the set of elements of order prime to p in $[G, \Theta]$ is a subgroup.*

(iii) *If the homomorphic image H of G possesses one and only one minimal normal subgroup M, and if the order of M is a multiple of p, then every maximal subgroup S of H whose core $S_H = 1$ is a Θ-group and its index $[H:S]$ is a power of p.*

(iv) *$G/\Phi(G)$ is a p-separated (Θ, p)-group.*

Remark 3. The second part of (ii) states the Pp-closure of $[G, \Theta]$. If Θ-groups happen to be p-separated, then condition (ii) may be stated more simply as follows: $[G, \Theta]$ is Pp-closed.

Proof. Assume firstly the validity of (i). Then $G/[G, \Theta]$ and $[G, \Theta]$ are both p-separated. If the order of the minimal normal subgroup U of the homomorphic image V of $[G, \Theta]$ is a multiple of p, then U is a p-group, has characteristic p. We apply Theorem 1, (C) to see that $U \leqq Z(V)$. Thus condition (iii) of Proposition 2 is satisfied by the group $[G, \Theta]$; and this implies that the set of elements of order prime to p in $[G, \Theta]$ is a subgroup. Hence (i) implies (ii).

Assume next the validity of (ii). Then one sees without difficulty that G is p-separated. To verify the validity of (iii) consider a homomorphic image H of G which possesses one and only one minimal normal subgroup M; and assume furthermore that the order of M is a multiple of p and that there exist maximal subgroups of H whose core is 1. Since G is p-separated, H is p-separated; and thus M is a p-group. Application of §2, Lemma 1 shows that M is an elementary abelian p-group; and $H = MS$, $1 = M \cap S$, $H/M \simeq S$ for every maximal subgroup S of H which does not contain M. The existence of maximal subgroups of H whose core is 1 assures us of the existence of such an S; and application of §2, Lemma 2 shows that M is its own centralizer. This implies in particular that H/M is essentially the same as the group of automorphisms, induced in M by elements in H.

By (ii), the set T of elements of order prime to p in $[H, \Theta]$ is a subgroup. Since T is of order prime to p, $T \cap M = 1$. Since T is a characteristic subgroup of a characteristic subgroup, T is a characteristic subgroup of H. Since M is part of every normal subgroup, not 1, of H, it follows that $T = 1$ and that therefore $[H, \Theta]$ is a p-group. If $[H, \Theta] \neq 1$, so is $Z([H, \Theta])$; and the center of a characteristic subgroup is likewise a characteristic subgroup so that $M \leqq Z([H, \Theta])$ in this case. Hence $[H, \Theta]$ is part of the centralizer of M which has been shown to be M itself; and thus we have shown in either case that $[H, \Theta] \leqq M$. But Θ is strictly homomorphism-invariant; and this implies now that H/M is a Θ-group.

If S is a maximal subgroup of H and if $S_H = 1$, then it follows from previous remarks that $H = SM$, $1 = S \cap M$, $S \simeq H/M$. Hence S is a Θ-group and the index $[H:S]$ is the order of M, a power of p. Thus we have shown that (iii) is a consequence of (ii).

Assume now by way of contradiction that (iii) does not imply (i). Then there exists a group G of minimal order with the following properties:

(1) G is not a p-separated (Θ, p)-group.

(2) G meets requirement (iii).

It is an immediate consequence of (2) and Proposition 3 that

(3) G is p-separated;

and this shows in combination with (1) that

(1′) G is not a (Θ, p)-group.

Every homomorphic image of G meets requirement (iii). Hence it follows from the minimality of G that

(4) every proper homomorphic image of G is a (Θ, p)-group.

The property (Θ, p) is strictly homomorphism-invariant by Proposition 1. Application of (4), (1′), and §3, Proposition 2 shows therefore that

(5) there exists one and only one minimal normal subgroup M of G.

Next note that every proper homomorphic image of G is a (Θ, p)-group, that G itself is not a (Θ, p)-group, and that M is the one and only one minimal normal subgroup of G. It follows that

(6) M is a p-group and G does not induce a Θ-group of automorphisms in$M$

Recall that (Θ, p) is strictly homomorphism-invariant [Proposition 1]. Thus $[G, \Theta, p] \neq 1$, since G is not a (Θ, p)-group. Combine this with (5) and an inequality verified before to see that

(7) $M \leqq [G, \Theta, p] \leqq [G, \Theta]$.

Assume by way of contradiction that $[G, \Theta]$ is a p-group. Since the center of a characteristic subgroup is a characteristic subgroup, since $[G, \Theta]$ is, by (7) a p-group, not 1, and since therefore its center is different from 1, it follows from (5) that $M \leqq Z([G, \Theta])$. Then $[G, \Theta]$ is part of the centralizer M^* of M in G. Since Θ is strictly homomorphism-invariant, G/M^* is a Θ-group so that G induces in M a Θ-group of automorphisms, contradicting (6). Thus we have shown that

(8) $[G, \Theta]$ is not a p-group.

It is a consequence of (7) that $[G/M, \Theta] = [G, \Theta]/M$; and this group is not a p-group by (6) and (8). We note the fact, already verified, that every p-separated (Θ, p)-group has property (ii). By (3) and (4), G/M is a p-separated (Θ, p)-group. The set T/M of elements of order prime to p in $[G/M, \Theta]$ is consequently a characteristic subgroup of order prime to p; and $T/M \neq 1$ is a consequence of (8). Application of (5) and §1, Lemma 3 shows now that

(9) $\Phi(G) = 1$.

Consequently there exists a maximal subgroup S of G which does not contain M. Application of §2, Lemma 1 shows that $G = MS$, $1 = M \cap S$, $S \simeq G/M$, since M is, by (6), a p-group. Since M is, by (5), contained in every normal subgroup, not 1, of G, and since M is not part of S, the core of S is $S_G = 1$. Because of (2), (5), and (6) we may apply (iii) on S. Hence S is a Θ-group so that G/M is likewise a Θ-group. Consequently $M = [G, \Theta]$ by (7) so that (6) and (8) contradict each other. Thus we have arrived at a contradiction which completes the proof of the equivalence of the first three conditions.

That (iv) and (i) are equivalent, is easily derived from the equivalence of conditions (i) and (iii), Q.E.D.

DEFINITION. *The group theoretical property* Δ *is subgroup-inherited, if subgroups of* Δ*-groups are always* Δ*-groups.*

Though the properties of interest to us will usually be subgroup-inherited, there are important examples of properties which are not subgroup-inherited, for instance, simplicity.

COROLLARY 4. *If* Θ *is strictly homomorphism-invariant and subgroup-inherited, then the p-separated* (Θ, p)*-groups form a subgroup-inherited class.*

Remark 4. The property Θ is, by §3, Corollary 1, strictly homomorphism-invariant and subgroup-inherited if, and only if, subgroups, homomorphic images, and direct products of Θ-groups are Θ-groups.

Proof. Assume that S is a subgroup of the p-separated (Θ, p)-group G. One verifies without difficulty that S is p-separated. Next we note the isomorphy $S/(S \cap [G, \Theta]) \simeq [G, \Theta]S/[G, \Theta]$. The latter group is a Θ-group, since $G/[G, \Theta]$ is a Θ-group (as Θ is strictly homomorphism-invariant) and since subgroups of Θ-groups are supposed to be Θ-groups. It follows that $[S, \Theta] \leqq S \cap [G, \Theta]$.

The set T of elements of order prime to p in $[G, \Theta]$ is a subgroup, since Θ is strictly homomorphism-invariant and G is a p-separated (Θ, p)-group [Theorem 2]. The set $T \cap [S, \Theta]$ of elements of order prime to p in $[S, \Theta]$ is therefore a subgroup too. Hence condition (ii) of Theorem 2 is satisfied by S so that S is a p-separated (Θ, p)-group, as we wanted to show.

The following result will prove useful in applications.

COROLLARY 5. *If* Θ *is strictly homomorphism-invariant, and if* N *is a normal p-subgroup of the p-separated group* G *whose centralizer in* G *is a p-group, then the following properties are equivalent:*

(i) G *is a* (Θ, p)*-group.*

(ii) $[G, \Theta]$ *is a p-group.*

(iii) $[G/N, \Theta]$ *is a p-group.*

Proof. If the p-separated group G is a (Θ, p)-group, then the set T of elements of order prime to p in $[G, \Theta]$ is a characteristic subgroup of G whose order is prime to p [Theorem 2]. Since N is a p-group, $N \cap T = 1$ so that T is part of the centralizer of N in G. But the centralizer of N is a p-group so that $T = 1$. Hence $[G, \Theta]$ is a p-group, proving that (ii) is a consequence of (i).

By §3, Lemma 1, $[G/N, \Theta] = N[G, \Theta]/N \simeq [G, \Theta]/(N \cap [G, \Theta])$. This shows the equivalence of (ii) and (iii), since N is a p-group.

That finally (ii) implies (i), is an immediate consequence of Theorem 2.

If we denote by $\Sigma(p)$ the property that the set of elements of order prime to p be a subgroup ($= Pp$-closure), then one derives from Theorem 2 without difficulty that

$$[G, \Theta, p] = [[G, \Theta], \Sigma(p)] \textit{ for every p-separated group } G,$$

since $[G, \Theta, p] \leqq [G, \Theta]$ has been shown to be true for every group G.

7. Extension and localisation

If Δ is a group theoretical property and n an integer, then we formulate the following extension property of Δ.

(E.n) *If N is a normal subgroup of the group G, and if G/N and $\{N, x_1, \cdots, x_n\}$ for every n-tuplet of elements x_i in G are Δ-groups, then G is a Δ-group.*

The condition that G/N be a Δ-group is necessary whenever Δ is homorphism-invariant; and the condition that every $\{N, x_1, \cdots, x_n\}$ be a Δ-group is necessary whenever Δ is subgroup-inherited.

Extensions of soluble groups by soluble groups are soluble; and so solubility meets requirement (E.0). It is a consequence of §4, Proposition 1 that Σ-closure meets requirement (E.1); and it is quite easy to construct examples showing that Σ-closure does not satisfy (E.0), except if Σ is vacuous or the set of all primes. In §11 it will be shown that supersolubility meets requirement (E.2), but not (E.1). Moreover to every positive integer n there exists a property satisfying (E.$n+1$), but not (E.n).

PROPOSITION 1. *Assume that the group theoretical property Δ is homomorphism-invariant and meets requirement* (E.n). *Then every subgroup of the group G is a Δ-group, if* (*and only if*)

(a) *there exists a soluble normal subgroup N of G such that every subgroup of G/N is a Δ-group and*

(b) *every* ($n + 1$)*-tuplet of elements in G generates a Δ-subgroup of G.*

Proof. If our proposition were false, then there would exist a group G of minimal order with the following properties:

(1) Not every subgroup of G is a Δ-group.

(2) Conditions (a) and (b) are satisfied by G.

As in (a) we shall denote by N a soluble normal subgroup of G such that every subgroup of G/N is a Δ-group. If S is a subgroup of G, then $N \cap S$ is a soluble normal subgroup of S; and $S/(S \cap N)$ is isomorphic to the subgroup NS/N of G/N. Thus (a) and (b) are satisfied by S too. If next σ is a homomorphism of G onto a group H, then N^σ is a soluble normal subgroup of $G^\sigma = H$; and H/N^σ is a homomorphic image of G/N. Since Δ is homomorphism-invariant, one sees that (a) and (b) are satisfied by H too. Because of the minimality of G it follows now that

(3) every subgroup of every proper subgroup of G and every subgroup of every proper homomorphic image of G is a Δ-group.

By (3) every proper subgroup of G is a Δ-group. By (1) therefore

(4) G is not a Δ-group.

Since every subgroup of G/N is a Δ-group, (4) implies $N \neq 1$. Consequently there exists a minimal normal subgroup M of G which is part of N. Then M is soluble; and application of §2, Lemma 1 shows that

(5) M is an elementary abelian p-group.

Assume by way of contradiction that G/M is generated by n elements. Then there exists a subgroup S of G which is generated by n elements such that $G = MS$. If $t \neq 1$ is an element in M, then the minimal normal subgroup $M = \{t^G\} = \{t^{MS}\} = \{t^S\}$, since M is abelian. Consequently $G = MS = \{t^S\}S = \{t, S\}$ so that G is generated by $n + 1$ elements. By (b), G is a Δ-group, contradicting (4). Hence

(6) G/M is not generated by n elements.

By (3), G/M is a Δ-group. If $x_1, \cdots, x_n$ are elements in G, then, by (6), $\{M, x_1, \cdots, x_n\} < G$. By (3), $\{M, x_1, \cdots, x_n\}$ is therefore a Δ-group. We apply the extension property (E.n) to show that G itself is a Δ-group, contradicting (4). Thus we have arrived at a contradiction which proves the validity of our proposition.

Remark 1. Solubility meets requirement (E.0). On the other hand every cyclic subgroup of every group is soluble. Thus it is impossible to omit condition (a). The indispensability of (b) need hardly be mentioned.

Remark 2. It is a famous conjecture of Burnside that every simple group may be generated by two elements. If this conjecture were true, then a group would be soluble if each pair of elements generates a soluble subgroup. Thus the truth of Burnside's conjecture would imply that condition (a) is a consequence of condition (b) whenever $0 < n$ and Δ-groups are soluble.

Condition (b) is just the special case $j = n + 1$ of the following localisation property:

(L.j) *The group G is a Δ-group, if every j-tuplet of elements in G generates a Δ-subgroup of G.*

One verifies easily that Δ is subgroup-inherited and meets requirement (L.j) if, and only if, Δ meets the following slightly stricter requirement:

(L'.j) The group G is a Δ-group if, and only if, every j-tuplet of elements in G generates a Δ-subgroup of G.

The class of Σ-groups, for Σ a set of primes, meets requirement (L.1); the classes of abelian and Σ-closed groups meet requirement (L.2), but not (L.1);

and supersolubility is an example of a property meeting requirement (L.3), but not (L.2); see §11. Moreover to every positive integer n there exists a property satisfying (L.$n+1$), but not (L.n).

If Δ is subgroup-inherited, then (L.n) clearly implies (E.n); and Proposition 1 is a weak form of a criterion asserting that (E.n) implies (L.$n+1$).

PROPOSITION 2. *If the strictly homomorphism-invariant and subgroup-inherited property* Θ *meets requirement* (L.n), *and if the characteristic* $\mathfrak{K}$ *is not a prime, then* (E.n) *is satisfied by* $(\Theta, \mathfrak{K})$.

Proof. If this were false, then there would exist a group G of minimal order with the following two properties:

(1) G is not a $(\Theta, \mathfrak{K})$-group.

(2) There exists a normal subgroup N of G such that G/N and every subgroup $\{N, x_1, \cdots, x_n\}$ of G is a $(\Theta, \mathfrak{K})$-group.

Since $(\Theta, \mathfrak{K})$ is homomorphism-invariant, every homomorphic image of G meets requirement (2). Because of the minimality of G it follows that

(3) every proper homorphic image of G is a $(\Theta, \mathfrak{K})$-group.

Because of (1) and (3) we may apply §6, Proposition 1 and §3, Proposition 2 to show that

(4) there exists one and only one minimal normal subgroup M of G.

Combining (1) and (2) we see that $N \neq 1$. Applying (1), (3), and (4), it follows that

(5) $M = [G, \Theta, \mathfrak{K}] \leqq N$.

A combination of (1), (3), and (4) shows next that

(6) the characteristic of M is $\mathfrak{K}$ and the group of automorphisms, induced in M by elements in G, is not a Θ-group.

Since $\mathfrak{K}$ is not a prime, M is, by (6), not abelian. Hence M is not contained in its centralizer in G; and this implies, by (4), that the centralizer of M in G i 1, since centralizers of normal subgroups are normal subgroups. Consequently G is essentially the same as the group of automorphisms, induced in M by elements in G. By (6), G is consequently not a Θ-group. Thus we have shown that

(7) the centralizer of M in G is 1 and G is not a Θ-group.

Suppose now that the subgroup S of G is generated by n elements. Application of (2) shows that NS is a $(\Theta, \mathfrak{K})$-group. Since M is part of NS, and since $[NS, \Theta]$ is a characteristic subgroup of NS, $M \cap [NS, \Theta]$ is a normal subgroup of NS and of M. Since the characteristic of M is $\mathfrak{K}$, M and its normal subgroups are direct products of simple groups of type $\mathfrak{K}$. Thus $M \cap [NS, \Theta] \neq 1$ would imply the existence of a minimal normal subgroup of characteristic $\mathfrak{K}$ of $[NS, \Theta]$. Thus condition (ii) of §6, Corollary 3 would not be satisfied by NS so that NS would not be a $(\Theta, \mathfrak{K})$-group, contradicting (2). Hence $M \cap [NS, \Theta] = 1$. Since M and $[NS, \Theta]$ are normal subgroups of NS, $[NS, \Theta]$ is part of the centralizer of M. By (7), $[NS, \Theta] = 1$ so that NS is a Θ-group. Since Θ is subgroup-inherited, S is a Θ-group. Thus every n-tuplet of elements in G generates a Θ-subgroup of G. Since (L.n) is satisfied by Θ, G is a Θ-group, contradicting (7). Consequently we have arrived at the desired contradiction which proves our proposition.

LEMMA 1. *If the strictly homomorphism-invariant and subgroup-inherited property Θ meets requirement* (L.n), *then*

$$[G, \Theta] = \prod_S [S[G, \Theta], \Theta] \text{ for every group } G$$

where the product ranges over all subgroups S of G which are generated by n elements.

Proof. If S is a subgroup of G, then $[G, \Theta]S/[G, \Theta]$ is a Θ-group as a subgroup of the Θ-group $G/[G, \Theta]$. Consequently $[S[G, \Theta], \Theta] \leqq [G, \Theta]$. Since $[S[G, \Theta], \Theta]$ is a characteristic sugroup of $S[G, \Theta]$, it follows now that $[S[G, \Theta], \Theta]$ is a normal subgroup of $[G, \Theta]$ for every subgroup S of G. Consequently we may form the product

$$W = \prod [S[G, \Theta], \Theta]$$

where S ranges over all the subgroups S of G which are generated by n elements; and it is clear that W is a normal subgroup of $[G, \Theta]$. If σ is an automorphism of G, then $[S[G, \Theta], \Theta]^\sigma = [S^\sigma[G, \Theta], \Theta]$ for every subgroup S of G, since $[G, \Theta]$ is a characteristic subgroup of G; and now it is clear that W is a characteristic subgroup of G.

If the subgroup U of G/W is generated by n elements, then there exists a subgroup S of G which is generated by n elements such that $U = WS/W$. Since

$$[S[G, \Theta], \Theta] \leqq W \leqq WS \leqq [G, \Theta]S,$$

$U = WS/W$ is a homomorphic image of a subgroup of the Θ-group $S[G, \Theta]/[S[G, \Theta], \Theta]$. Hence U itself is Θ-group. Thus every n-tuplet of elements in G/W generates a Θ-subgroup of G/W. Application of (L.n) shows that G/W is a Θ-group. Hence $[G, \Theta] \leqq W$ and consequently $[G, \Theta] = W$, Q.E.D.

Remark 3. It is worth recalling that each of the $[S[G, \Theta], \Theta]$ is a normal subgroup of $[G, \Theta]$.

COROLLARY 1. *Assume that the strictly homomorphism-invariant and subgroup-inherited property Θ meets requirement* (L.n) *and that G is p-separated. Then*

(a) *G is a (Θ, p)-group if, and only if, $S[G, \Theta]$ is a (Θ, p)-group whenever the subgroup S of G is generated by n elements*; *and*

(b) *$[G, \Theta]$ is a p-group if, and only if, $[S[G, \Theta], \Theta]$ is a p-group whenever the subgroup S of G is generated by n elements.*

Remark 4. It is an immediate consequence of §6, Theorem 2 that G is a (Θ, p)-group in case $[G, \Theta]$ is a p-group.

Proof. It is a consequence of §6, Corollary 4 that subgroups of p-separated (Θ, p)-groups are p-separated (Θ, p)-groups.

If G is a (Θ, p)-group, then all its subgroups are (Θ, p)-groups, proving the necessity of the condition given under (a). If conversely $S[G, \Theta]$ is a (Θ, p)-group whenever S is generated by n elements in G, then the elements of order prime to p in $[S[G, \Theta], \Theta]$ form a subgroup of order prime to p [§6, Theorem 2]. The same is true then of the product of all these subgroups $[S[G, \Theta], \Theta]$; and this product is, by Lemma 1, just $[G, \Theta]$. By §6, Theorem 2 therefore, G is a (Θ, p)-group.

That (b) is a consequence of Lemma 1, is almost obvious.

PROPOSITION 3. *If Θ is strictly homomorphism-invariant and subgroup-inherited, then the following two properties (of Θ and the prime p) are equivalent*:

(a) (E.n) *is satisfied by the class of p-separated (Θ, p)-groups.*

(b) *If G is a p-separated (Θ, p)-group, and if $[S, \Theta]$ is a p-group whenever the subgroup S of G is generated by n elements, then $[G, \Theta]$ is a p-group.*

Proof. Assume first the validity of (a); and consider a p-separated (Θ, p)-group G such that $[S, \Theta]$ is a p-group whenever the subgroup S of G is generated by n elements. It is easy to construct elementary abelian p-groups possessing a group of automorphisms isomorphic to G; and consequently there exists an extension E of an elementary abelian p-group A such that A is its own centralizer in E and $E/A \simeq G$. In particular therefore E/A is a p-separated (Θ, p)-group; and this implies that E and all its subgroups are p-separated. Consider now a subgroup S of E which contains A and which is generated modulo A by n elements. This is equivalent to saying that the subgroup S/A of E/A is generated by n elements. Since $E/A \simeq G$, this implies that $[S/A, \Theta]$ is a p-group. Since A is its own centralizer in the p-separated group S and A is a p-group, we may apply §6, Corollary 5 to show that S is a p-separated (Θ, p)-group. Hence we may apply (E.n) on E to show that E is a p-separated (Θ, p)-group. Since the normal p-subgroup A of E equals its own centralizer in E, we may apply §6, Corollary 5 again to show that $[E/A, \Theta]$ is a p-group. But $E/A \simeq G$ so that $[G, \Theta]$ is a p-group; and thus we have shown that (b) is a consequence of (a).

Assume next the validity of (b). If it were not true that then (E.n) is satisfied by the class of p-separated (Θ, p)-groups, then there would exist a group G of minimal order with the following two properties:

(1) G is not a p-separated (Θ, p)-group.

(2) There exists a normal subgroup N of G such that G/N and $\{N, x_1, \cdots, x_n\}$ for x_i in G is a p-separated (Θ, p)-group.

It is a consequence of (2) that N and G/N are p-separated. Since extensions of p-separated groups by p-separated groups are p-separated,

(3) G is p-separated;

and this implies in combination with (1) that

(1′) G is not a (Θ, p)-group.

It is clear that every homomorphic image of G satisfies condition (2) too; and thus it follows from the minimality of G and (1′) that

(4) every proper homomorphic image of G is a (p-separated) (Θ, p)-group.

Because of (1) and (4) we may apply §6, Proposition 1 and §3, Proposition 2 to show that

(5) there exists one and only one minimal normal subgroup M of G.

Combining (1) and (2) we see that $N \neq 1$. Applying (1′), (4), and (5), it follows that

(6) $M = [G, \Theta, p] \leqq N$;

and a combination of (1′), (4), and (5) shows next that

(7) the characteristic of M is p and the group of automorphisms, induced in M by elements in G, is not a Θ-group.

$\Phi(G) \neq 1$ would imply (by (4)) that $G/\Phi(G)$ is a (Θ, p)-group; and this would imply, by §6, Theorem 2, that the p-separated group G is a (Θ, p)-group. This contradicts (1′). Hence

(8) $\Phi(G) = 1$.

Consequently there exists a maximal subgroup U of G which does not contain M. This implies $U_G = 1$, since M is, by (5), the one and only one minimal normal subgroup of G. We deduce from §2, Lemma 1 and (7) that M is an elementary abelian p-group; and thus it follows from §2, Lemma 2 that

(9) M is its own centralizer in G.

Suppose now that the subgroup S/M of G/M is generated by n elements. Then the subgroup NS/N of G/N is generated by n elements, since $M \leqq N$. We apply (2) to see that NS is a p-separated (Θ, p)-group. Since Θ is strictly homomorphism-invariant and subgroup-inherited, the same is true of the property of being a p-separated (Θ, p)-group [§6, Corollary 4]. Thus the subgroup S of NS is a p-separated (Θ, p-group. By (9), M is its own centralizer in S; and clearly M is a normal p-subgroup of S; see (7). Hence we may apply §6, Corollary 5 to see that $[S/M, \Theta]$ is a p-group. Thus we have shown that G/M is a p-separated (Θ, p)-group (by (4)) and $[U, \Theta]$ is a p-group whenever the subgroup U of G/M is generated by n elements. We apply condition (b) to show that

(10) $[G/M, \Theta]$ is a p-group.

Since M is a normal p-subgroup of the p-separated group G, and since M is its own centralizer in G, we may apply §6, Corollary 5; and it follows because of (10) that G is a p-separated (Θ, p)-group, contradicting (1). This contradiction shows that (a) is a consequence of (b), as we wanted to prove.

8. Soluble groups with nilpotent Θ-commutator group

Combination of the various elementary properties discussed in the preceding sections leads to new properties of groups. Combining the properties of the elementary classes of groups we obtain properties of the more complex ones; and these combinations lead to sharper results than might be expected offhand. The classes of groups to be discussed in the present section, though still of considerable generality, provide a quite striking example for the applicability of this principle.

THEOREM 1. *Assume that* Θ *is strictly homomorphism-invariant.*

(a) *The group* G *is a* $(\Theta, \mathfrak{K})$*-group for every characteristic* $\mathfrak{K}$ *if, and only if,* $[G, \Theta]$ *is nilpotent.*

(b) *Nilpotency of* $[G, \Theta]$ *is a strictly homomorphism-invariant property.*

The proof of (a) is easily derived from §6, Theorem 1, if one remembers only the characteristic property of nilpotency: The group N is nilpotent if, and only if, every minimal normal subgroup of every homomorphic image of N is contained in the center. (*b*) is readily deduced from (a), since, by §6, Proposition 1, every property $(\Theta, \mathfrak{K})$ is strictly homomorphism-invariant.

It might be well to remember that a group G is, by definition, a $(\Theta, \mathfrak{K})$-group for every characteristic $\mathfrak{K}$ if, and only if, every homomorphic image of G induces in each of its minimal normal subgroups a Θ-group of automorphisms.

THEOREM 2. *If* Θ *is strictly homomorphism-invariant, then the following properties of the group* G *are equivalent:*

(i) $[G, \Theta]$ *is nilpotent and* $G/[G, \Theta]$ *is soluble.*

(ii) G *is, for every prime* p*, a* p*-separable* (Θ, p)*-group.*

(iii) *If the homomorphic image* H *of* G *possesses one and only one minimal normal subgroup, then every maximal subgroup* S *of* H *with core* $S_H = 1$ *is a* Θ*-group; and there exists a prime* p *such that* $[H:S]$ *is a multiple of* p *whenever* S *is a maximal subgroup of* H *whose core* $S_H = 1$.

(iv) *If S is a maximal subgroup of G, then S/S_G is a Θ-group; and maximal subgroups with equal core are conjugate in G.*

(v) $[G/\Phi(G), \Theta]$ *is nilpotent and* $G/\Phi(G)$ *is soluble.*

Remark 1. If we recall that the Fitting subgroup $F(G)$ of G is nilpotent and contains every nilpotent normal subgroup of G, then we may restate (i) in the following more striking form.

(i′) $G/F(G)$ *is a soluble* Θ-*group.*

Remark 2. If we assume that every Θ-group is soluble, then we may omit the solubility requirements from (i), (i′), and (v). Similarly the second half of condition (iv) may be dropped if we make the blanket hypothesis that G be soluble.

Proof. Since groups are soluble if, and only if, they are p-separable for every prime p, and since extensions of soluble groups by soluble groups are soluble, the equivalence of (i) and (ii) is a consequence of Theorem 1, (a).

Assume next the validity of the equivalent conditions (i) and (ii); and consider a homomorphic image H of G possessing maximal subgroups with core 1. Then H is soluble and its minimal normal subgroups are abelian [§2, Lemma 1]. Consequently H possesses one and only one minimal normal subgroup [§2, Corollary 2, (c)]. We apply §6, Theorem 2, (iii) to show that the maximal subgroups of H with core 1 are Θ-groups. Hence S/S_G is a Θ-group for every maximal subgroup S of G. That maximal subgroups of a soluble group are conjugate, if their cores are equal, is a consequence of §5, Corollary 1; and thus we have shown that (iv) is implied by the equivalent conditions (i) and (ii).

It is fairly obvious that (iv) implies (iii); and that (iii) implies (ii), may be deduced from §2, Lemma 3 and §6, Theorem 2. The equivalence of (i) and (iv) implies finally the equivalence of (i) and (v).

COROLLARY 1. *If Θ is strictly homomorphism-invariant, and if Θ-groups are nilpotent, then the following properties of the group G are equivalent*:

(i) $[G, \Theta]$ *is nilpotent.*

(ii) *Every homomorphic image of G induces in each of its minimal normal subgroups a Θ-group of automorphisms.*

(iii) *S/S_G is a Θ-group for every maximal subgroup S of G.*

(iv) $[G/\Phi(G), \Theta]$ *is nilpotent.*

Proof. Since $G/[G, \Theta]$ is a Θ-group and consequently nilpotent, our present condition (i) is equivalent to condition (i) of Theorem 2.

Consider next a minimal normal subgroup M of the homomorphic image H of G. If H induces in M a Θ-group of automorphisms, then H induces in M a nilpotent group of automorphisms. Thus in particular the group of inner automorphisms of M is nilpotent so that the minimal normal subgroup

M of H is nilpotent. Application of §2, Lemma 1 shows that M is an elementary abelian p-group. Now one sees without difficulty the equivalence of our present condition (ii) to condition (ii) of Theorem 2.

It is clear that condition (iv) of Theorem 2 implies our present condition (iii). If conversely our present condition (iii) is satisfied by G, then S/S_G is nilpotent for every maximal subgroup S of G. Application of §2, Lemma 5 shows the solubility of G/S_G for every maximal subgroup S of G. We apply §2, Lemma 4, (b) to show that maximal subgroups of equal core are conjugate in G. Consequently our present condition (iii) is equivalent to condition (iv) of Theorem 2.

On the basis of the preceding remarks the equivalence of conditions (i), (ii), and (iii) is an immediate consequence of Theorem 2; and the equivalence of (i) and (iii) implies the equivalence of (i) and (iv).

To obtain a rather striking application of this result, we need the following simple

LEMMA 1. *If an abelian group of automorphisms is induced by the elements of the group G in the minimal normal subgroup M of G, then M is an elementary abelian p-group, for p a prime, and the induced group of automorphisms is cyclic of order prime to p.*

Remark. This result and its proof may also be found in Huppert [1]. We include them here for the reader's convenience.

Proof. The centralizer M^* of M in G is a normal subgroup of G, and G/M^* is essentially the same as the group of automorphisms of M which are induced in M by elements in G. By hypothesis, G/M^* is abelian. If M were not abelian, then $M \not\leqq M^*$ so that $M \cap M^* = 1$ because of the minimality of M. But then $M \simeq M^*M/M^*$ would be abelian; and this would imply $M \leqq M^*$. This contradiction shows that M is abelian and that M is part of M^*. Application of §2, Lemma 1 shows that $M^p = 1$ for some prime p.

The automorphisms induced in the abelian group M by elements in G span a ring E of endomorphisms of M. Since the group of automorphisms, induced by G, is abelian, the ring E is commutative. Since M is a minimal normal subgroup of G, there do not exist E-admissible subgroups of M, except 1 and M. Thus we may apply Schur's Lemma on the commutative ring E and find that E is a finite field. The characteristic of E is p, since M is an elementary abelian p-group. Thus the multiplicative group of E is a cyclic group of order prime to p. Since the group of automorphisms, induced in M by G, is a subgroup of the multiplicative group of E, the group of automorphisms is likewise a cyclic group of order prime to p.

COROLLARY 2. *The following properties of the group G are equivalent:*

(i) *G' is nilpotent.*

(ii) *Every homomorphic image of G induces in each of its minimal normal subgroups a cyclic group of automorphisms.*

(iii) *If S is a maximal subgroup of G, then S/S_G is cyclic.*

(iv) *If S is a maximal subgroup of G, then S/S_G is abelian.*

(v) $(G/\Phi(G))'$ *is nilpotent.*

Proof. Denote by $\mathfrak{A}$ the class of abelian groups. Then $[G, \mathfrak{A}] = G'$. The equivalence of conditions (i), (iv), and (v) is therefore an immediate consequence of Corollary 1.

If $[G, \mathfrak{A}] = G'$ is nilpotent, then, by Corollary 1, (ii), an abelian group of automorphisms is induced by every homomorphic image of G in each of its minimal normal subgroups; and an immediate application of Lemma 1 shows that these groups of automorphisms are cyclic. Thus (ii) is a consequence of (i).

Assume next the validity of (ii); and consider a maximal subgroup S of G. There exists a minimal normal subgroup M of G/S_G ; and G/S_G induces in M a cyclic group of automorphisms, by (ii). The group of inner automorphisms of M is therefore cyclic too; and this implies the commutativity of M. Since M is not part of S/S_G , the maximal subgroup S/S_G of G/S_G is a complement of the abelian minimal normal subgroup M [§2, Lemma 1]; and since the core of S/S_G is 1, M is its own centralizer [§2, Lemma 2]. Thus S/S_G , $(G/S_G)/M$, and the cyclic group of automorphisms, induced in M by G/S_G, are isomorphic groups; and this shows that (iii) is a consequence of (ii). Since (iv) is a consequence of (iii), and (i), (iv), (v) are equivalent, we have completed the proof of the equivalence of properties (i) to (v).

Remark 3. If A and B are normal subgroups of the group G, then AB and its commutator subgroup are normal subgroups of G too. We note next that $(AB)' = A' [A, B] B'$. The commutator subgroup of AB is consequently nilpotent if, and only if, A', B', *and* $[A, B]$ are nilpotent. It is clear now that products of normal subgroups with nilpotent commutator subgroup need not have nilpotent commutator subgroups; and it is not difficult to construct examples substantiating this remark; see §11, Examples 1, 2. The above remark makes it clear, however, that the commutator subgroup of AB is nilpotent in case A and B' are nilpotent; and this shows in particular that the Fitting subgroup $F(G)$ is contained in every normal subgroup of G which is maximal with respect to the property of having a nilpotent commutator subgroup. If we denote by $P(G)$ the product of all normal subgroups of G whose commutator subgroup is nilpotent, then $F(G) \leqq P(G)$ and $P(G)/F(G)$ is the product of all abelian normal subgroups of $G/F(G)$. If we denote by $J(G)$ the intersection of all normal subgroups of G which are maximal with respect to the property of having a nilpotent commutator subgroup, then $F(G) \leqq J(G)$ and $J(G)/F(G) = Z[P(G]/F(G)]$, since the center of a product of abelian normal subgroups is exactly the intersection of its maximal abelian normal subgroups.

If the property Θ is strictly homomorphism-invariant and subgroup-inherited, then the class of soluble groups with nilpotent Θ-commutator subgroup is likewise strictly homomorphism-invariant [Theorem 1,(b)] and subgroup-inherited [§6, Corollary 4].

THEOREM 3. *If* Θ *is strictly homomorphism-invariant and subgroup-inherited, then the following properties of* Θ *are equivalent:*

(i) (E.n) *is satisfied by the class of soluble groups with nilpotent* Θ*-commutator subgroup.*

(ii) *If* G *is soluble,* $[G, \Theta]$ *nilpotent, and every* n*-tuplet of elements in* G *generates a* Θ*-subgroup of* G*, then* G *is a* Θ*-group.*

(iii) *If* G *is soluble and every* n*-tuplet of elements in* G *generates a* Θ*-subgroup of* G*, then* G *is a* Θ*-group.*

Proof. Assume first the validity of (i) and consider a soluble group G whose Θ-commutator subgroup $[G, \Theta]$ is nilpotent and whose n-tuplets generate Θ-subgroups. If p is a prime, then one constructs easily (and in many ways) a group E possessing a normal subgroup P with the following properties:

P is an elementary abelian p-group;

P is its own centralizer in E;

$E/P \simeq G$.

It is clear that E is soluble, since P and G are soluble. If the subgroup S/P of E/P is generated by n elements, then S/P is a Θ-group, since n-tuplets of elements in $G \simeq E/P$ generate Θ-subgroups. Hence $[S, \Theta] \leqq P$ so that $[S, \Theta]$ is abelian and consequently nilpotent. Thus we have shown that E/P and $\{P, x_1, \cdots, x_n\}$ for x_i in E are soluble groups with nilpotent Θ-commutator subgroup. But the class of soluble groups with nilpotent Θ-commutator subgroup meets requirement (E.n) by (i); and this implies that $[E, \Theta]$ is nilpotent. The elements of order prime to p in $[E, \Theta]$ form consequently a characteristic subgroup of E; and this implies that they belong to the centralizer of the normal p-subgroup P of E. But P is its own centralizer in E; and thus it follows that $[E, \Theta]$ is a p-group. But then $P[E, \Theta]/P$ is a p-group too. This group is, by §3, Lemma 1, identical with $[E/P, \Theta]$; and now it follows that $[G, \Theta]$ is a p-group. But the only group which is a p-group for every prime p is the identity. Hence $[G, \Theta] = 1$ so that G is a Θ-group, since Θ is strictly homomorphism-invariant. Hence (ii) is a consequence of (i).

Assume next the validity of (ii); and consider a soluble group G whose n-tuplets generate Θ-subgroups. Assume by way of contradiction that $[G, \Theta] \neq 1$. Since G is soluble, so is $[G, \Theta]$; and this implies that $[G, \Theta]' < [G, \Theta]$. The commutator subgroup of the Θ-commutator subgroup is a characteristic subgroup; and thus we may form $H = G/[G, \Theta]'$. Then every n-tuplet of elements in H generates a Θ-subgroup, since Θ is homomorphism-invariant; and H is soluble, since G is soluble. By §3, Lemma 1, we have $[H, \Theta] = [G, \Theta]/[G, \Theta]'$ so that $[H, \Theta]$ is an abelian group, not 1. We apply condition (ii) to see that H is a Θ-group; and this implies that $[H, \Theta] = 1$, a contradiction. Thus $[G, \Theta] = 1$ so that G is a Θ-group, since Θ is strictly homomorphism-invariant. Hence (iii) is a consequence of (ii).

Assume now the validity of (iii). If (i) were not a consequence of (iii), then there would exist a group G of minimal order with the following properties:

(1) G is not a soluble group with nilpotent Θ-commutator subgroup.

(2) There exists a normal subgroup N of G such that G/N and $\{N, x_1, \cdots, x_n\}$ for x_i in G are soluble groups with nilpotent Θ-commutator subgroups.

This implies in particular that N and G/N are soluble; and thus it follows that

(3) G is soluble, whereas $[G, \Theta]$ is not nilpotent.

It is clear that every homomorphic image of G has property (2). This implies, because of the minimality of G, that

(4) the Θ-commutator subgroup of every proper homomorphic image of G is nilpotent.

Since the class of soluble groups with nilpotent Θ-commutator subgroup is strictly homomorphism-invariant, application of §3, Proposition 2 shows that

(5) there exists one and only one minimal normal subgroup M of G.

By (2) and (3) both N and $[G, \Theta]$ are different from 1. By (5), consequently

(6) $M \leqq [G, \Theta] \cap N$.

If $\Phi(G)$ were not 1, then $G/\Phi(G)$ would be a soluble group with nilpotent Θ-commutator subgroup (by (3) and (4)). By Theorem 2, (v), G itself would be a soluble group with nilpotent Θ-commutator subgroup, contradicting (1). Hence

(7) $\Phi(G) = 1$.

Since M is a soluble minimal normal subgroup of G, and since $\Phi(G) = 1$, application of (5) and §2, Lemmas 1 and 2 shows that

(8) M is an elementary abelian p-group, for p a prime; there exist maximal subgroups of G whose core is 1 and which are complements of M in G; M is its own centralizer in G.

Thus G/M is essentially the same as the group of automorphisms induced in M by elements in G. If this group were a Θ-group, then a combination of

(4), (5), and Theorem 2 would show that G is a soluble group with nilpotent Θ-commutator subgroup, contradicting (1). Hence

(9) G/M is not a Θ-group.

Assume now that the subgroup S/M of G/M is generated by n elements. Since $M \leqq N$, by (6), NS is, by (2), a soluble group with nilpotent Θ-commutator subgroup. Since the class of soluble groups with nilpotent Θ-commutator subgroup is subgroup-inherited, $[S, \Theta]$ is nilpotent. The elements of order prime to p in $[S, \Theta]$ form therefore a characteristic subgroup T of S whose order is prime to p. Since M is a normal p-subgroup of S, and since $M \cap T = 1$, T is part of the centralizer of M which is, by (8), equal to M. Hence $T = 1$; and we have shown that $[S, \Theta]$ is a p-group. Since S/M is a subgroup of G/M, and since Θ is subgroup-inherited and strictly homomorphism-invariant, $[S/M, \Theta] \leqq [G/M, \Theta]$. By §3, Lemma 1, $[S/M, \Theta] = M[S, \Theta]/M$ is a p-group. By (4), $[G/M, \Theta]$ is nilpotent so that the set of p-elements in $[G/M, \Theta]$ is a characteristic p-subgroup W/M of G/M. Since M is a p-group, so is W. Hence $M \cap Z(W) \neq 1$. But $Z(W)$ is a normal subgroup of G as a characteristic subgroup of W. Consequently $M \leqq Z(W)$ so that W is part of the centralizer of M. This implies $M = W$ by (8). Since the p-subgroup $[S/M, \Theta]$ of $[G/M, \Theta]$ is a subgroup of $W/M = 1$, we have shown that $[S/M, \Theta] = 1$ and that therefore S/M is a Θ-group. Hence every n-tuplet of elements in the soluble group G/M generates a Θ-subgroup; and this implies, by (iii), that G/M is a Θ-group, contradicting (9). Thus we have arrived at a contradiction which completes the proof of the equivalence of properties (i) to (iii).

COROLLARY 3. *If* Θ *is strictly homomorphism-invariant and subgroup-inherited, and if* (L.n) *is satisfied by the class of soluble* Θ*-groups, then* (E.n) *is satisfied by the class of soluble groups with nilpotent* Θ*-commutator subgroup.*

This is an immediate consequence of Theorem 3. Cp. §7, Proposition 1 in this context.

Remark 4. The class $\mathfrak{A}$ of abelian groups is strictly homomorphism-invariant, subgroup-inherited; and a group is abelian if, and only if, every pair of its elements generates an abelian subgroup. Thus Corollary 3 may be applied, proving that (E.2) is satisfied by the class of groups with nilpotent commutator subgroup. A similar remark may be made concerning the class of nilpotent groups etc.

Derived properties

If Θ is a group theoretical property, then we denote by Θ' *the class of groups with nilpotent* Θ*-commutator subgroups.* If Θ is strictly homomorphism-invariant and subgroup-inherited, and if every Θ-group is soluble, then Θ' is strictly homomorphism-invariant and subgroup-inherited and every Θ'-group is soluble. This process may be iterated in an obvious fashion. We want to illustrate this general principle by discussing shortly a particularly interesting instance.

Denote by Δ^0 the identity class, consisting of the group of order 1 only; and define Δ^n inductively by the rule: $\Delta^{n+1} = (\Delta^n)'$. Since $\Delta^0 = 1$ is strictly homomorphism-invariant, subgroup-inherited, and a class of soluble groups, the same is true of every Δ^n.

To obtain interesting characterisations of these classes, we define the iterated Fitting subgroups of a group G by the rules:

$$F_0(G) = 1, \qquad F_{n+1}(G)/F_n(G) = F[G/F_n(G)].$$

It is clear that the $F_n(G)$ form an ascending chain of characteristic subgroups of G; and that G is soluble if, and only if, $G = F_t(G)$ for some t. One proves now without too much difficulty the equivalence of the following properties of the group G and the positive integer n:

(i) $G = F_n(G)$.

(ii) G is a Δ^n-group.

(iii) Every homomorphic image of G induces in each of its minimal normal subgroups a Δ^{n-1}-group of automorphisms.

(iv) If S is a maximal subgroup of G, then S/S_G is a Δ^{n-1}-group; and maximal subgroups of equal core in G are conjugate (are isomorphic, have equal index).

(v) $G/\Phi(G)$ is a Δ^n-group.

9. Dispersed groups

If Θ is a set of sets of primes, then we may term Θ-group any group G which is Σ-closed for every set Σ in Θ. Applying the results of §4 one notes the following properties of this class Θ: it is strictly homomorphism-invariant and subgroup-inherited; it meets requirements (E.1) and (L.2); G is a Θ-group if, and only if, $G/\Phi(G)$ is a Θ-group and products of normal Θ-subgroups are Θ-groups. In the present section we are going to investigate a specialization of this concept which permits a more intensified study. The class of group-theoretical properties which we shall obtain shares to an astonishing degree the properties of the class of nilpotent groups.

Suppose that a partial order σ has been defined in the set Σ of primes. We shall write $p \sigma q$ whenever p and q are different primes in Σ and p precedes q in the partial order σ; if we permit equality, then we shall write $p \overset{\sigma}{=} q$. As Σ is the range of the partial order σ, it will often suffice to refer to this partially ordered set of primes as σ. We admit all possibilities: Σ may be vacuous; Σ may be the set of all primes and $p \not\sigma q$ for every pair of primes; Σ may be the set of all primes and σ just the natural ordering of the primes, and so on.

It will be convenient to term *segment of* σ any subset Θ of Σ which contains with any prime p every prime x satisfying $x \sigma p$. Two special types of segments will prove useful. If p is a prime in Σ, then $\sigma(p)$ is the totality of all primes x in Σ such that $p \not\sigma x$. It is clear that $\sigma(p)$ is a segment and that p belongs to $\sigma(p)$. A second segment arises by omitting p from $\sigma(p)$.

DEFINITION. *If σ is a partially ordered set of primes, and if the group G is Ξ-closed for every segment Ξ of σ, then G is σ-dispersed.*

We mention a few extreme examples of σ-dispersion. If firstly the range Σ of σ is empty, then every group is σ-dispersed; and more generally every group is σ-dispersed whose order is prime to every prime in the range Σ of σ. If next Σ is the set of all primes, and if the partial order σ is the trivial one—i.e. $p \not\sigma q$ for every pair of primes—then σ-dispersion and nilpotency are equivalent concepts. If finally Σ is the set of all primes and σ some complete ordering of the primes, then it may be verified easily—and will be contained in some of our results below—that σ-dispersion amounts to what is also called the Sylow Tower Property [Huppert]; and Ore has termed a group dispersed if it is σ-dispersed (for σ the inverted natural ordering of the primes).

If σ is a partially ordered set of primes and G a group, and if the prime p in Σ is a divisor of the order of G, though for every prime divisor x of the order of G which belongs to Σ we have $x \not\sigma p$, then we say that p is a *σ-minimal G-relevant prime.* The σ-maximal G-relevant primes are defined similarly. It will often be possible to omit σ from this term without danger of confusion; and this we shall do. Finally we denote by $\Sigma(G)$ the set of all *σ-G-relevant* primes; these are the primes in Σ which divide the order of G. Since $\Sigma(G)$ is always finite, the existence of σ-minimal and σ-maximal G-relevant primes is assured, as soon as $\Sigma(G)$ is not vacuous.

THEOREM 1. *If σ is a partial ordering of the set Σ of primes, then the following properties of the group G are equivalent:*

(i) *G is σ-dispersed.*

(ii) *G is Ξ-dissolved for every σ-segment Ξ.*

(iii) *$G/\Phi(G)$ is σ-dispersed.*

(iv) *Every homomorphic image H of G is p-closed for every σ-minimal H-relevant prime p.*

(v) *If H is a homomorphic image of G and p a σ-minimal H-relevant prime, then there exists a normal p-subgroup, not 1, of H.*

(vi) *If H is a homomorphic image of G, then $[H:F(H)]$ is prime to every σ-minimal H-relevant prime.*

(vii) *If H is a homomorphic image of G and $\Sigma(H)$ not vacuous, then there exists an element $h \neq 1$ in H such that $hx = xh$ for every element x in H whose order is divisible by σ-minimal H-relevant primes only.*

(viii) *If the homomorphic image H of G possesses one and only one minimal normal subgroup M, and if $\Sigma(H)$ is not vacuous, then there exists one and only one σ-minimal H-relevant prime p and M is a p-group.*

(ix) *If S is a maximal subgroup of G and p a σ-minimal (G/S_G)-relevant prime, then $[G:S]$ is a power of p and $[S:S_G]$ is prime to p.*

(x) *If S is a maximal subgroup of G and $\Sigma(G/S_G)$ not vacuous, then there exists one and only one σ-minimal (G/S_G)-relevant prime p, and $[G:S]$ is a multiple of p.*

(xi) *G is Σ-dissolved and a $(P\sigma(p), p)$-group for every prime p in $\Sigma(G)$.*

(xii) *G is Σ-dissolved, and $[G, P\sigma(p)]$ is Pp-closed for every prime p in $\Sigma(G)$.*

(xiii) *G is Σ-closed; and if S is a Σ-subgroup of G, then $[S:S']$ is a multiple of every σ-maximal S-relevant prime.*

(xiv) *G is Σ-closed; and if S is a characteristic Σ-subgroup of G, then $[S:S']$ is a multiple of every σ-maximal S-relevant prime.*

(xv) *G is Σ-closed; and if N is a normal Σ-subgroup of G and p a σ-maximal N-relevant prime, then $[N:[N, G\sigma(p)]]$ is a multiple of p.*

(xvi) *G is Σ-closed; and if N is a characteristic Σ-subgroup of G and p a σ-maximal N-relevant prime, then $[N, G\sigma(p)] < N$.*

(xvii) *Pairs of elements of relatively prime prime power order generate σ-dispersed subgroups of G.*

(xviii) *If the orders of the elements x and y in G are powers of the same σ-G-relevant prime, then $\{x, y\}$ is a σ-dispersed subgroup of G.*

Here as always we have used the following *notations*: The group X is a Σ-group, if every prime divisor of the order of X belongs to Σ; and X is a $P\Sigma$-group, if none of the prime divisors of the order of X belongs to Σ, i.e. if the order of X is prime to Σ. Consequently a group X will be termed $P\Sigma$-closed, if the set of elements of order prime to Σ in X is a $P\Sigma$-subgroup of X. If X is a group and Λ a set of primes, then $X\Lambda$ is the subgroup of X which is generated by the Λ-elements in X.

To connect the properties discussed here with those investigated in §6 it might be worth noting that a group G is Σ-dissolved if, and only if, G is Σ-closed and p-separated for every p in Σ.

If every prime divisor of the order of G is contained in Σ, then several of the above properties may stated in a simpler form. For instance, the requirement of Σ-closure may then be dropped from conditions (xiii) to (xvi); and for the requirement of Σ-dissolution, solubility may be substituted in (xi) and (xii).

Proof. If G is σ-dispersed, then every homomorphic image H of G is σ-dispersed too. If p is a σ-minimal H-relevant prime, then we form the segment $\bar{p}$ of all the primes x in Σ which satisfy $x \overset{\sigma}{\leqq} p$. It is clear that p is the only prime divisor of the order of H which belongs to $\bar{p}$. Since H is σ-dispersed, the set T of $\bar{p}$-elements in H is a characteristic subgroup of H; and since p is the only prime divisor of the order of H which belongs to $\bar{p}$, T is a p-group. Hence H is p-closed; and we have shown that (i) implies (iv).

It is obvious that (iv) implies (v). If (v) is satisfied by G, if p is a σ-minimal H-relevant prime for the homomorphic image H of G, if N is a normal p-subgroup of H and p a divisor of $[H:N]$, then p is a σ-minimal (H/N)-relevant prime; and there exists, by (v), a normal p-subgroup $K/N \neq 1$ of the homomorphic image H/N of G. Thus N is not a maximal normal p-subgroup of H; and now it is clear that (iv) and (v) are equivalent properties.

Since p-groups are nilpotent, and since the Fitting subgroup is the nilpotent characteristic subgroup which contains every nilpotent normal subgroup, it is clear that (iv) implies (vi). Since the center of nilpotent groups, not 1, is different from 1, and since the Fitting subgroup is nilpotent, (vi) implies (vii).

Assume now the validity of (vii) and consider a homomorphic image H of G which possesses one and only one minimal normal subgroup M. Assume furthermore that $\Sigma(H)$ is not vacuous. Then there exist σ-minimal H-relevant primes. Consider one of them, say p; and assume by way of contradiction that M is not a p-group. Form the set A of all elements in H which commute with every p-element in H. It is clear that A is a characteristic subgroup of H; and it is a consequence of (vii) that $A \neq 1$. Since M is the one and only one minimal normal subgroup of H, we have $M \leqq A$. The p-elements in A belong to the p-component W of the center of A. Since W is a characteristic subgroup of the characteristic subgroup A of H, W is a characteristic p-subgroup of H which cannot contain M. Since M is part of every normal subgroup, not 1, of H, $W = 1$; and this implies that the order of A is prime to p. It follows that p is a σ-minimal (H/A)-relevant prime. We apply (vii) to prove the existence of an element $Au \neq 1$ in H/A which commutes with all p-elements in the homomorphic image H/A of G. If x is a p-element in H, then the commutator $[u, x]$ belongs to A so that in particular $x[u, x] = [u, x]x$ and the order of $[u, x]$ is prime to p. However, it follows successively that

$$x^{-1}ux = u[u, x], \qquad x^{-i}ux^{i} = u[u, x]^{i}, \qquad u = x^{-p^k}ux^{p^k} = u[u, x]^{p^k}$$

where p^k is the order of x. Thus $[u, x]^{p^k} = 1$; and this implies $[u, x] = 1$, since the order of $[u, x]$ is prime to p. Hence u commutes with every p-element in H. Thus u belongs to A, contradicting $Au \neq 1$. We have arrived at a contradiction which proves that M is a p-group, if p is any σ-minimal H-relevant prime. Hence (viii) is a consequence of (vii).

Assume next the validity of (viii). To show that (x) is a consequence of (viii), consider a maximal subgroup S of G such that $\Sigma(G/S_G)$ is not vacuous. Then $U = S/S_G$ is a maximal subgroup of $H = G/S_G$ and the core $U_H = 1$. Assume by way of contradiction that A and B are two different minimal normal subgroups of H. It is a consequence of §2, Corollary 2 that A and B are isomorphic non-abelian groups and that A is the centralizer of B in H. If N is a normal subgroup of H such that $A \leqq N$ and $B \not\leqq N$, then $B \cap N = 1$ so that N is part of the centralizer A of B, and hence $A = N$. It follows that AB/A is the one and only one minimal normal subgroup of H/A. Since $A \simeq B \simeq AB/A$, the orders of H and H/A are divisible by the same primes so that in particular $\Sigma(H/A) = \Sigma(H) = \Sigma(G/S_G)$ is not vacuous. We may apply condition (viii) to see that AB/A is a p-group; and this implies, by §2, Lemma 1, that the isomorphic groups A, B, and AB/A are abelian, an impossibility. Consequently H itself possesses one and only one minimal normal subgroup M. We apply condition (viii) to see that there exists one and only one σ-minimal H-relevant prime p and that M is a p-group. Since M is not part of the maximal subgroup U of core 1, and since M is a minimal normal subgroup of H, we deduce from §2, Lemma 1 that M is an elementary abelian p-group and that U is a complement of M. It follows in particular that $[G:S] = [H:U]$ is the order of M and as such a power of p; and thus we have verified the validity of (x).

Assume next the validity of (x); and consider a segment Ξ of σ. To verify the validity of condition (iii) of §4, Proposition 3 we consider a maximal subgroup S of G such that $[G:S_G]$ is a multiple of some prime x in Ξ and such that G/S_G possesses one and only one minimal normal subgroup M. Since x is in Σ, $\Sigma(G/S_G)$ is not vacuous. Application of (x) shows two facts: firstly there exists one and only one σ-minimal (G/S_G)-relevant prime p; and secondly $[G:U]$ is a multiple of p for every maximal subgroup U of G which satisfies $U_G = S_G$. Since the prime x belongs to $\Sigma(G/S_G)$, $p \overset{\sigma}{\leqq} x$; and since x belongs to the σ-segment Ξ, p is in Ξ. Thus condition (ii) of §2, Lemma 3 is satisfied by G/S_G; and this implies the solubility of the one and only one minimal normal subgroup M of G/S_G as well as the fact that $[G:S]$ is a power of p. By §2, Lemma 1, the maximal subgroup S/S_G is a complement of M in G/S_G; and this implies that M is a p-group and hence a Ξ-group. Thus we have verified the validity of condition (iii) of §4, Proposition 3 (with respect to Ξ). Hence G is Ξ-closed for every σ-segment Ξ; and this proves that G is σ-dispersed. The equivalence of conditions (i), (iv), (v), (vi), (vii), (viii) and (x) is now completely verified.

The equivalence of (i) and (iii) is a consequence of §4, Proposition 3, as has been noted before. If N is a normal subgroup of the σ-dispersed group G, and if $\Sigma(G/N)$ is not vacuous, then there exists, by (v), a prime p in $\Sigma(G/N) \leqq \Sigma(G)$ and a normal subgroup M of G such that $N < M$ and M/N is a p-group. If in particular N is a maximal normal soluble Σ-subgroup of G, then it follows from the previous remark that $\Sigma(G/N)$ is vacuous; and thus we have shown that

σ-dispersed groups are Σ-dissolved.

A special consequence of this fact is the equivalence of properties (i) and (ii).

Assume again that G is σ-dispersed. Then G is in particular Σ-dissolved; and this is clearly equivalent to the following property:

G is Σ-closed and p-separated for every p in Σ.

Suppose now that p is in Σ and M a minimal normal subgroup of characteristic p of the homomorphic image H of G. The set T of $\sigma(p)$-elements in the σ-dispersed group H is a $\sigma(p)$-subgroup of H, since $\sigma(p)$ is a σ-segment; and it is clear that this characteristic subgroup T of H contains the p-group M. The primes different from p in $\sigma(p)$ form likewise a σ-segment; and thus it follows that the set S of elements of order prime to p in T is a subgroup of T and H. Clearly S is a characteristic subgroup of order prime to p. Hence $S \cap M = 1$; and this implies that S is part of the centralizer of M in H. The elements in the characteristic subgroup T of H induce therefore in the minimal normal p-subgroup M of H a p-group of automorphisms—note that $[T:S]$ is a power of p—and this implies that T is part of the centralizer of M. Recalling that T contains every $\sigma(p)$-element in H, it follows now that a $P\sigma(p)$-group of automorphisms is induced in M by the elements in H; in other words: G is a $(P\sigma(p), p)$-group for every prime p in $\Sigma(G)$. Hence (xi) is a consequence of (i). The equivalence of conditions (xi) and (xii) may be deduced from §6, Theorem 2.

Assume now the validity of (xii). Suppose that the homomorphic image H of G possesses one and only one minimal normal subgroup M and that $\Sigma(H)$ is not vacuous. Since G is Σ-dissolved, so is H. The set T of all Σ-elements in H is consequently a soluble characteristic Σ-subgroup of H. Since $\Sigma(H)$ is not vacuous, $T \neq 1$; and this implies $M \leqq T$, since M is part of every normal subgroup, not 1, of H. Hence M is a soluble Σ-group. By §2, Lemma 1, M is an elementary abelian p-group for p a prime in $\Sigma(H)$. It is a consequence of §3, Lemma 1 that (xii) is satisfied by the homomorphic image H of G. The elements of order prime to p in $[H, P\sigma(p)]$ form therefore a characteristic subgroup W of order prime to p. Since M is part of every normal subgroup, not 1, of H, $W = 1$. Thus $[H, P\sigma(p)]$ is a p-group and $H/[H, P\sigma(p)]$ is $P\sigma(p)$-group. It follows that p is the one and only one σ-minimal H-relevant prime. Hence (viii) is a consequence of (xii); and thus we have verified the equivalence of (i), (xi), and (xii).

If G is σ-dispersed, S a maximal subgroup of G and p a σ-minimal (G/S_G)-relevant prime, then the set T of p-elements in $H = G/S_G$ is, by (iv), a characteristic p-subgroup of H. Since $T \neq 1$, there exists a minimal normal subgroup of H which is contained in the center of T (as the center of a p-group, not 1, is likewise different from 1). Since S/S_G is maximal subgroup of H whose core is 1, application of §2, Lemmas 1 and 2 and §2, Corollary 1 shows that T itself is a minimal normal p-subgroup of H whereas S/S_G is a complement of T in H. It is clear then that $[G:S]$ is the order of T and hence a power of p, whereas S/S_G is isomorphic to H/T and consequently of order prime to p. Hence (ix) is a consequence of (i). It is almost obvious that (x) is a consequence of (ix); and thus we have completed the proof of the equivalence of conditions (i) to (xii).

If G is σ-dispersed, then G is certainly Σ-closed. If S is a Σ-subgroup of G, then S too is σ-dispersed. Suppose that p is a σ-maximal S-relevant prime. The totality $\bar{p}$ of primes x in Σ which do not satisfy $p \overset{\sigma}{\leqq} x$ is a segment so that the set T of $\bar{p}$-elements in S is a subgroup of S. Clearly $T < S$ and S/T is a p-group; and this implies that $[S:S']$ is a multiple of p. Hence (xiii) is a consequence of (i); and it is trivial that (xiv) is a consequence of (xiii).

Assume again the validity of (xiii); and consider a normal Σ-subgroup N of G and a σ-maximal N-relevant prime p. It is a consequence of (xiii) that $[N:N']$ is a multiple of p. Since N' is a characteristic subgroup of the normal subgroup N of G, N' is a normal subgroup of G; and now one verifies easily the existence of a normal subgroup K of G with the following properties: $N' \leqq K < N$ and N/K is a minimal normal p-subgroup of G/K. It is a consequence of (xi) that the elements in G induce in N/K a $P\sigma(p)$-group of automorphisms. Thus every $\sigma(p)$-element in G/K belongs to the centralizer of N/K. Hence $[N, G\sigma(p)] \leqq K$ so that $[N:K]$ is a divisor of $[N:[N, G\sigma(p)]]$. Thus we have shown that (xv) is a consequence of (xiii); and by a similar argument one sees that (xvi) is a consequence of (xiv). Furthermore it is clear that (xvi) is a consequence of (xv).

Assume now the validity of (xvi). If C is a characteristic Σ-subgroup of G, then we denote by $D(C)$ the intersection of all the normal subgroups X of C with nilpotent quotient group C/X. As a characteristic subgroup of a characteristic subgroup $D(C)$ is a characteristic Σ-subgroup of G. It is clear that $C/D(C)$ is nilpotent; as a matter of fact $D(C)$ is just the terminal member of the descending central chain of C. Consider now a σ-maximal C-relevant prime p. If x is a prime in $\Sigma(C)$, then certainly $p \not\sigma\, x$ so that x belongs to $\sigma(p)$. It follows that C is, as a Σ-group, a $\sigma(p)$-group. Hence $C \leqq G\sigma(p)$. Since $D(C)/[D(C), C] \leqq Z(C/[D(C), C])$, and since $C/D(C)$ is nilpotent, $C/[D(C), C]$ is likewise nilpotent; and this implies $D(C) = [D(C), C]$, since $D(C)$ is contained in every normal subgroup of C with nilpotent quotient group. It follows that

$$D(C) = [D(C), C] \leqq [D(C), G\sigma(p)] \leqq D(C) \text{ or } D(C) = [D(C), G\sigma(p)].$$

If p were a divisor of the order of the characteristic subgroup $D(C)$ of G, then p would be a σ-maximal $D(C)$-relevant prime; and this would imply $[D(C), G\sigma(p)] \neq D(C)$ by (xvi). Hence p is prime to the order of $D(C)$; and thus we have derived from (xvi) the following condition:

(xvi′) *If C is a characteristic Σ-subgroup of G, and if $D(C)$ is the intersection of all the normal subgroups X of C with nilpotent quotient group C/X, then the order of $D(C)$ is prime to every σ-maximal C-relevant prime.*

We note next that (xvi) implies furthermore

(xvi″) *G is Σ-closed.*

We assume now the validity of the conditions (xvi′) and (xvi″). The set S of Σ-elements is, by (xvi″), a characteristic Σ-subgroup of G. Condition (xvi′) implies in particular the following fact: If C is a characteristic Σ-subgroup of G, then $D(C)$ is a characteristic Σ-subgroup of G and $C/D(C)$ is nilpotent; if $C \neq 1$, then $D(C) < C$. By an obvious inductive argument it follows now that S is soluble; and thus we have shown that G is Σ-dissolved.

Consider now some prime p in $\Sigma(G)$. The characteristic subgroup $[G, P\sigma(p)]$ is, by definition, the intersection of all normal subgroups X of G whose index $[G:X]$ is prime to every prime in $\sigma(p)$, i.e. to all primes x in Σ which satisfy $p \not\sigma x$. Recall that S is a characteristic Σ-subgroup of G and that $[G:S]$ is prime to every prime in Σ. Consequently $[G, P\sigma(p)] \leqq S$ so that $[G, P\sigma(p)]$ is a characteristic Σ-subgroup of G.

If q is a prime divisor of the order of the nilpotent group $[G, P\sigma(p)]/D([G, P\sigma(p)])$, then there exists a characteristic subgroup Q of G such that $D([G, P\sigma(p)]) \leqq Q < [G, P\sigma(p)]$ and $[[G, P\sigma(p)]:Q]$ is a power of q whereas $[Q:D([G, P\sigma(p)])]$ is prime to q, since nilpotent groups are direct products of characteristic primary groups. Since G/Q is not a $P\sigma(p)$-group as $Q < [G, P\sigma(p)]$, it follows that q belongs to $\sigma(p)$; and thus we have shown that $p \not\sigma q$ for every prime divisor q of $[[G, P\sigma(p)]:D([G, P\sigma(p)])]$. This index is, by (xvi′), a multiple of every σ-maximal $[G, P\sigma(p)]$-relevant prime m so that $p \not\sigma m$ for all these primes m. This implies the following alternative: either p is not a divisor of the order of $[G, P\sigma(p)]$, or else p is a σ-maximal $[G, P\sigma(p)]$-relevant prime. But in the latter case p is, by (xvi′), not a divisor of the order of $D([G, P\sigma(p)])$; and thus we have shown that p is in neither case a divisor of the order of $D([G, P\sigma(p)])$. Since $C/D(C)$ is, as a nilpotent group, a direct product of characteristic primary groups, it is now easily seen that $[G, P\sigma(p)]$ contains a characteristic subgroup of order prime to p and index a power of p; in other words: $[G, P\sigma(p)]$ is Pp-closed. Thus we have shown that (xii) is a consequence of conditions (xvi′) and (xvi″); and this completes the proof of the equivalence of the conditions (i) to (xvi).

Since subgroups of σ-dispersed groups are likewise σ-dispersed, condition (i) implies both (xvii) and (xviii). If conversely (xviii) is satisfied by the group G, then the same is true for every homomorphic image H of G. If p is a σ-minimal H-relevant prime, and if x and y are p-elements in H, then $\{x, y\}$ is, by (xviii), σ-dispersed. The set of p-elements in $\{x, y\}$ is consequently a subgroup [by (iv)] so that $\{x, y\}$ is a p-group. Hence xy^{-1} is a p-element. The set of p-elements in H is consequently a subgroup of H. Thus (xviii) implies (iv).

Assume finally the validity of (xvii); and consider a homomorphic image H of G and a σ-minimal H-relevant prime p. If x is a p-element in H and y a q-element in H for q a prime different from p, then p is a σ-minimal $\{x, y\}$-relevant prime. Since $\{x, y\}$ is, by (xvii), σ-dispersed, $\{x, y\}$ is p-closed by (iv). Thus condition (a) of §4, Proposition 2 is satisfied by H (and $\Sigma = p$). Since condition (b) of §4, Proposition 2 is automatically satisfied in our case [§4, Remark 1], we may apply §4, Proposition 2 to prove p-closure of H. Thus we have shown that (xvii) implies (iv); and this completes the proof.

Remark. If Δ and Λ are segments of Σ, and if $\Delta \cap \Lambda$ is free of G-relevant primes, then the set D of Δ-elements in G and the set L of Λ-elements in G are characteristic subgroups of the σ-dispersed group G such that $D \cap L = 1$. It follows that every Δ-element in G commutes with every Λ-element in G. This property of σ-dispersed groups will not lead us to a criterion for σ-dispersibility, as may be seen from the special case where σ is a complete ordering of the set of all primes and where our property is satisfied by all groups.

COROLLARY 1. *If σ is a partial ordering of the set of all primes, and if every proper subgroup of the group G is σ-dispersed, then G is soluble.*

Remark. This is a slight generalization of the theorems of Schmidt [1], Iwasawa [1] and Huppert [1]. The following proof is an adaptation of Huppert's proof; see Huppert [1; p. 429, Satz 22].

Proof. If Corollary 1 were false, then there would exist a group G of minimal order with the following two properties:

(1) G is not soluble.

(2) Every proper subgroup of G is σ-dispersed.

Since σ is a partial ordering of the set of all primes, σ-dispersed groups are soluble. In particular every proper subgroup of G is soluble. Every homomorphic image of G has property (2); and thus it follows from the minimality of G that every proper homomorphic image of G is soluble. Since extensions of soluble groups by soluble groups are soluble, one deduces now from (1) that

(3) G is simple.

Since σ is a partial ordering of the set of all primes, there exists a σ-maximal G-relevant prime p. Suppose now that the order of the proper subgroup S of G is divisible by p. The totality $\bar{p}$ of primes $q \neq p$ satisfying $p \not\sigma q$ is a segment that contains every prime divisor, not p, of the order of G. The totality of $\bar{p}$-elements in S is therefore a characteristic subgroup of S; and this implies the following fact, since S is, by (2), σ-dispersed:

(4) If S is a proper subgroup of G, then the set of elements of order prime to p in S is a subgroup $\bar{S}$ of S.

We distinguish now two cases.

Case 1. G is p-normal.

This signifies that the center $Z(S)$ of a p-Sylow subgroup S of G is the center of every p-Sylow subgroup of G which contains $Z(S)$. Consider now a p-Sylow subgroup S of G. Then its center $Z(S) \neq 1$, and it is a consequence of (1) and (3) that $Z(S)$ is not a normal subgroup of G. The normalizer T of $Z(S)$ in G is consequently different from G; and it is clear that $Z(S) \leqq S \leqq T$. We may apply (4). Hence the totality $\overline{T}$ of elements of order prime to p in T is a characteristic subgroup of T and $T/\overline{T} \simeq S$. We apply a Theorem of Grün to show that S is a homomorphic image of G; see Zassenhaus [1; p. 135, Satz 6]. This contradicts (1) and (3).

Case 2. G is not p-normal.

Then we apply a Theorem of Burnside to show the existence of a p-subgroup $W \neq 1$ of G and of an element t in the normalizer of W which induces in W an automorphism, not 1, of order prime to p; see Zassenhaus [1; p. 103, Satz 8]. It is a consequence of (1) and (3) that W is not a normal subgroup of G. The normalizer V of W is therefore a proper subgroup G. Application of (4) shows that the set $\bar{V}$ of elements of order prime to p in V is a characteristic subgroup of V. Since W is a normal subgroup of V and since $\bar{V} \cap W = 1$, it follows that a p-group of automorphisms is induced in W by its normalizer V. This contradicts the presence of the element t in V. Thus we have arrived at a contradiction in either case, completing the proof of our result.

Appendix I: Nilpotent groups

Denote by ν the trivial partial ordering of the set of all primes so that $p \; \nu \; q$ for no pair of primes. Then every set of primes is a segment of ν; and every prime is both ν-minimal and ν-maximal. We have mentioned before that the group G is nilpotent if, and only if, G is ν-dispersed. By an almost immediate specialization of §9, Theorem 1, one obtains now the following theorem—in one instance one has to make use of the Remark at the end of §9; and we have noted each time the condition of §9, Theorem 1 whose specialization is the present criterion.

The following properties of the group G are equivalent:

(1) G is nilpotent [(i)].

(2) If p is a prime divisor of the order of the homomorphic image H of G, then there exists a normal p-subgroup, not 1 of H [(v)].

(3) If $H \neq 1$ is a homomorphic image of G, then $Z(H) \neq 1$ [(vii)].

(4) If the homomorphic image H of G possesses one and only one minimal normal subgroup, then H is a p-group [(viii)].

(5) Maximal subgroups of G are normal [(ix)].

(6) If S is a maximal subgroup of G, then G/S_G is a p-group [(x)].

(7) If M is a minimal normal subgroup of the homomorphic image H of G, then $M \leqq Z(H)$[(xi)].

(8) $[S:S']$ is, for every subgroup S of G, a multiple of every prime divisor of the order of S [(xiii)].

(9) $[C:C']$ is, for every characteristic subgroup C of G, a multiple of every prime divisor of the order of C [(xiv)].

(10) If N is a normal subgroup of G and p a prime divisor of the order of N, then $[N:[N, G]]$ is a multiple of p [(xv)].

(11) If $N \neq 1$ is a characteristic subgroup of G, then $[N, G] < N$ [(xvi)].

(12) Pairs of elements of relatively prime prime power order commute [(xvii) and the Remark at the end of §9].

(13) If x and y are p-elements in G, then $\{x, y\}$ is nilpotent [(xviii)].

Most of these criteria are more or less well known and have been restated here only for the purpose of showing that they are special cases of §9, Theorem 1.

Appendix II: *σ-dispersed Δ^n-groups*

Thoughout this appendix we assume that σ is a partial ordering of the set of *all* primes. If G is a σ-dispersed group, and if p is a σ-minimal G-relevant prime, then the set of all p-elements in G is a characteristic p-subgroup of G which is necessarily contained in the Fitting subgroup $F(G)$ of G. An obvious inductive argument shows now that $G = F_n(G)$ if the order of G is divisible by at most n different primes. Thus σ-dispersed groups whose orders are divisible by at most n different primes are Δ^n-groups; cp. the end of §8 for the definitions of F_n and Δ^n.

LEMMA. *If Δ is strictly homomorphism-invariant and subgroup-inherited, and if* (L.n), *for* $2 \leqq n$, *is satisfied by the class of σ-dispersed Δ-groups, then* (E.n) *and* (L.$n+1$) *are satisfied by the class of σ-dispersed groups with nilpotent Δ-commutator subgroups* (= *σ-dispersed Δ'-groups*).

Proof. We note first that σ-dispersion is strictly homomorphism-invariant and subgroup-inherited, that σ-dispersed groups are soluble, and that σ-dispersion meets requirements (E.1) and (L.2). An immediate application of §8, Corollary 3 shows that (E.n) is satisfied by the class of σ-dispersed Δ'-groups. Suppose next that every $(n + 1)$-tuplet of elements in the group G generates a σ-dispersed Δ'-group. Then every pair of elements in G generates a σ-dispersed subgroup of G so that G is σ-dispersed and in particular soluble. Consequently we may apply §7, Proposition 1 to show that (every subgroup of) G is a σ-dispersed Δ'-group.

By an obvious inductive argument it follows from this Lemma that (E.n) *and* (L.$n+1$) *are satisfied by the class of σ-dispersed Δ^n-groups*, if we only remember that (E.1) and (L.2) are satisfied by the class Δ^1 of nilpotent groups.

10. Strictly dispersed groups

The strictly dispersed groups are going to be dispersed groups meeting a further requirement. For a convenient enunciation of the properties to be discussed we recall first the concept of a Steinitz E-number. This is a formal product

$$e = \prod_p p^{e(p)}$$

where the product ranges over all the primes p and where $e(p)$ is 0 or a positive integer or the symbol ∞. One says that the E-number e is a divisor of the E-number e', or e' is a multiple of e, or e/e', if $e(p) \leqq e'(p)$ for every prime p (where naturally every integer is smaller than the symbol ∞). Without danger of confusion factors of the form p^0 may be omitted; and this we shall do quite often. Thus positive integers may be considered as E-numbers too and may be treated accordingly.

If the E-number m is a multiple of the order of every element in the group G, then G is termed an $\varepsilon(m)$-*group*. Thus a group G is certainly an $\varepsilon(m)$-group if the order of G is a divisor of m, though the converse is not true. In particular every p-group is an $\varepsilon(p^\infty)$-group. More generally: if Σ is a set of primes, then the class of Σ-groups is exactly the class of $\varepsilon(\Pi_{p \text{ in } \Sigma}\, p^\infty)$-groups.

Subgroups, homomorphic images, and direct products of $\varepsilon(m)$-groups are likewise $\varepsilon(m)$-groups, i.e. $\varepsilon(m)$ is strictly homomorphism-invariant and subgroup-inherited. The property $\varepsilon(m)$ clearly and trivially meets the requirements (L.1) and (E.1) whereas extensions of $\varepsilon(m)$-groups by $\varepsilon(m)$-groups need not be $\varepsilon(m)$-groups.

If G is any group, then we may form the $\varepsilon(m)$-commutator subgroup $[G, \varepsilon(m)]$. If m happens to be an ordinary positive integer, then one sees easily that $[G, \varepsilon(m)]$ is the subgroup of G which is generated by all the m^{th} powers of elements in G. If, however, m is not an ordinary integer, then we form the G.C.D. m^* of m and the order of G. One verifies again that $[G, \varepsilon(m)]$ is the subgroup of G which is generated by the $m^{*\text{th}}$ powers of elements in G, since the order of G is a multiple of the order of every element in G and since $G = G^i$ for every positive integer i which is prime to the order of G. These remarks may be expressed shortly by the (symbolic) equation

$$[G, \varepsilon(m)] = G^m.$$

Suppose now that σ is a partial ordering of the set Σ of primes and that to every prime p in Σ we have assigned an E-number $s(p)$ of the form

$$s(p) = \prod_{x \notin \sigma(p)} x^{s(p,x)}.$$

Recall that $\sigma(p)$ is the set of primes y in Σ such that $p \not\sigma y$. Hence x is not in $\sigma(p)$ if, and only if, either x is not in Σ, or else $p\ \sigma\ x$.

DEFINITION. *The group G is a σ-s-dispersed group, if for every homomorphic image H of G and for every σ-minimal H-relevant prime p*

(a) *the set H_p of p-elements in H is a subgroup of H and*

(b) *the elements in H induce in H_p an $\varepsilon(p^\infty s(p))$-group of automorphisms.*

Comparison of condition (a) with §9, Theorem 1, (iv) shows that σ-s-dispersed groups are σ-dispersed. It is furthermore not difficult to see that subgroups, homomorphic images, and direct products of σ-s-dispersed groups are likewise σ-s-dispersed.

If in particular every exponent $s(p, x) = \infty$, then the class of σ-s-dispersed groups may be shown to be identical with the class of σ-dispersed groups.

THEOREM 1. *If σ is a partial ordering of the set Σ of primes, then the following properties of the group G are equivalent:*

(i) *G is σ-s-dispersed.*

(ii) *G is σ-dispersed; if N is a normal p-subgroup of the homomorphic image H of G, and if p belongs to Σ, then H induces an $\varepsilon(p^\infty s(p))$-group of automorphisms in N.*

(iii) *If H is a homomorphic image of G and p a σ-minimal H-relevant prime, then there exists a minimal normal subgroup M of H whose order is a multiple of p and in which H induces an $\varepsilon(s(p))$-group of automorphisms.*

(iv) *G is Σ-dissolved; and if the order of the minimal normal subgroup M of the homomorphic image H of G is a multiple of the prime p in Σ, then H induces an $\varepsilon(s(p))$-group of automorphisms in M.*

(v) *G is Σ-dissolved and an $(\varepsilon(s(p)), p)$-group for every prime p in Σ.*

(vi) *G is Σ-closed and $G^{s(p)}$ is Pp-closed for every prime p in Σ.*

(vii) *If H is a homomorphic image of G and p a σ-minimal H-relevant prime, then $H^{s(p)}$ is the direct product of a p-group and a Pp-group.*

(viii) *If S is a maximal subgroup of G, and if $\Sigma(G/S_G)$ is not vacuous, then there exists a prime p in Σ such that $[G:S]$ is a power of p and S/S_G is an $\varepsilon(s(p))$-group.*

(ix) *$G/\Phi(G)$ is σ-s-dispersed.*

(x) *Pairs of elements of relatively prime prime power order in G generate σ-s-dispersed subgroups of G.*

Remark. In (viii), the prime p is uniquely determined as the one and only one σ-minimal (G/S_G)-relevant prime.—If the order of the minimal normal subgroup M of H is a multiple of p and if H induces an $\varepsilon(s(p))$-group of automorphisms in M, then M is an elementary abelian p-group. Hence it suffices to assume Σ-closure in (iv).

Proof. It is an almost immediate consequence of the definition of σ-s-dispersion and of §9, Theorem 1, (iv) that (ii) is a consequence of (i). If (ii) is satisfied by G, and if H is a homomorphic image of G and p a σ-minimal H-relevant prime, then the set P of p-elements in H is a characteristic p-subgroup of H. Since $P \neq 1$, so is $Z(P)$; and it is clear that $Z(P)$ is likewise a characteristic p-subgroup of H. Consequently there exists a minimal normal subgroup M of H which is part of $Z(P)$. Then M is a p-group. Since P is part of the centralizer of M, the order of the group of automorphisms, induced in M by elements in H, is prime to p. Application of (ii) shows then that H induces in M an $\varepsilon(s(p))$-group of automorphisms. (iii) is therefore a consequence of (ii).

Assume next the validity of (iii); and consider a minimal normal subgroup M of the homomorphic image H of G such that the order of M is a multiple of some prime p in Σ. Among the normal subgroups X of H satisfying $X \cap M = 1$, there exists a maximal one, say K. Then $H^* = H/K$ is a homomorphic image of H and G. The normal subgroup

$$M^* = KM/K \simeq M/(M \cap K) = M$$

of H^* is the one and only one minimal normal subgroup of H^*, since every normal subgroup, not 1, of H^* has the form U/K where U is a normal subgroup of H and $K < U$ so that $M \cap U \neq 1$ because of the maximality of K and hence $M \leqq U$ because of the minimality of M. If q is a σ-minimal H^*-relevant prime, then there exists, by (iii), a minimal normal subgroup W of H^* whose order is a multiple of q and in which H^* induces an $\varepsilon(s(q))$-group of automorphisms. Thus W induces in particular an $\varepsilon(s(q))$-group of automorphisms in W; and since $s(q)$ is prime to q, it follows that the center of W is not 1 and that therefore the minimal normal subgroup W is abelian. By §2, Lemma 1, W is an elementary abelian q-group. We recall next that M^* is the one and only one minimal normal subgroup of H^*. Hence $M^* = W$ so that M and W are isomorphic. Since the order of M is a multiple of p, and since W is a q-group, $p = q$. Thus H^* induces an $\varepsilon(s(p))$-group of automorphisms in M^*; and this is equivalent to saying that

$$[M^*, H^{*s(p)}] = 1.$$

This implies

$$[M, H^{s(p)}] \leqq M \cap K = 1$$

so that $H^{s(p)}$ is part of the centralizer of M in H. It follows that H induces in M an $\varepsilon(s(p))$-group of automorphisms.

The result of the preceding paragraph implies in particular that the second part of (iv) is a consequence of (iii). To prove that G is Σ-dissolved we form the product S of all normal soluble Σ-subgroups of G. Then S is a characteristic soluble Σ-subgroup of G. If $\Sigma(G/S)$ were not vacuous, then there would exist a σ-minimal (G/S)-relevant prime r. By (iii) there exists a minimal normal subgroup R/S of G/S whose order is a multiple of r. By the result of the preceding paragraph of our proof R/S is an r-group so that R is a soluble Σ-group. Since R is a normal subgroup of G, we have $S < R \leqq S$, a contradiction. Hence $[G:S]$ is prime to every prime in p; and it follows that the characteristic soluble Σ-subgroup S of G is the set of all Σ-elements in G. Hence G is Σ-dissolved; and we have shown that (iv) is a consequence of (iii).

It is almost obvious that (v) is a consequence of (iv); and application of §6, Theorem 2, (ii) shows that (vi) is a consequence of (v).

Assume next the validity of (vi); and consider a homomorphic image H of G. Then every subgroup of H is Σ-closed, since G is Σ-closed. If q is a prime in $\Sigma(H)$, then $H^{s(q)}$ is a homomorphic image of $G^{s(q)}$; and this implies, by (vi), that $H^{s(q)}$ is Pq-closed. The set $T(q)$ of elements of order prime to q in $H^{s(q)}$ is consequently a characteristic subgroup of $H^{s(q)}$ and H whose order is prime to q. Since every subgroup of H is Σ-closed, the set $S(q)$ of Σ-elements in $T(q)$ is a characteristic Σ-subgroup of H. The order of $S(q)$ is likewise prime to q; and $S(q)$ is the set of Σ-elements of order prime to q in $H^{s(q)}$.

If p is a σ-minimal H-relevant prime, then denote by P the intersection of all the $S(q)$ with $q \neq p$ in $\Sigma(H)$:

$$P = \bigcap_{q \in \Sigma(H) - p} S(q).$$

It is clear that P is a characteristic subgroup of H and that P is a Σ-group as the intersection of Σ-groups. If $q \neq p$ is a prime, then the order of P is prime to q whenever q is not in Σ; and if q is in Σ, then P is part of the group $S(q)$ of order prime to q. Thus the order of P is prime to every prime $q \neq p$; and this implies that P is a p-group.

Suppose next that x is a p-element in H and that q belongs to $\Sigma(H)$. Because of the minimality of p we have then $q \not\sigma p$ so that p belongs to $\sigma(q)$. Consequently p is prime to $s(q)$; and this implies that the p-element x belongs to $H^{s(q)}$. If we assume in addition that $p \neq q$, then the order of x is prime to q so that x belongs to $T(q)$; and since p is in Σ, x belongs even to $S(q)$. It follows that x belongs to the intersection P of all the $S(q)$ with $q \neq p$ in $\Sigma(H)$; and this implies that P is the totality of all the p-elements in H.

Since $s(p)$ is prime to p, P is part of $H^{s(p)}$; and now it is clear that $H^{s(p)}$ is the direct product of the p-group P and the Pp-group $T(p)$. Hence (vii) is a consequence of (vi).

Assume next the validity of (vii); and consider a homomorphic image H of G and a σ-minimal H-relevant prime p. Then $H^{s(p)}$ is the direct product of a p-group P and of a group Q of order prime to p. It is clear that P is the totality of p-elements and Q the totality of Pp-elements in $H^{s(p)}$. Hence P and Q are characteristic subgroups of a characteristic subgroup of H; and as such they are characteristic subgroups of H. Since $s(p)$ is prime to p, $H^{s(p)}$ contains every p-element in H so that P is the totality of p-elements in H. Since Q is part of the centralizer of P, the group of automorphisms of

P which are induced in P by elements in H is a homomorphic image of H/Q. Since $H^{s(q)}/Q \simeq P$ is a p-group, H/Q is an $\varepsilon(p^\infty s(p))$-group so that H induces in P an $\varepsilon(p^\infty s(p))$-group of automorphisms. Thus we have verified that G is σ-s-dispersed; and this completes the proof of the equivalence of conditions (i) to (vii).

Assume again that G is σ-s-dispersed; and consider a maximal subgroup S of G such that $\Sigma(G/S_G)$ is not vacuous. Then there exists a σ-minimal (G/S_G)-relevant prime p. By (iii)—the equivalence of (i) and (iii) has already been established—there exists a minimal normal subgroup M of G/S_G whose order is a multiple of p and in which G/S_G induces an $\varepsilon(s(p))$-group of automorphisms. By (v), G is Σ-dissolved. Hence M is a p-group. If we note that M is not part of the maximal subgroup S/S_G of G/S_G, then it follows from §2, Lemma 1 that S/S_G is a complement of M in G/S_G and from §2, Lemma 2 that M is its own centralizer in G/S_G. Thus S/S_G, $(G/S_G)/M$, and the group of automorphisms induced in M by elements in G/S_G are isomorphic. Since $[G:S]$ equals the order of M, we see that (viii) is a consequence of the equivalent conditions (i) to (vii).

Assume conversely the validity of (viii). Noting that every prime divisor q of $s(x)$ satisfies $p \,\sigma\, q$ it follows that the prime p occurring in (viii) is the one and only one σ-minimal (G/S_G)-relevant prime. Hence (viii) implies in particular condition (x) of §9, Theorem 1. Hence G is σ-dispersed. It is now easy to see that condition (iii) of §6, Theorem 2 is satisfied by every prime p in Σ together with $\Theta = \varepsilon(s(p))$. Thus we see that G is σ-dispersed and hence Σ-dissolved; and that G is an $(\varepsilon(s(p)), p)$-group for every p in Σ. Hence (v) is a consequence of (viii); and we have shown the equivalence of conditions (i) to (viii). The equivalence of conditions (i) and (viii) implies the equivalence of conditions (i) to (ix).

Let us note that property (vi) is trivially subgroup-inherited. The equivalence of (i) and (vi) shows now that σ-s-dispersion is a subgroup-inherited property—we pointed out before that this fact admits of easy direct verification. It implies in particular that (x) is a consequence of (i). If conversely (x) is satisfied by G, then condition (xvii) of §9, Theorem 1 is satisfied by G, since every σ-s-dispersed group is σ-dispersed. It follows that G is σ-dispersed. Consider next a normal p-subgroup N of the homomorphic image H of G; and assume that p belongs to Σ. Suppose that h is a q-element in H and that $q \neq p$. If x is an element in N, then x is a p-element; and it is a consequence of (x) that $U = \{x, h\}$ is σ-s-dispersed. Since $N \cap U$ is a normal subgroup of U, and since (ii) holds in U, the element h induces in the normal p-subgroup $N \cap U$ of the σ-s-dispersed group U an automorphism whose order is a divisor of $s(p)$. The minimal positive integer i such that $x = x^{h^i}$ is consequently a divisor of $s(p)$. Now it is easy to see that H induces in N an $\varepsilon(p^\infty s(p))$-group of automorphisms. Thus (ii) is a consequence of (x); and this completes the proof of our theorem.

COROLLARY 1. *Property* (E.1) *is satisfied by σ-s-dispersion.*

Proof. Assume that N is a normal subgroup of the group G and that G/N and every $\{N, x\}$ for x in G is a σ-s-dispersed group. Since σ-s-dispersed groups are σ-dispersed, and since σ-dispersion meets requirement (E.1) [§4, Proposition 1], G is σ-dispersed; and this implies that G is Σ-dissolved.

Consider now a minimal normal subgroup M of the homomorphic image H of G; and assume that the order of M is a multiple of the σ-minimal H-relevant prime p. Since G is Σ-dissolved, G is p-separated; and consequently M is a p-group—actually an elementary abelian p-group [§2, Lemma 1]. The homomorphism mapping G onto H maps N upon a normal subgroup K of H such that H/K and $\{K, x\}$ for every x in H is σ-s-dispersed.

Case 1. $M \cap K = 1$.

Then $M \simeq KM/K$; and the group of automorphisms induced in M by elements in H is essentially the same as the group of automorphisms induced in KM/K by elements in H/K. Thus KM/K is a minimal normal p-subgroup of the σ-s-dispersed group H/K; and this implies that the group of automorphisms induced in M and MK/K by elements in H is an $\varepsilon(s(p))$-group.

Case 2. $M \cap K \neq 1$.

This implies $M \leqq K$, since M is a minimal normal subgroup of H. Consider now an element t in H whose order is prime to p. Then $\{K, t\}$ is σ-s-dispersed; and the group of automorphisms induced by $\{K, t\}$ in its normal p-subgroup M is, by Theorem 1, (ii), an $\varepsilon(p^{\infty}s(p))$-group. If i is the order of the automorphism induced by t in M, then i is a divisor of $s(p)$. Next we recall that G is σ-dispersed. Since p is a σ-minimal H-relevant prime, the set P of p-elements in H is a characteristic p-subgroup of H [§9, Theorem 1, (iv)]. Since M is a minimal normal p-subgroup of H, $M \leqq P$. Hence $M \cap Z(P) \neq 1$, and consequently $M \leqq Z(P)$. Thus P is part of the centralizer of M. We have shown therefore that $H^{s(p)}$ is part of the centralizer of M; and this implies that an $\varepsilon(s(p))$-group of automorphisms is induced in M by H.

This completes the verification of the validity of condition (iv) of Theorem 1; and G is consequently a σ-s-dispersed group.

We terminate this section by a short discussion of a special case which will prove important in the next section. Denote by α the inverted natural ordering of the set of all primes so that $p \; \alpha \; q$ if, and only if, $q < p$. Furthermore let $a(p) = p - 1$ for every prime p. Then the pair α, a meets the requirements imposed upon pairs σ, s in Theorem 1. Consequently the following results are obtained by straightforward specialization of Theorem 1 and Corollary 1.

THEOREM 2. *The following properties of the group G are equivalent:*

(i) *G is α-a-dispersed.*

(ii) *G is α-dispersed; if N is a normal p-subgroup of the homomorphic image H of G, then H induces an $\varepsilon(p^\infty(p-1))$-group of automorphisms in N.*

(iii) *If H is a homomorphic image of G and p the maximal prime divisor of the order of H, then there exists a minimal normal subgroup M of H whose order is a multiple of p and in which H induces an $\varepsilon(p-1)$-group of automorphisms.*

(iv) *If M is a minimal normal subgroup of the homomorphic image H of G, and if the order of M is a multiple of the prime p, then H induces an $\varepsilon(p-1)$-group of automorphisms in M.*

(v) *G is soluble and an $(\varepsilon(p-1), p)$-group for every prime p.*

(vi) *G^{p-1} is Pp-closed for every prime p.*

(vii) *If H is a homomorphic image of G and p the maximal prime divisor of the order of H, then H^{p-1} is the direct product of a p-group and a Pp-group.*

(viii) *If S is a maximal subgroup of G, and if p is the maximal prime divisor of $[G:S_G]$, then $[G:S]$ is a power of p and $(S/S_G)^{p-1} = 1$.*

(ix) *$G/\Phi(G)$ is α-a-dispersed.*

(x) *Pairs of elements of relatively prime prime power order in G generate α-a-dispersed subgroups of G.*

Note that (iv) may also be stated in the following equivalent form:

If M is a minimal normal subgroup of the homomorphic image H of G, then $M' = M^p = [M, H^{p-1}] = 1$ for some prime p.

COROLLARY 2. *Property* (E.1) *is satisfied by α-a-dispersion.*

11. Supersoluble groups

We begin by proving some simple facts needed in the derivation of the principal result of this section.

LEMMA 1. *If S is a maximal subgroup of G whose core $S_G = 1$ and whose index $[G:S]$ is a prime p, then $q < p$ for every prime divisor q of the order of S.*

Proof. There is nothing to prove if $S = 1$. If $S \neq 1$, then $S_G = 1$ implies that S is not a normal subgroup of G. Since S is a maximal subgroup of G, S is its own normalizer in G. Since $[G:S] = p$, this implies that S possesses exactly p conjugate subgroups in G. Every element x in S induces a permutation in the set of $p-1$ subgroups which are conjugate to S, but different from S. If i is the order of this permutation, then x^i belongs to the intersection $S_G = 1$ of all the subgroups conjugate to S, since each subgroup conjugate to S is equal to its own normalizer. Hence $x^i = 1$ and i is the order of x. Thus the order of x is a divisor of $(p - 1)!$; and this implies that every prime divisor of the order of S is smaller than p, Q.E.D.

LEMMA 2. *The following properties of the group G are equivalent:*

(i) *There exists a normal subgroup of G whose order is a prime and which is equal to its own centralizer.*

(ii) *There exists a maximal subgroup S of G whose core $S_G = 1$, whose index $[G:S]$ is a prime, and which is abelian.*

(iii) *There exists a maximal subgroup S of G whose core $S_G = 1$, whose index $[G:S]$ is a prime, and which satisfies $S \cap S^x = 1$ whenever $S \neq S^x$.*

(iv) $\Phi(G) = 1$; *there exists one and only one minimal normal subgroup M of G, and the order of M is a prime.*

(v) *There exist maximal subgroups whose core is* 1; *and every maximal subgroup with core* 1 *has index a prime.*

Proof. Assume first the existence of a normal subgroup N of G whose order is a prime p and which is equal to its own centralizer. Then G/N is essentially the same as a group of automorphisms of the cyclic group N of order p; and this implies that G/N is a cyclic group whose order is a divisor of $p - 1$. Since the orders of the cyclic groups N and G/N are relatively prime, there exists an element w in G whose order is exactly $[G:N]$. Then $S = \{w\}$ is a complement of N in G; and now it is clear that (ii) is a consequence of (i).

Assume next the existence of a maximal subgroup S of G whose core $S_G = 1$, whose index $[G:S]$ is a prime, and which is abelian. If $S \neq S^x$, then $G = \{S, S^x\}$, since S and S^x are distinct maximal subgroups. Since S and S^x are both abelian, the centralizer of $S \cap S^x$ contains both S and S^x so that $S \cap S^x \leqq Z(G)$. But subgroups of the center are normal. Hence $S \cap S^x \leqq S_G = 1$; and we see that (ii) implies (iii).

Assume next the validity of (iii). Then it is clear that $\Phi(G) = 1$. There exists a maximal subgroup S of G whose core $S_G = 1$, whose index $[G:S] = p$ is a prime, and which satisfies $S \cap S^x = 1$ whenever $S \neq S^x$. If $S = 1$, then the order of G is p. If $S \neq 1 = S_G$, then S is not a normal subgroup of G. But S is maximal; and consequently S equals its normalizer in G. It follows that the number of subgroups conjugate to S in G is exactly $p = [G:S]$. It is a consequence of Lemma 1 that the order n of S is prime to p. Since pn is the order of G, it follows that p is the order of every p-Sylow subgroup of G. The number of elements contained in S and its conjugate subgroups is exactly $1 + p(n - 1)$; and this implies that the number of elements of order p in G is exactly $p - 1$. Hence there exists one and only one subgroup M of order p; and now it is easy to deduce (iv) from (iii) [use §2, Corollary 1].

Assume next the existence of one and only one minimal normal subgroup M of G; and assume furthermore that the order of M is a prime p and that $\Phi(G) = 1$. The last hypothesis implies the existence of a maximal subgroup which does not contain M. If X is a maximal subgroup of G which does not contain M, then it is clear that $X_G = 1$ and that X is a complement of M in G so that $[G:X] = p$. Hence (v) is a consequence of (iv).

Assume finally the validity of (v). If S is a maximal subgroup of G and if $S_G = 1$, then $[G:S]$ is, by hypothesis, a prime; and we deduce from Lemma 1, that $[G:S]$ is the maximal prime divisor p of the order of G and that furthermore p is prime to the order of S. By §2, Lemma 3, there exists a soluble normal subgroup $N \neq 1$ of G; and this implies the existence of a soluble minimal normal subgroup M of G. Application of §2, Lemma 1 shows that M is an elementary abelian group; and that every maximal subgroup of core 1 is a complement of M in G. The order of M equals the index $[G:S] = p$ for every maximal subgroup S of core 1; and application of §2, Lemma 2 shows that M is its own centralizer in G. Hence (i) is a consequence of (v), completing the proof.

Remark 1. The derivation of (iv) from (iii) could have been shortened slightly by applying a celebrated Theorem of Frobenius. But this theorem and its proof are rather deep, whereas our derivation was quite elementary.

Of the various possible definitions of supersolubility we select the following one which fits best into the general mood of our discussion.

DEFINITION. *The group G is supersoluble, if every minimal normal subgroup of every homomorphic image of G is cyclic (of order a prime).*

It is well known, and easily verified, that subgroups, homomorphic images, and direct products of supersoluble groups are supersoluble; in short: the property of supersolubility is subgroup-inherited and strictly homomorphism-invariant.

THEOREM 1. *The following properties of the group G are equivalent:*

(i) *G is supersoluble.*

(ii) *Every homomorphic image, not 1, of G possesses a cyclic normal subgroup, not 1.*

(iii) *Every maximal subgroup of G has index a prime.*

(iv) *If S is a maximal subgroup of G, then G/S_G possesses a cyclic normal subgroup, not 1.*

(v) *If S is a maximal subgroup of G, then $[G:S]$ is a power of a prime p and S/S_G is an abelian $\varepsilon(p-1)$-group.*

(vi) *If S is a maximal subgroup of G, then $[G:S]$ is a power of a prime p and S/S_G is a cyclic group whose order is a divisor of $p-1$.*

(vii) *G' is nilpotent and G is an α-a-dispersed group.*

(viii) *If the order of the minimal normal subgroup M of the homomorphic image H of G is a multiple of p, then $[M, H'H^{p-1}] = 1$.*

(ix) *G is soluble; and if the maximal subgroup S of G does not contain the normal subgroup N of G, then $N \cap S$ is a maximal subgroup of N.*

(x) *$G/\Phi(G)$ is supersoluble.*

Remark 2. The equivalence of properties (i) and (iii) has been discovered by Huppert [1]. The present proof, however, appears to be essentially different from Huppert's.

Remark 3. Combining (vii) with §8, Corollary 2 and §10, Theorem 2 a great number of further supersolubility criteria may be obtained.

Remark 4. The indispensability of the first half of condition (ix) may be seen from the remark that the second part of this condition is vacuously satisfied by all simple groups.

Remark 5. That supersoluble groups are α-dispersed and that their commutator subgroups are nilpotent, has already been noted by Ore and Zappa.

Proof. It is clear that (i) implies (ii). If (ii) is true and S is a maximal subgroup of G, then G/S_G contains a cyclic normal subgroup different from 1; and this implies the existence of a normal subgroup M of G/S_G such that the order of M is a prime p. It is clear then that S/S_G is a complement of M in G/S_G and that consequently the order p of M equals the index $[G:S]$. Thus (iii) is a consequence of (ii).

It is an immediate consequence of Lemma 2 that (iii) implies (iv). If next (iv) is satisfied by G and S is a maximal subgroup of G, then G/S_G possesses a normal subgroup M whose order is a prime p. It is clear then that S/S_G is a complement of M in G/S_G and that therefore the order of M equals the index $[G:S]$. One deduces furthermore from §2, Lemma 2 that M is its own centralizer in G/S_G; and $(G/S_G)/M$ is therefore essentially the same as the group of automorphisms induced in M by elements in G/S_G. Since M is cyclic of order p, its group of automorphisms is cyclic of order $p - 1$ and this implies that S/S_G is a cyclic group whose order is a divisor of $p - 1$. Hence (iv) implies (vi); and it is obvious that (vi) implies (v).

It is an immediate consequence of §8, Corollary 2 and §10, Theorem 2 that (v) implies (vii). If (vii) is satisfied by G, then G is soluble; and if the order of the minimal normal subgroup M of the homomorphic image H of G is a multiple of p, then M is, by §2, Lemma 1, an elementary abelian p-group. H consequently induces in M an abelian [by §8, Corollary 2] $\varepsilon(p - 1)$-group of automorphisms. Hence $H'H^{p-1}$ is contained in the centralizer of M so that $[M, H^{p-1}H'] = 1$. Hence (viii) is a consequence of (vii).

Assume next the validity of (viii); and consider a minimal normal subgroup M of the homomorphic image H of G. Then $[M, H'] = 1$ so that H' is part of the centralizer of M. Thus H induces in M an abelian group of automorphisms; and this implies by §8, Corollary 2 the nilpotency of G' so that in particular G is soluble. It follows in particular that M is soluble; and this implies, by §2, Lemma 1, that M is an elementary abelian p-group. As noted before H induces in M an abelian group Θ of automorphisms. By (viii), we have $[M, H^{p-1}] = 1$. Hence H^{p-1} is part of the centralizer of M in H so that $\Theta^{p-1} = 1$. Next we form the ring $\mathfrak{P}$ of endomorphisms which is spanned by Θ. Since Θ is abelian, $\mathfrak{P}$ is commutative. Since Θ is the group of automorphisms, induced by H in its minimal normal subgroup M, 1 and M are the only $\mathfrak{P}$-admissible subgroups of M. We apply Schur's Lemma to see that $\mathfrak{P}$ is a (finite and commutative) field whose characteristic is p, since $M^p = 1$. Since $\Theta^{p-1} = 1$, the elements in Θ belong to the prime field of characteristic p; and since $\mathfrak{P}$ is spanned by Θ, $\mathfrak{P}$ is the prime field of characteristic p. But then every element in $\mathfrak{P}$ (and in Θ) is a multiple of the identity so that every subgroup of M is $\mathfrak{P}$-admissible. It follows that M is cyclic of order p, since 1 and M are the only $\mathfrak{P}$-admissible subgroups of M. Thus (i) is a consequence of (viii); and we have completed the proof of the equivalence of conditions (i) to (viii).

If G is supersoluble, then G is certainly soluble. If the normal subgroup N of G is not contained in the maximal subgroup S of G, then $G = NS$ so that $[G:S] = [N:N \cap S]$. But this index is, by (iii), a prime so that $N \cap S$ is a maximal subgroup of N. Thus (ix) is a consequence of (i). If conversely (ix) is satisfied by G, then consider a maximal subgroup S of G. There exists a minimal normal subgroup M of $H = G/S_G$. Since G is soluble, so is M; and this implies by §2, Lemma 1 that M is an elementary abelian p-group and that S/S_G is a complement of M in G/S_G. But, by (ix), $1 = M \cap (S/S_G)$ is a maximal subgroup of M; and this implies that M is cyclic of order p. Hence (iv) is a consequence of (ix), showing the equivalence of conditions (i) to (ix).

The equivalence of conditions (i) and (x) is a consequence of the equivalence of conditions (i) and (iii). Q.E.D.

COROLLARY 1. *Properties* (E.2) *and* (L.3) *are satisfied by supersolubility.*

Proof. It is a consequence of §10, Corollary 1 that property (E.1) is satisfied by α-a-dispersion. Since a group is abelian if, and only if, every pair of elements generates an abelian subgroup, we may deduce from §8, Theorem 3 that the class of groups with nilpotent commutator subgroup meets requirement (E.2). Application of Theorem 1 shows now that the class of supersoluble groups meets requirement (E.2).

Assume next that every triplet of elements in the group G generates a supersoluble subgroup. Then every pair of elements in G generates an α-dispersed subgroup of G; and this implies clearly that G itself is α-dispersed. But α-dispersed groups are soluble. Since supersolubility meets requirement (E.2), we may now use §7, Proposition 1 to show that G is supersoluble, proving the validity of (L.3).

COROLLARY 2. *If G is the product of its normal supersoluble subgroups $N_1, \cdots, N_k$, then the following properties of G are equivalent:*

(a) *G is supersoluble.*

(b) *G' is nilpotent.*

(c) *$[N_i, N_j]$ is nilpotent for $i \neq j$.*

Proof. It is a consequence of Theorem 1 that (a) implies (b); and it is clear that (b) implies (c). Since every N_i is supersoluble, every N_i' is nilpotent [Theorem 1]; and now it is clear that (b) is a consequence of (c). Since every N_i is as a supersoluble group α-a-dispersed, G is a α-a-dispersed; and it is a consequence of Theorem 1 that (b) implies (a), Q.E.D.

Example 1. Denote by p an odd prime which is $\not\equiv 1$ modulo 4; and denote by M a direct product of two cyclic groups of order p. Let a, b be a basis of M. Then automorphisms r and s of M may be defined by the rules:

$$a^r = b^{-1}, \qquad b^r = a;$$

$$a^s = b \ , \qquad b^s = a.$$

It is clear that s is of order 2, that $r^2 = -1$ and that $srs = r^{-1}$. The group A of automorphisms of M which is generated by r and s has consequently order 8; and the element r generates a normal subgroup of order 4 of A. We note furthermore that A is the product of its normal subgroups $S = \{s, r^2\}$ and $T = \{sr, r^2\}$ each of which normal subgroups is a direct product of two cyclic groups of order 2.

Denote now by G the subgroup of the holomorph of M which is generated by M and A. This group G is obtained by adjoining to M elements r and s, subject to the relations:

$$r^4 = 1, \quad s^2 = 1, \quad r^{-1}ar = b^{-1}, \quad r^{-1}br = a, \quad sas = b, \quad sbs = a.$$

This group G has the following properties:

1. $G' = \{M, r^2\}$ so that G' is not nilpotent.

2. $\{M, r^2, s\} = MS$ and $\{M, r^2, sr\} = MT$ are normal subgroups of G which both contain the commutator subgroup as a subgroup of index 2, and whose product is just G. Each of these normal subgroups of G is supersoluble in spite of the fact that the commutator subgroup of G is not nilpotent.

3. It is easily seen that G is α-dispersed, but not α-a-dispersed, since G induces in its normal p-subgroup M an automorphism of order 4, though 4 is not a divisor of $p - 1$.

Example 2. Modify Example 1 by assuming that 4 is a divisor of $p - 1$; but leave everything else unchanged. Then G will be α-a-dispersed, though its commutator subgroup is not nilpotent. Furthermore every subgroup

$\{M, x\}$ for x in G is then supersoluble; and G/M is even nilpotent (a 2-group). It follows that supersolubility does not meet requirement (E.1); and the same is true of the property of having a nilpotent commutator subgroup. Every pair of elements generates a supersoluble subgroup; and so it follows that property (L.2) is satisfied neither by supersolubility nor by the property of having a nilpotent commutator subgroup.

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