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A STURM-TYPE COMPARISON THEOREM BY A GEOMETRIC STUDY OF PLANE MULTIHEDGEHOGS

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ABSTRACT. We prove a Sturm-type comparison theorem by a geometric study of plane (multi)hedgehogs. This theorem implies that for every 2π-periodic smooth real function h, the number of zeros of h in $[0,2\pi[$ is not bigger than the number of zeros of $h+h''$ plus 2. In terms of N-hedgehogs, it can be interpreted as a comparison theorem between number of singularities and maximal number of support lines through a point. The rest of the paper is devoted to a series of geometric consequences.

1. Introduction and statement of results

The main result of this paper may be stated as follows.

THEOREM. *Let h be a real $2N\pi$-periodic function of class C^2 on $\mathbb{R}$ ($N \in \mathbb{N}^*$). The number $S \in \mathbb{N} \cup \{\infty\}$ of zeros of $h+h''$ in $[0,2N\pi[$ satisfies*

$$n_h \leq S + 4N - 2,$$

where $n_h \in \mathbb{N} \cup \{\infty\}$ is the number of zeros of h in $[0,2N\pi[$.

In particular, this Sturm-type comparison theorem ensures that:

If h is a real 2π-periodic function of class C^2 on $\mathbb{R}$ then, on the circle $\mathbb{R}/2\pi\mathbb{Z}$, the number of zeros of h is not bigger than the number of zeros of $h+h''$ plus 2 (of course, these numbers may be infinite).

Sturm theory and geometry of curves. Sturm theory is closely related to the geometry of curves. In particular, Sturm-type oscillation theorems enable us to minorate the number of certain special points or concurrent lines for different types of closed curves. Although our results do not fall in the

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category of oscillation theorems, let us digress for a moment to recall a central result of Sturm oscillation theory.

Sturm–Hurwitz theorem. *Every continuous real function of the form* $h(\theta) = \sum_{n\geq N}(a_n \cos n\theta + b_n \sin n\theta)$ *has at least as many zeros on the circle* $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ *as its first nonvanishing harmonics:*

$$\#\big(\{\theta(\operatorname{mod} 2\pi) \in \mathbb{S}^1 \mid h(\theta) = 0\}\big) \geq 2N.$$

This result was rediscovered by Tabachnikov [13] who applied it to the four vertex theorem, which claims that every smooth convex curve $\mathcal{C} \subset \mathbb{R}^2$ has at least 4 vertices (i.e., 4 critical points of its curvature). Let us recall the relationship between these two theorems. This will give us the opportunity of introducing some basic notations and properties that will be essential in the sequel.

Preliminaries and relation to the four vertex theorem. In the Euclidean vector plane $\mathbb{R}^2$ oriented by its canonical basis, let us consider a convex curve $\mathcal{C}$ of class C^2_+. Let us recall that:

(i) $\mathcal{C}$ is determined by its support function h, which is defined on the unit circle $\mathbb{S}^1$ by $h(u) = \max_{x\in\mathcal{C}}\langle x, u\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^2$; indeed, h is of class C^2 on $\mathbb{S}^1$ and $\mathcal{C}$ can be regarded as the envelope of the family of lines $(D_h(u))_{u\in\mathbb{S}^1}$ with equation $\langle x, u\rangle = h(u)$.

(ii) The curve $\mathcal{C}$ can be parametrized by

$$\begin{aligned} x_h : \mathbb{S}^1 \subset \mathbb{R}^2 &\longrightarrow \mathbb{R}^2, \\ u(\theta) &\longmapsto x_h(\theta) := h(\theta)u(\theta) + h'(\theta)u'(\theta), \end{aligned}$$

where $u(\theta)$ is the point with coordinates $(\cos\theta, \sin\theta)$ in the canonical basis of $\mathbb{R}^2$ and where h is regarded as a function of θ.

(iii) The algebraic curvature $k(\theta)$ of $\mathcal{C}$ at $x_h(\theta)$ is given by

$$\frac{1}{k(\theta)} = (h + h'')(\theta).$$

From now on, $\mathcal{C}$ is assumed to be of class C^3_+, which implies that h is of class C^3 on $\mathbb{S}^1$. Vertices of $\mathcal{C}$ correspond to the values of θ for which $(h' + h''')(\theta) = 0$. In the Hilbert space of square integrable functions on $[0, 2\pi]$, $h' + h''' = (h + h'')'$ is orthogonal to constants (as a derivative) and to $\cos(\theta)$ and $\sin(\theta)$ (because the operator $\partial^2/\partial\theta^2 + 1$ kills these harmonics). Since the first nonvanishing harmonics of $h' + h'''$ is of order $N \geq 2$, the Sturm–Hurwitz theorem ensures that $\mathcal{C}$ has at least 4 vertices. For more details on geometric applications of the Sturm–Hurwitz theorem, the reader may refer for instance to [2] and [13].

Comparison theorems. Hedgehogs and multihedgehogs. The results of this paper are of different nature since they fall in the category of comparison theorems: they do not enable us to minorate the number of certain special points or concurrent lines for different types of closed curves but to compare such numbers. For instance, Theorem 1 will enable us to compare number of vertices and maximal number of concurrent normal lines for a convex curve of class C^3_+ in $\mathbb{R}^2$.

The proof of these results is based on the notions of hedgehogs and multihedgehogs, which were already used by the author to give a geometric proof of the Sturm–Hurwitz theorem for C^2 functions [9]. Let us recall briefly these notions. We know that the support function of a convex curve of class C^2_+ is C^2 on $\mathbb{S}^1$. Of course, a function $h \in C^2(\mathbb{S}^1;\mathbb{R})$ is not necessarily the support function of a convex curve of class C^2_+. However, we can always associate to h the envelope of the family of lines $(D_h(u))_{u\in\mathbb{S}^1}$ with equation

$$\langle x, u\rangle = h(u).$$

This envelope is denoted by $\mathcal{H}_h$ and called *hedgehog with support function h*. It can always be parametrized by the map $x_h : \mathbb{S}^1 \to \mathbb{R}^2$, $u(\theta) = (\cos\theta, \sin\theta) \longmapsto x_h(\theta)$, where $x_h(\theta) = (x^1_h(\theta), x^2_h(\theta))$ is the unique solution to the system

$$\begin{cases} x_1 = h(\theta)\cos\theta - h'(\theta)\sin\theta, \\ x_2 = h(\theta)\sin\theta + h'(\theta)\cos\theta, \end{cases}$$

that is $x_h(\theta) := h(\theta)u(\theta) + h'(\theta)u'(\theta)$, h being regarded as a function of θ. For every $u \in \mathbb{S}^1$, the line $D_h(u)$ is called the *support line cooriented by* u and $h(u)$ can be interpreted as the signed distance from the origin to $D_h(u)$. Note that $x_h : \mathbb{S}^1 \to \mathcal{H}_h$ can be regarded as the inverse of the Gauss map of $\mathcal{H}_h$, in the sense that at each regular point $x_h(u)$, u is normal to $\mathcal{H}_h$. Differentiation of x_h gives

$$x'_h(\theta) = (h + h'')(\theta)u'(\theta)$$

for all $\theta \in \mathbb{R}$. Therefore, the hedgehog $\mathcal{H}_h$ is regular at $x_h(\theta)$ if and only if $(h + h'')(\theta) \neq 0$. Its algebraic curvature $k(\theta)$ at this point is then given by

$$\frac{1}{k(\theta)} = |(h + h'')(\theta)|.$$

The function $R_h(\theta) := (h + h'')(\theta)$ is called *curvature function* of $\mathcal{H}_h$. Note that $R_h(\theta)$ is well-defined for all $\theta \in \mathbb{R}$ and that $R_h(\theta) = 0$ if and only if $x_h(\theta)$ is a singular point of $\mathcal{H}_h$.

A regular (i.e., singularity-free) hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ is a convex curve of class C^2_+ (Figure 1(a)). A hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ is said to be not too singular if it has a well-defined (geometric) tangent line at every point. A not too singular hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ is a curve that has exactly one oriented tangent line in each direction (Figure 1(b)). For every $u \in \mathbb{S}^1$, the signed distance $h(u) + h(-u)$ between the cooriented support lines $D_h(u)$ and $D_h(-u)$ is called the width

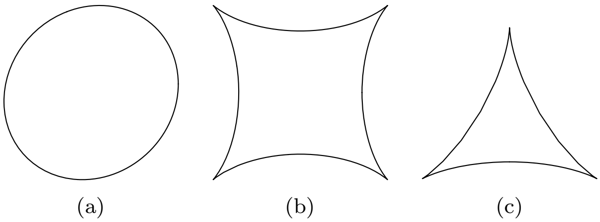


FIGURE 1. (a) A regular hedgehog. (b) A hedgehog. (c) A projective hedgehog.

of $\mathcal{H}_h$ *in direction* u. A hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ is said to be *projective* if it is of constant width 0 that is, if: $\forall u \in \mathbb{S}^1$, $h(-u) = -h(u)$. A not too singular projective hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ is a curve that has exactly one nonoriented tangent line in each direction (Figure 1(c)).

In the Euclidean vector plane $\mathbb{R}^2$, N-hedgehogs are defined in the same way as hedgehogs, except that their support functions are $2N\pi$-periodic instead of being 2π-periodic ($N \in \mathbb{N}^*$). The integer N is just the number of full rotations of the coorienting normal vector $u(\theta) = (\cos\theta, \sin\theta)$ as θ describes $[0, 2N\pi[$. Therefore, an N-hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ has exactly N cooriented support lines with a given normal vector $u \in \mathbb{S}^1$ (counted with their multiplicity). Naturally, hedgehogs of $\mathbb{R}^2$ are simply 1-hedgehogs of $\mathbb{R}^2$. Here are some examples of "multihedgehogs" of $\mathbb{R}^2$: for every $n \geq 2$, the hypocycloid (resp. the epicycloid) with support function $h_n(\theta) = \sin(n\theta)$ (resp. $e_n(\theta) = \sin((n-1)\theta/n\theta)$) is a 1-hedgehog (resp. an n-hedgehog) with $2n$ (resp. $2(n-1)$) cusps (cf. Figure 2); when n is odd, the cusps of the hypocycloid are counted twice because $\mathcal{H}_{h_n}$ is described twice by $x_{h_n}(\theta)$ as θ describes $[0, 2\pi[$.

Hedgehog theory. The notion of hedgehog with support function C^2 extends to the Euclidean vector space $\mathbb{R}^{n+1}$ ($n \in \mathbb{N}^*$) [5]. Every hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ with support function C^2 can be regarded as a difference of convex

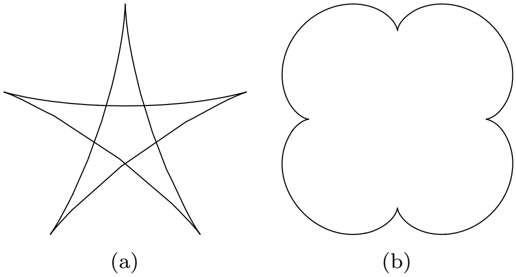


FIGURE 2. (a) The hypocycloid $\mathcal{H}_{h_5}$. (b) The epicycloid $\mathcal{H}_{e_3}$.

bodies of class C^2_+ (i.e., bounded by a C^2 surface with nonvanishing Gauss curvature). To this end, it suffices to write h in the form $(h+r)-r$, where $r \in \mathbb{R}^*_+$ is large enough. We can also associate a hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ to any function $h \in C^1(\mathbb{S}^1;\mathbb{R})$. But in general, such a hedgehog cannot be interpreted as a difference of convex bodies (and may even be a fractal) [10].

More generally, this interpretation of hedgehogs with a C^2 support function as differences of convex bodies of class C^2_+ leads to associate an appropriate geometric realization to any formal difference of convex bodies of $\mathbb{R}^{n+1}$ [10]. Hedgehog theory then consists in: 1. regarding each formal difference $K-L$ of convex bodies $K, L \subset \mathbb{R}^{n+1}$ as a (possibly singular and self-intersecting) hypersurface of $\mathbb{R}^{n+1}$, called a *hedgehog* and determined by the difference $h = h_K - h_L$ of the support functions; 2. extending the mixed volume $V : (\mathcal{K}^{n+1})^{n+1} \to \mathbb{R}$, where $\mathcal{K}^{n+1}$ denotes the set of convex bodies of $\mathbb{R}^{n+1}$, to a symmetric $(n+1)$-linear form on the vector space $\mathcal{H}^{n+1}$ of hedgehogs of $\mathbb{R}^{n+1}$; 3. considering the Brunn–Minkowski theory in $\mathcal{H}^{n+1}$. For $n \le 2$, the idea goes back to a paper by Geppert [4, 1937]. Its relevance comes from the two following principles:

1. The study of convex bodies by splitting them judiciously (that is, according to the problem under consideration) into a sum of hedgehogs;
2. The geometrization of analytical problems by regarding real functions as support functions of hedgehogs or multihedgehogs.

The first one enabled the author to disprove a characterization of the 2-sphere conjectured by Alexandrov in the thirties [7] and the second one to give a geometrical proof of the Sturm–Hurwitz theorem [9]. Both principles will be used in this paper. The reader will find a short introduction to the theory in [10]. A discrete version of hedgehogs has been studied since the nineties under the name "virtual polytopes" (see the paper by Panina [11], which gives new counterexamples to the Alexandrov conjecture). For a survey on the Brunn–Minkowski theory, we refer the reader to the book by Schneider [12].

Introduction of results. The main result is the following theorem.

THEOREM 1. *Let h be a real $2N\pi$-periodic function of class C^2 on $\mathbb{R}$ ($N \in \mathbb{N}^*$). The number $S \in \mathbb{N} \cup \{\infty\}$ of zeros of $h + h'$ in $[0, 2N\pi[$ satisfies the inequality*

$$\sup_{x \in \mathbb{R}^2} n_h(x) \le S + 4N - 2, \tag{1}$$

where $n_h(x) \in \mathbb{N} \cup \{\infty\}$ denotes the number of zeros of $h_x(\theta) = h(\theta) - \langle x, u(\theta) \rangle$ in $[0, 2N\pi[$.

It can naturally be interpreted in terms of N-hedgehogs.

GEOMETRICAL INTERPRETATION. *Let $\mathcal{H}_h$ be an N-hedgehog of $\mathbb{R}^2$. The number $S \in \mathbb{N} \cup \{\infty\}$ of its singular points (counted on $[0, 2N\pi[$) satisfies the* *inequality*

$$\sup_{x\in\mathbb{R}^2} n_h(x) \le S + 4N - 2,$$

where $n_h(x) \in \mathbb{N} \cup \{\infty\}$ *denotes the number of cooriented support lines of* $\mathcal{H}_h$ *through* x *(counted with their multiplicity).*

Note that Theorem 1 allows us to estimate from above the number of zeros that the solutions of certain differential equations may have.

Reformulation of Theorem 1. *Let* R *be a real* $2N\pi$*-periodic continuous function on* $\mathbb{R}$ *such that*

$$\int_0^{2N\pi} R(\theta)u(\theta)\,d\theta = 0_{\mathbb{R}^2},$$

and let $S \in \mathbb{N} \cup \{\infty\}$ *be the number of its zeros in* $[0, 2N\pi[$. *Then we have*

$$\sup_{h\in\mathcal{S}_R} n(h) \le S + 4N - 2,$$

where $n(h) \in \mathbb{N} \cup \{\infty\}$ *is the number of zeros of* h *in* $[0, 2N\pi[$ *and* $\mathcal{S}_R$ *the set of solutions of the differential equation* $h''(\theta) + h(\theta) = R(\theta)$.

Remark. The inequality of Theorem 1 can be reinforced in various particular cases (see below). However, under assumptions of Theorem 1, this inequality is sharp: for every $(N, S) \in \mathbb{N}^* \times 2\mathbb{N}$, there exists an N-hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ having exactly S cusps and satisfying the inequality

$$\sup_{x\in\mathbb{R}^2} n_h(x) = S + 4N - 2. \tag{2}$$

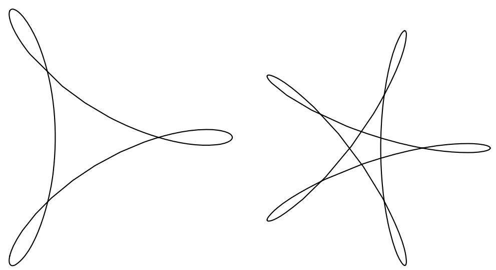
For every $N = k + 1 \ge 2$, the two N-hedgehogs having the same geometrical realization as the hypotrochoid Γ_k parametrized by $\gamma_k : [0, 2(k+1)\pi] \to \mathbb{R}^2$, $t \mapsto (\gamma_k^1(t), \gamma_k^2(t))$, where

$$\begin{cases} \gamma_k^1(t) = 2k\cos t + (2k+1)\cos\frac{kt}{k+1}, \\ \gamma_k^2(t) = 2k\sin t - (2k+1)\sin\frac{kt}{k+1} \end{cases}$$

are singularity-free and satisfy (2). Figure 3 represents Γ_k for $k = 1$ and $k = 2$.

For every $S \in 2\mathbb{N}$, we can then make pairs of cusps to appear successively (as shown on Figure 4) in the region of the plane where the index achieves its minimum value in order to obtain an N-hedgehog having exactly S cusps and still satisfying (2).

For $N = 1$, every regular N-hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ satisfies (2) and, for every $S \in 2\mathbb{N}$, we can then make pairs of cusps to appear successively in the region of the plane where the index achieves its minimum value in order to obtain a hedgehog having exactly S cusps and still satisfying (2). For $S = 2$, one obtains a hedgehog such that the hedgehog $\mathcal{H}_h$ represented in Figure 5, where $h(\theta) = -7 - 4\cos(2\theta) - \cos(3\theta)$.



(a) (b)

FIGURE 3. (a) $(N,S)=(2,0)$. (b) $(N,S)=(3,0)$.

The conclusion of Theorem 1 can be reinforced in the case where h is a Möbius function, i.e., a 2π-periodic function such that $h(\theta+\pi)=-h(\theta)$ for every $\theta\in\mathbb{R}$.

THEOREM 2. *Let h be a real 2π-periodic function of class C^2 on $\mathbb{R}$ such that $h(\theta+\pi)=-h(\theta)$ for every $\theta\in\mathbb{R}$. The number $S\in\mathbb{N}\cup\{\infty\}$ of zeros of $h+h''$ in $[0,2\pi[$ satisfies the inequality*

$$\sup_{x\in\mathbb{R}^2} n_h(x)\le S, \tag{3}$$

where $n_h(x)\in\mathbb{N}\cup\{\infty\}$ is the number of zeros of $h_x(\theta)=h(\theta)-\langle x,u(\theta)\rangle$ in $[0,2\pi[$.

This result can naturally be interpreted in terms of projective hedgehogs.

GEOMETRICAL INTERPRETATION. *Let $\mathcal{H}_h$ be a projective hedgehog of $\mathbb{R}^2$. The number $S\in\mathbb{N}\cup\{\infty\}$ of its singular points (counted on $\mathbb{S}^1$) satisfies the inequality*

$$\sup_{x\in\mathbb{R}^2} n_h(x)\le S,$$

where $n_h(x)\in\mathbb{N}\cup\{\infty\}$ is the number of cooriented support lines of $\mathcal{H}_h$ through x that is, the number of zeros of $h_x(\theta)=h(\theta)-\langle x,u(\theta)\rangle$ in $[0,2\pi[$.

REMARK. The equality

$$\sup_{x\in\mathbb{R}^2} n_h(x)=S$$

holds for every projective hedgehog that is the geometrical realization of a hypocycloid with an odd number of cusps, i.e., with a support function of the

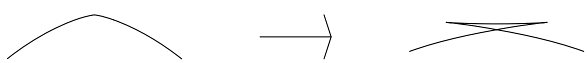


FIGURE 4. Creation of pairs of cusps.

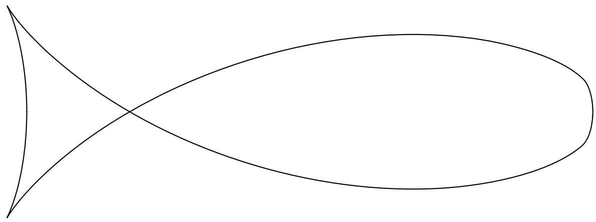


FIGURE 5. Case $(N,S)=(1,2)$.

form

$$h(\theta)=a\cos\theta+b\sin\theta+c\,\cos[(2n+1)\theta]+d\,\sin[(2n+1)\theta],$$

where $(a,b)\in\mathbb{R}^2$, $(c,d)\in\mathbb{R}^2-\{(0,0)\}$ and $n\in\mathbb{N}^*$.

Other geometric applications. These results also allow us to compare number of vertices and maximal number of concurrent normal lines for a convex curve of class C^3_+ in $\mathbb{R}^2$.

COROLLARY 1. *Let $\mathcal{H}_h\subset\mathbb{R}^2$ be an N-hedgehog of class C^3_+. The number $S'\in\mathbb{N}\cup\{\infty\}$ of its vertices satisfies the inequality*

$$\sup_{x\in\mathbb{R}^2} n'_h(x)\leq S'+4N-2, \tag{4}$$

where $n'_h(x)\in\mathbb{N}\cup\{\infty\}$ is the number of oriented normal lines to $\mathcal{H}_h$ through x (counted with their multiplicity) that is, the number of zeros of the derivative of $h_x(\theta)=h(\theta)-\langle x,u(\theta)\rangle$ in $[0,2N\pi[$.

In the case of a curve of constant width, the conclusion can be reinforced.

COROLLARY 2. *Let $\mathcal{H}_h\subset\mathbb{R}^2$ be a convex curve of class C^3_+. If $\mathcal{H}_h$ is of constant width (i.e., if there exists $L\in\mathbb{R}$ such that: $\forall\theta\in\mathbb{R}$, $h(\theta+\pi)+h(\theta)=L$), then the number $S'\in\mathbb{N}\cup\{\infty\}$ of its vertices satisfies the inequality*

$$\sup_{x\in\mathbb{R}^2} n'_h(x)\leq S', \tag{5}$$

where $n'_h(x)\in\mathbb{N}\cup\{\infty\}$ is the number of oriented normal lines to $\mathcal{H}_h$ through x (i.e., the number of zeros of the derivative of $h_x(\theta)=h(\theta)-\langle x,u(\theta)\rangle$ in $[0,2\pi[$).

Since the first nonvanishing harmonics of $h'+h'''$ is then of order $N\geq 3$, $\mathcal{H}_h$ has at least 6 vertices from the Sturm–Hurwitz theorem. The author: (i) proved that for every convex curve of constant width and of class C^2_+ of $\mathbb{R}^2$, there exists a point of $\mathbb{R}^2$ through which infinitely many normals pass or an open set of points through each of which pass at least 6 normals [6];

(ii) studied under which necessary and sufficient condition(s) on $\mathcal{H}_h$, we have $S' = \sup_{x\in\mathbb{R}^2} n'_h(x) = 6$ [8].

Lastly, by projective duality, Theorems 1 and 2 allow to compare, for any spherical curve that is transverse to the meridians, the number of inflection points to the maximal number of intersection points with a great circle. Recall that, by virtue of the Arnold Tennis Ball Theorem, "*Every closed simple smooth spherical curve dividing the sphere $\mathbb{S}^2$ into two regions of equal areas has at least* 4 *inflection points*" ([1], [3]). Here, an inflection point is simply a zero of the geodesic curvature that is, a point of at least second order tangency of the curve with a great circle. See [2] for the case where the spherical curve is invariant under the antipodal map.

Corollary 3. *Let $\mathcal{C}$ be a closed simple smooth curve on the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$. If $\mathcal{C}$ is everywhere transverse to the meridians then the number $I \in \mathbb{N}\cup\{\infty\}$ of its inflection points (i.e., of zeros of its geodesic curvature) satisfies the inequality*

$$\sup_{v\in\mathbb{S}^2} I(v) \le I + 2, \tag{6}$$

where $I(v) \in \mathbb{N}\cup\{\infty\}$ is the number of intersection points of $\mathcal{C}$ with the great circle $\mathbb{S}^2 \cap v^\perp$, $v^\perp \subset \mathbb{R}^3$ denoting the vector plane that is orthogonal to v.

If moreover the curve $\mathcal{C}$ is invariant under the antipodal map, then

$$\sup_{v\in\mathbb{S}^2} I(v) \le I. \tag{7}$$

2. Proof of results and further remarks

Proof of Theorem 1. Let us demonstrate that the inequality

$$n_h(x) \le S + 4N - 2 \tag{8}$$

holds for every $x \in \mathbb{R}^2$. As the equality is obvious when $S = +\infty$, we may assume that S is finite. The proof will rely on the following lemma.

Lemma. *For every line D of $\mathbb{R}^2$, we have:*

$$\#\big(\{\theta \in [0, 2N\pi[\ |\ x_h(\theta) \in D\}\big) \le S + 2N.$$

Proof. We may assume without loss of generality that D is the x_1-axis, where (x_1, x_2) are the coordinates of x in the canonical basis of $\mathbb{R}^2$. In this case, we have $x_h(\theta) = (x_h^1(\theta), x_h^2(\theta)) \in D$ if and only if $x_h^2(\theta) = 0$. Since Rolle's theorem ensures that the zeros of $(x_h^2)'(\theta) = R_h(\theta)\cos\theta$ separate those of x_h^2, we thus have indeed

$$S \ge \#\big(\{\theta \in [0, 2N\pi[\ |\ x_h(\theta) \in D\}\big) - 2N,$$

from the fact that the equation $\cos\theta = 0$ has exactly $2N$ solutions in the interval $[0, 2N\pi[$.

Let us distinguish two cases.

First case: $x \in \mathbb{R}^2 - \mathcal{H}_h$. Let $i_h(x)$ be the index of x with respect to the oriented curve $\mathcal{H}_h$, that is, the winding number of $\mathcal{H}_h$ around x (defined in terms of degree or using complex analysis). Let us recall that:

(i) The orientation of an N-hedgehog such as $\mathcal{H}_h$ is determined by the direction of the concavity of its regular subpaths. For this reason, the hedgehog $\mathcal{H}_{h_3}$ (resp. the 3-hedgehog $\mathcal{H}_{e_3}$) represented in Figure 1(c) (resp. Figure 2(b)) is oriented in the positive (resp. negative) sense.

(ii) The index $i_h(x)$ can be regarded as the algebraic intersection number of $\mathcal{H}_h$ and almost every oriented half-line with origin x. More precisely, for every oriented half-line D with origin x that does not meet the singular locus of $\mathcal{H}_h$, we have:

$$i_h(x) = \sum_{\theta \in x_h^{-1}(D)} \operatorname{sign}[\langle x_h(\theta) - x, u(\theta)\rangle R_h(\theta)].$$

(iii) The number $n_h(x)$ of cooriented support lines of $\mathcal{H}_h$ through x satisfies $n_h(x) = 2(N - i_h(x))$ [9].

Therefore, inequality (8) is obvious if $i_h(x) \geq 0$. Thus, let us assume $i_h(x) < 0$. As we have then

$$2\big(1 + |i_h(x)|\big) = 2\big(1 - i_h(x)\big) = n_h(x) - 2(N-1),$$

the lemma ensures that it suffices to prove the existence of a line $D \subset \mathbb{R}^2$ such that

$$\#\big(\{\theta \in [0, 2N\pi[\mid x_h(\theta) \in D\}\big) \geq 2\big(1 + |i_h(x)|\big).$$

To this end, we may assume that $h_x R_{h_x}$ changes sign on $\mathbb{R}$. Indeed, since $x \in \mathbb{R}^2 - \mathcal{H}_h$ the function h_x changes sign at each of its zero so that if $h_x R_{h_x}$ does not change sign then R_{h_x} (i.e., R_h) changes sign at each zero of h_x and thus $n_h(x) \leq S$, which implies that inequality (8) holds.

Now, the sign of $h_x R_{h_x}$ indicates the direction of the concavity of $\mathcal{H}_h$ with respect to x that is, the direction in which

$$\gamma(\theta) = \frac{x_h(\theta) - x}{|\, x_h(\theta) - x\|}$$

is moving on the circle $\mathbb{S}^1$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. Therefore, if $h_x R_{h_x}$ changes sign on $\mathbb{R}$, then there exists a line D through x for which

$$\#\big(\{\theta \in [0, 2N\pi[\mid x_h(\theta) \in D\}\big) \geq 2\big(1 + |i_h(x)|\big).$$

Second case: $x \in \mathcal{H}_h$. By virtue of the first case, it suffices to prove that there exists a $y \in \mathbb{R}^2 - \mathcal{H}_h$ such that $n_h(x) \leq n_h(y)$. To this end, we are going to study variations of n_h on a line D that passes through x and is distinct from support lines of $\mathcal{H}_h$. We may assume without loss of generality that D is the x_1-axis and that x is the origin $0_{\mathbb{R}^2}$ of $\mathbb{R}^2$. The lemma ensures that the set

$$\{\theta \in [0, 2N\pi[\mid x_h(\theta) \in D, \text{ i.e., } x_h^2(\theta) = 0\}$$

is finite and therefore that there exists an $\varepsilon > 0$ such that $(]-\varepsilon,\varepsilon[\times\{0\})\cap\mathcal{H}_h = \{0_{\mathbb{R}^2}\}$. For every $\theta\in\mathbb{R}$ such that $\cos\theta\neq 0$, the support line with equation $\langle x,u(\theta)\rangle = h(\theta)$ cuts the x_1-axis at a point $(x_1(\theta),0)$. The map $x_1:\mathbb{R}-\{\frac{\pi}{2}+k\pi | k\in\mathbb{Z}\}\to\mathbb{R}$ is given by

$$x_1(\theta) = \frac{h(\theta)}{\cos\theta},$$

so that

$$x_1'(\theta) = \frac{h'(\theta)\cos\theta + h(\theta)\sin\theta}{\cos^2\theta} = \frac{x_h^2(\theta)}{\cos^2\theta}.$$

Therefore, on every connected component C of $x_1^{-1}(]-\varepsilon,\varepsilon[)$, the map $x_1 : C\to\mathbb{R}$ is never constant on a nontrivial segment and its variation direction changes only at isolated points at which x_h^2 vanishes and changes of sign (i.e., at points for which $\mathcal{H}_h$ crosses the x_1-axis at $0_{\mathbb{R}^2}$). The map n_h is thus of the following form on $]-\varepsilon,\varepsilon[\times\{0\}$:

$$n_h(x_1,0) = \begin{cases} 2(p+r), & \text{if } -\varepsilon < x_1 < 0, \\ p+q+2r, & \text{if } x_1 = 0, \\ 2(q+r), & \text{if } 0 < x_1 < \varepsilon, \end{cases}$$

where $r\in\mathbb{N}$ and where

$$p = \#(\{\theta\in[0,2N\pi[\,|\, x_1 \text{ achieves a maximum equal to 0 at } \theta\})$$

and

$$q = \#(\{\theta\in[0,2N\pi[\,|\, x_1 \text{ achieves a minimum equal to 0 at } \theta\}).$$

As $p+q\leq 2\max(p,q)$, it follows that there exists indeed a $y\in\mathbb{R}^2-\mathcal{H}_h$ such that $n_h(y)\geq n_h(x)$. □

□

Proof of Theorem 2. By assumption, the hedgehog $\mathcal{H}_h$ is projective that is, its support function h satisfies: $\forall\theta\in\mathbb{R}, h(\theta+\pi) = -h(\theta)$. The parametrization x_h is then π-periodic so that $x_h(\theta)$ describes $\mathcal{H}_h$ twice as θ describes the segment $[0,2\pi]$. Therefore, for every $x\in\mathbb{R}^2-\mathcal{H}_h$ such that $h_x R_{h_x}$ changes sign on $\mathbb{R}$, there exists a line D through x for which

$$\#\big(\{\theta\in[0,2N\pi[\,|\, x_h(\theta)\in D\}\big)\geq 2\big(2+|i_h(x)|\big).$$

Repeating the proof of Theorem 1 step by step, it follows that h satisfies (3). □

Proofs of Corollaries 1 and 2. The following remarks enable us to consider these corollaries as applications of Theorems 1 and 2 to the evolute of $\mathcal{H}_h$:

(i) Vertices of $\mathcal{H}_h$ correspond to singular points of its evolute. Now, we know (cf. [9]) that this evolute is the N-hedgehog with support function

$$\partial h(\theta) = h'\left(\theta - \frac{\pi}{2}\right) \quad (\theta \in \mathbb{R}).$$

The number of vertices of $\mathcal{H}_h$ is thus equal to the number of zeros of $R_{\partial h}$ in $[0, 2N\pi[$.

(ii) For every $x \in \mathbb{R}^2$, we have:

$$\forall \theta \in \mathbb{R}, \quad (\partial h)_x\left(\theta + \frac{\pi}{2}\right) = (h_x)'(\theta),$$

where $(\partial h)_x$ is given by $(\partial h)_x(\theta) = \partial h(\theta) - \langle x, u(\theta)\rangle$ for every $\theta \in \mathbb{R}$. □

Proof of Corollary 3. Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$. Note that upper bounds of the sets $\{I(v) \mid v \in \mathbb{S}^2\}$ and $\{I(v) \mid v \in \mathbb{R}^2 \times \{-1\}\}$ are equal by the fact that $\mathcal{C}$ is transverse to the meridians (i.e., to the great circles through e_3). By virtue of assumptions, $\mathcal{C}$ admits a parametrization of the form

$$\begin{aligned} \gamma_h : \mathbb{S}^1 &\longrightarrow \mathbb{S}^2 \subset \mathbb{R}^2 \times \mathbb{R}, \\ u &\longmapsto \frac{1}{\sqrt{1 + h(u)^2}}(u, h(u)), \end{aligned}$$

where $h : \mathbb{S}^1 \to \mathbb{R}$ is the support function of a hedgehog of $\mathbb{R}^2$. The curves $\mathcal{C}$ and $\mathcal{H}_h$ can be regarded as two projectively dual curves: the intersection of $\mathbb{S}^2$ (resp. of $\mathbb{R}^2 \times \{-1\}$) with the vector plane that is orthogonal to $(x_h(u), -1)$ (resp. $\gamma_h(u)$) is the great circle of $\mathbb{S}^2$ that is tangent to $\mathcal{C}$ at $\gamma_h(u)$ (resp. the support line of the hedgehog $\mathcal{H}_h \times \{-1\} \subset \mathbb{R}^2 \times \{-1\}$ with normal vector $(u, -1)$). Therefore, inflection points of $\mathcal{C}$ correspond exactly to singular points of $x_h : \mathbb{S}^1 \to \mathbb{R}^2$. Besides, for every $v = (x, -1) \in \mathbb{R}^2 \times \{-1\}$, points of the intersection of $\mathcal{C}$ with the great circle $\mathbb{S}^2 \cap v^\perp$ correspond exactly to cooriented support lines of $\mathcal{H}_h$ through x (i.e., to zeros of $h_x(u) = h(u) - \langle x, u\rangle$ on $\mathbb{S}^1$). Inequalities (6) and (7) are thus straightforward consequences of those of Theorems 1 and 2. □

References

[1] V. I. Arnold, *A branched covering of $\mathbb{C}P^2 \to \mathbb{S}^4$, hyperbolicity and projectivity topology*, Sib. Math. J. **29** (1988), 717–726. MR 0971226

[2] V. I. Arnold, *Topological invariants of plane curves and caustics*, University Lecture Series, vol. 5, Amer. Math. Soc., Providence, 1994. MR 1286249

[3] V. I. Arnold, *Topological properties of Legendre projections in contact geometry of wave fronts*, St. Petersburg Math. J. **6** (1995), 439–452. MR 1301827

[4] H. Geppert, *Über den Brunn–Minkowskischen Satz*, Math. Z. **42** (1937), 238–254. MR 1545673

[5] R. Langevin, G. Levitt and H. Rosenberg, *Hérissons et multihérissons (enveloppes paramétrées par leur application de Gauss)*. Singularities, Banach Center Publ. **20** (1988), 245–253. MR 1101843

[6] Y. Martinez-Maure, *Indice d'un hérisson: étude et applications*, Publ. Mat. **44** (2000), 237–255. MR 1775763

[7] Y. Martinez-Maure, *Contre-exemple à une caractérisation conjecturée de la sphère*, C. R. Acad. Sci. Paris, Sér. I **332** (2001), 41–44. MR 1805625

[8] Y. Martinez-Maure, *Sommets et normales concourantes des courbes convexes de largeur cons tante et singularités des hérissons*, Arch. Math. **79** (2002) 489–498. MR 1967267

[9] Y. Martinez-Maure, L*es multihérissons et le théorème de Sturm–Hurwitz*, Arch. Math. **80** (2003), 79–86. MR 1968290

[10] Y. Martinez-Maure, *Geometric study of Minkowski differences of plane convex bodies*, Canad. J. Math. **58** (2006), 600–624. MR 2223458

[11] G. Panina, *New counterexamples to A. D. Alexandrov's hypothesis*, Adv. Geom. **5** (2005), 301–317. MR 2131822

[12] R. Schneider, *Convex bodies: The Brunn–Minkowski theory*, Cambridge Univ. Press, Cambridge, 1993. MR 1216521

[13] S. Tabachnikov, *Around four vertices*, Russian Math. Surveys **45** (1990), 229–230. MR 1050943

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