

SYMMETRIZATION AND HARMONIC MEASURE

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Dedicated to Matti Vuorinen on the occasion of his sixtieth birthday.

ABSTRACT. We prove the equality statements for the classical symmetrization estimates for harmonic measure. In fact, we prove more general results for α -harmonic measure. The α -harmonic measure is the hitting distribution of symmetric α -stable processes upon exiting an open set in \mathbb{R}^n ($0 < \alpha < 2$, $n \geq 2$). It can also be defined in the context of Riesz potential theory and the fractional Laplacian. We prove polarization and symmetrization inequalities for α -harmonic measure. We give a complete description of the corresponding equality cases. The proofs involve analytic and probabilistic arguments.

1. Introduction

The classical symmetrization estimates for harmonic measure were proved in 1974 by Baernstein [1] (for planar harmonic measure) and in 1976 by Baernstein and Taylor [4] (in higher dimensions). Equality statements were proved by Essén and Shea [20] and by Solynin [25] under certain regularity conditions. We will prove equality statements without regularity assumptions. In fact, we will work in the more general context of Riesz potential theory.

The α -harmonic functions, $0 < \alpha < 2$, are defined by a mean value property (involving the parameter α), analogous to the classical one. Equivalently, they are the solutions of the equation $\Delta^{\alpha/2}u = 0$, where $\Delta^{\alpha/2}$ is the *fractional Laplacian*, a nonlocal integro-differential operator.

A function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, $n \geq 2$, which is α -harmonic in an open set D is determined by its exterior values (its values in $D^c := \mathbb{R}^n \setminus D$). If B is a Borel set in D^c , the α -harmonic measure of B with respect to D is the α -harmonic

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function u in D with exterior values $u = \chi_B$ on D^c . The α -harmonic measure of B with respect to D , evaluated at the point $x \in \mathbb{R}^n$, will be denoted by $\omega_\alpha^D(x, B)$. For fixed $x \in D$, $\omega_\alpha^D(x, \cdot)$ is a Borel probability measure on D^c .

Both classical and α -harmonic measures have symmetry properties and satisfy the Carleman principle (domain monotonicity) and the Harnack principle. The latter implies that if $\omega_\alpha^D(x, B) = 0$ for some $x \in D$, then $\omega_\alpha^D(y, B) = 0$ for all $y \in D$; we say then that B is a D -null set. There are, however, essential differences. The classical harmonic measure is defined (as function) in a domain D and is supported (as measure) on the boundary of D . The α -harmonic measure is defined (as function) in whole \mathbb{R}^n and is supported (as measure) in the exterior of D . These properties become transparent when are viewed from the probabilistic point of view. The classical harmonic measure is the hitting distribution of Brownian motion upon exiting D , while the α -harmonic measure is the hitting distribution of symmetric α -stable process. This is a Hunt process with discontinuous paths. Thus, its paths may jump from one component of D to another, and may hit D^c (upon exiting D) at points of $(\overline{D})^c$ and not necessarily at points of ∂D .

The basic facts of Riesz potential theory are presented in the book of Landkof [22]. Recently, there has been a renewed interest on Riesz potential theory, mainly from the probabilistic point of view. Bogdan [11] proved the boundary Harnack principle for α -harmonic functions on Lipschitz open sets. Song and Wu [26] proved extensions of Bogdan's results. Bogdan [12] and Chen with Song [16] gave a Martin representation for nonnegative α -harmonic functions. Wu [27] found necessary or sufficient conditions for a boundary set to have zero α -harmonic measure. Bañuelos, Latala, and Méndez-Hernández [5] proved isoperimetric type inequalities for transition probabilities, Green functions, and eigenvalues associated with symmetric stable processes. Various other properties and applications of α -harmonic functions and the fractional Laplacian are presented in [15], [13], [7], and [8].

In the present article, we study the behavior of α -harmonic measure under symmetrization. There are various kinds of symmetrization. For the sake of concreteness, we will state and prove symmetrization results only for 1-dimensional Steiner symmetrization. The results, however, hold for all kinds of Steiner and cap symmetrization with the obvious modifications. The 1-dimensional Steiner symmetrization of a set $A \subset \mathbb{R}^n$ with respect to an $(n - 1)$ -dimensional hyperplane H is a set having the same volume as A , convex in the direction perpendicular to H , and symmetric under reflection in H . A rigorous definition appears in Section 2.

Let $\Pi = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}$. Every $(n - 1)$ -dimensional hyperplane in \mathbb{R}^n will be simply called *plane*. Every plane parallel to Π will be called *horizontal*. A line will be called *vertical* if it is perpendicular to Π . For an open or closed set $A \subset \mathbb{R}^n$, we denote by $S_H A$ the 1-dimensional Steiner symmetrization of A with respect to the plane H . If $H = \Pi$, we write $S_H A = A^\sharp$.

If $p = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n$, we denote by p^\sharp the orthogonal projection of p on Π , $p^\sharp = (x_1, \dots, x_{n-1}, 0)$.

In the theorem below, we assume that the open set D lies in a striplike set G . We say that an open set $G \subset \mathbb{R}^n$ is a *striplike set* if for every vertical line l that intersects G , we have $l \cap G = l$. This condition guarantees that if B is a Borel set in G^c , then B^\sharp lies also in G^c . By m_k , $k \in \mathbb{N}$, we denote the k -dimensional Lebesgue measure.

THEOREM 1. *Let D be an open set in \mathbb{R}^n lying in a striplike set G . Let Σ be a vertical line intersecting D . Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ be a nonconstant, convex, increasing function. Let B be a closed set in G^c . Then*

$$(1.1) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^D(p, B)) m_1(dp) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^\sharp}(p, B^\sharp)) m_1(dp),$$

$$(1.2) \quad \omega_{\alpha}^D(p, B) \leq \omega_{\alpha}^{D^\sharp}(p^\sharp, B^\sharp), \quad p \in D.$$

An illustration for Theorem 1 appears in Figure 1. For the classical planar harmonic measure ($n = 2$, $\alpha = 2$), the inequalities (1.1) and (1.2) were proved by Baernstein [1] who used his star-function method; see also [21, Chapter 9], [3]. Baernstein’s inequalities were extended to higher dimensions in [4]. Related inequalities for α -harmonic measure were proved in [7] under the additional assumption that the boundary of D is smooth.

The next two theorems deal with the equality cases for the inequalities (1.1) and (1.2). For $\alpha = 2$, the equality cases were studied by Essén and Shea [20] for $n = 2$ and by Solynin [25] for all dimensions; both these works use the additional regularity assumptions. The theorems say (roughly) that equality holds if and only if D and B are essentially symmetric. To make this statement rigorous, we use various types of exceptional sets: (a) Sets of zero Riesz capacity, (b) sets in $(\overline{G})^c$ of zero n -dimensional Lebesgue measure, (c) sets on $\partial G \cap \partial D$ which are D -null.

We do not make any regularity assumption for D or B and so we are obliged to introduce some special terminology and the corresponding notation. We

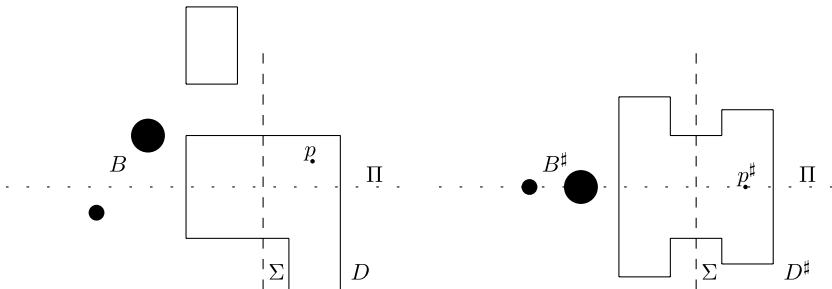


FIGURE 1. An illustration for Theorem 1.

say that an open set D is an *essentially striplike set* if there exists a striplike set $G \supset D$ such that $C_\alpha(G \setminus D) = 0$. Here, C_α denotes the capacity associated with the Riesz kernel; see [22, Chapter II] for more details about this capacity.

NOTATION 1. For an open set D and a plane H , the notation $S_H D \stackrel{C_\alpha}{\equiv} D$ means that there exists an open set Ω such that (i) $D \subset \Omega$, (ii) $C_\alpha(\Omega \setminus D) = 0$, and (iii) $\Omega = S_H \Omega$. For a closed set B , the notation $S_H B \stackrel{m_n}{\equiv} B$ means that the symmetric difference of B and $S_H B$ has m_n -measure zero. More generally, $A \stackrel{m_n}{\equiv} B$ means that the symmetric difference of B and A has m_n -measure zero. Suppose that B is a closed set on the boundary of an open set D with $S_H D \stackrel{C_\alpha}{\equiv} D$. The notation $S_H B \stackrel{D}{\equiv} B$ means that the symmetric difference of B and $S_H B$ is a D -null set.

THEOREM 2. Let D, G, B, Σ, Φ be as in Theorem 1. Let $B_1 = B \cap (\overline{G})^c$ and $B_2 = B \cap \partial G \cap \partial D$. Assume that B is not a D^\sharp -null set and that

$$\int_\Sigma \Phi(\omega_\alpha^D(p, B))m_1(dp) < \infty.$$

(a) Suppose that D is an essentially striplike set, B is bounded, and Φ is affine function (that is, of the form $\Phi(x) = ax + b$). Then

$$(1.3) \quad \int_\Sigma \Phi(\omega_\alpha^D(p, B))m_1(dp) = \int_\Sigma \Phi(\omega_\alpha^{D^\sharp}(p, B^\sharp))m_1(dp).$$

(b) Suppose that D is an essentially striplike set and Φ is not affine in any interval. Then

$$(1.4) \quad \int_\Sigma \Phi(\omega_\alpha^D(p, B))m_1(dp) = \int_\Sigma \Phi(\omega_\alpha^{D^\sharp}(p, B^\sharp))m_1(dp)$$

if and only if there exists a horizontal plane H such that $S_H B_1 \stackrel{m_n}{\equiv} B_1$ and $S_H B_2 \stackrel{D}{\equiv} B_2$.

(c) Suppose that D is not an essentially striplike set. Then

$$(1.5) \quad \int_\Sigma \Phi(\omega_\alpha^D(p, B))m_1(dp) = \int_\Sigma \Phi(\omega_\alpha^{D^\sharp}(p, B^\sharp))m_1(dp)$$

if and only if there exists a horizontal plane H such that $S_H D \stackrel{C_\alpha}{\equiv} D$, $S_H B_1 \stackrel{m_n}{\equiv} B_1$, and $S_H B_2 \stackrel{D}{\equiv} B_2$.

The assumption in (a) that B is bounded is necessary; for example, if $n = 2$, D is the union of two vertical strips with common boundary a vertical line l , and B is a half-line on l , then B^\sharp is the whole line l .

THEOREM 3. Let D, G, B be as in Theorem 1. Let $p \in D$. Let $B_1 = B \cap (\overline{G})^c$ and $B_2 = B \cap \partial G \cap \partial D$. Assume that B is not a D^\sharp -null set. Then

$$(1.6) \quad \omega_\alpha^D(p, B) = \omega_\alpha^{D^\sharp}(p^\sharp, B^\sharp)$$

if and only if there exists a horizontal plane H such that $p \in H$, $S_H D \stackrel{C_\alpha}{=} D$, $S_H B_1 \stackrel{m_\alpha}{=} B_1$, and $S_H B_2 \stackrel{D}{=} B_2$.

The proof of the above symmetrization results is based on the approach to symmetrization via polarization. Polarization is another geometric transformation which is simpler than symmetrization. Moreover, as a general principle, symmetrization can be approximated by a sequence of polarizations. We use concrete statements of this principle to reduce the symmetrization results to polarization results. The latter results are proved by using the probabilistic interpretation of α -harmonic measure. The strong Markov property and the geometric decomposition associated with polarization induce a probabilistic decomposition of the paths of the symmetric stable process. This decomposition reduces the polarization inequalities for α -harmonic measure to similar inequalities for sets of simpler geometry.

In Section 2, we review some facts about Steiner symmetrization, Riesz potential theory, and symmetric stable processes. In Section 3, we introduce polarization and state the related theorems for α -harmonic measure. Sections 4–6 contain the proofs of the polarization theorems. In Section 7, we prove the symmetrization results, Theorems 1–3. Finally in Section 8, we treat two special cases: the classical case ($\alpha = 2$) and the case of regular sets. Theorems 8 and 9 are the equality statements for Baernstein’s symmetrization inequalities for classical harmonic measure.

2. Background

2.1. Steiner symmetrization. We give here the definition of 1-dimensional Steiner symmetrization. Let H be a plane in \mathbb{R}^n . We define the symmetrization $S_H A$ of an open or closed set $A \subset \mathbb{R}^n$ by determining its intersections with every line perpendicular to H . Let $l(x)$ be the line which is perpendicular to H and passes through the point $x \in H$. Let r_x be the 1-dimensional Lebesgue measure of the set $l(x) \cap A$.

- If $0 < r_x < \infty$, let $(-r_x, r_x)$ be the open linear segment on $l(x)$ centered at x with length $2r_x$. Let $[-r_x, r_x]$ be the corresponding closed segment. Then

$$S_H A \cap l(x) := \begin{cases} (-r_x, r_x), & \text{if } A \text{ is open,} \\ [-r_x, r_x], & \text{if } A \text{ is closed.} \end{cases}$$

- If $r_x = 0$, then

$$S_H A \cap l(x) := \begin{cases} \emptyset, & \text{if } A \cap l(x) \text{ is empty,} \\ \{x\}, & \text{if } A \cap l(x) \text{ is nonempty.} \end{cases}$$

- If $r_x = \infty$, then

$$S_H A \cap l(x) = l(x).$$

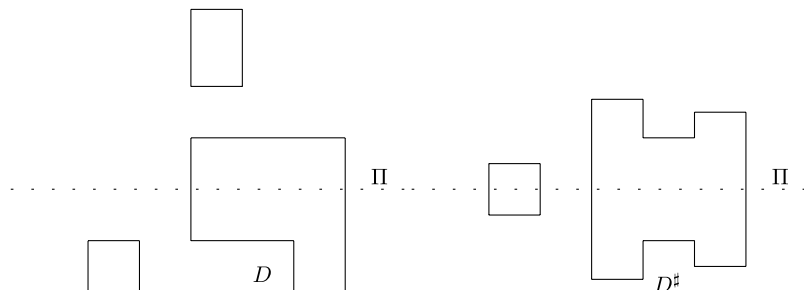


FIGURE 2. An open set D and its symmetrization $D^\#$ with respect to Π .

We refer to [2], [3], [14], [18], [23], [24] and references therein for more information about symmetrization. See Figure 2.

2.2. α -harmonic measure. The Perron–Wiener–Brelot method can be applied for the solution of the Dirichlet problem for α -harmonic functions; see [22, Chapter IV], [10, Chapter VII], [27]. Suppose D is an open set in \mathbb{R}^n . If B is a Borel set in D^c , the α -harmonic measure of B with respect to D , denoted by $\omega_\alpha^D(x, B)$, is defined via the Perron–Wiener–Brelot method. It is the α -harmonic function u in D with exterior values $u = \chi_B$ on D^c .

There is no known geometric characterization of null sets for α -harmonic measure. If a boundary set has zero α -capacity, then it has also zero α -harmonic measure; see [22]. Also, the α -harmonic measure and the Lebesgue measure m_n are mutually absolutely continuous in $(\bar{D})^c$. Some more refined necessary or sufficient conditions are given in [27].

2.3. Symmetric stable processes. (See [10], [11], [12], [13], [15], [16], [26], [27]). From now on, we assume that $0 < \alpha < 2$ (unless otherwise stated). The fractional Laplacian $\Delta^{\alpha/2}$ is the characteristic operator of the symmetric α -stable process $\{X_t, t \in [0, \infty)\}$ in \mathbb{R}^n . This is a Lévy process (homogeneous and with independent increments) with transition density $p_t(x, y) = p_t(y, x) = p_t(x - y)$ (relative to the Lebesgue measure) uniquely determined by its Fourier transform

$$(2.1) \quad \int_{\mathbb{R}^n} e^{ix \cdot \xi} p_t(x) dx = e^{-t|\xi|^\alpha}.$$

When $\alpha = 2$, we get a Brownian motion running at twice the speed.

The probability measures and the corresponding expectations of the process $\{X_t\}$ starting at $x \in \mathbb{R}^n$ will be denoted by \mathbf{P}^x and \mathbf{E}^x . The symmetric α -stable process $\{X_t\}$ is a strong Markov, a strong Feller, and a Hunt process. For $A \subset \mathbb{R}^n$, we put

$$(2.2) \quad T^A = \inf\{t > 0 : X_t \notin A\},$$

the first exit time from A . A Borel function u defined on \mathbb{R}^n is α -harmonic in an open set $D \subset \mathbb{R}^n$ if and only if

$$(2.3) \quad u(x) = \mathbf{E}^x u(\mathbf{X}_{T^U}), \quad x \in U,$$

for every bounded open set U with closure \bar{U} contained in D . If $D \subset \mathbb{R}^n$ is open and B is a Borel subset of D^c , then

$$(2.4) \quad \omega_\alpha^D(x, B) = \mathbf{P}^x(\mathbf{X}_{T^D} \in B), \quad x \in \mathbb{R}^n.$$

3. Polarization results for α -harmonic measure

For $E \subset \mathbb{R}^n$, we denote by \widehat{E} the reflection of E in the $(n - 1)$ -dimensional plane Π . Thus, we have

$$\widehat{E} = \{(x_1, \dots, x_{n-1}, x_n) : (x_1, \dots, x_{n-1}, -x_n) \in E\}.$$

We will also use the following notation: if $x = (x_1, \dots, x_{n-1}, x_n)$ then $\hat{x} := (x_1, \dots, x_{n-1}, -x_n)$; $E_+ := \{(x_1, \dots, x_{n-1}, x_n) \in E : x_n > 0\}$; $E_o := E \cap \Pi$; $E_- := \{(x_1, \dots, x_{n-1}, x_n) \in E : x_n < 0\}$.

Now, let E be any set in \mathbb{R}^n . We divide E into three subsets $S, U, V : S = S_E = \{x \in E : \hat{x} \in E\} = E \cap \widehat{E}$ (the *symmetric part* of E), $U = U_E = \{x \in E : x \in E_+, \hat{x} \notin E\} = E_+ \setminus S_E$ (the *upper nonsymmetric part* of E), $V = V_E = \{x \in E : x \in E_-, \hat{x} \notin E\} = E_- \setminus S_E$ (the *lower nonsymmetric part* of E). Then $E = S \cup U \cup V$. The *polarization* E^* of E is the set

$$E^* := S \cup U \cup \widehat{V}.$$

Equivalently, $E^* = (E \cup \widehat{E})_+ \cup (E \cap \widehat{E})_-$. See Figure 3.

If $x \in \mathbb{R}^n$, we set

$$x^* = \begin{cases} x, & \text{if } x \in \mathbb{R}_+^n \cup \Pi, \\ \hat{x}, & \text{if } x \in \mathbb{R}_-^n. \end{cases}$$

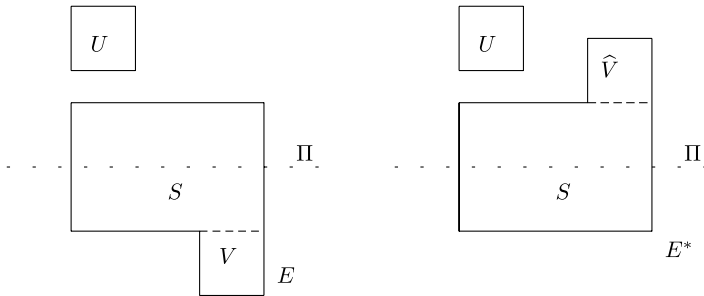


FIGURE 3. A set $E = S \cup U \cup V$ and its polarization $E^* = S \cup U \cup \widehat{V}$.

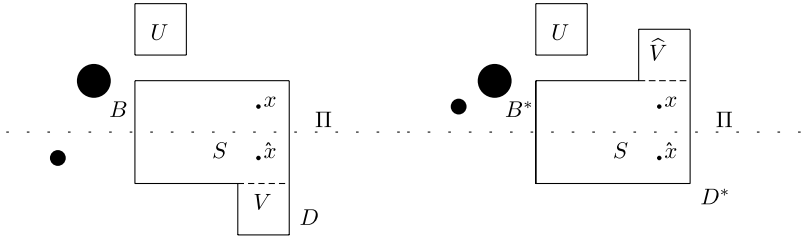


FIGURE 4. An illustration for Theorem 4.

It is clear that the polarization of an open set is open. For more properties of polarization and its relation to symmetrization, we refer to [2], [6], [7], [9], [14], [17], [18], [24], [25].

NOTATION 2. The polarization as defined above may be called polarization with respect to Π . In a similar way, one can define polarization with respect to any other oriented $(n - 1)$ -dimensional plane in \mathbb{R}^n . Let H be such a plane. We denote by $P_H E$ the polarization of E with respect to H . We also denote by $R_H E$ the reflection of E in H .

THEOREM 4. Let D be an open set in $\mathbb{R}^n, n \geq 2$, and let B be a Borel set in D^c such that $B^* \subset (D^*)^c$. Then for $x \in \mathbb{R}^n$,

$$(3.1) \quad \omega_\alpha^D(x, B) \leq \omega_\alpha^{D^*}(x^*, B^*)$$

and

$$(3.2) \quad \omega_\alpha^D(x, B) + \omega_\alpha^D(\hat{x}, B) \leq \omega_\alpha^{D^*}(x, B^*) + \omega_\alpha^{D^*}(\hat{x}, B^*).$$

In the following theorem, we determine the equality cases in the above inequalities. We need some more pieces of notation. An illustration for Theorem 4 appears in Figure 4.

NOTATION 3. Let D be an open set and B a Borel set. Let S, U, V be the symmetric, upper nonsymmetric, and lower nonsymmetric parts of D , respectively. We write $D \stackrel{C_\alpha}{\cong} D^*$ if $C_\alpha(V) = 0$. We write $D \stackrel{C_\alpha}{\cong} \widehat{D}^*$ if $C_\alpha(U) = 0$. Also, $D \stackrel{C_\alpha}{\cong} \widehat{D}$ means that $C_\alpha(V) = C_\alpha(U) = 0$.

THEOREM 5. Let D be an open set in $\mathbb{R}^n, n \geq 2$ and let B be a Borel set in D^c such that $B^* \subset (D^*)^c$. Denote by S, U , and V the symmetric, upper nonsymmetric, and lower nonsymmetric part of D , respectively. Suppose that B is not a D^* -null set.

(i) Assume that $B \subset \mathbb{R}_+^n$. Then

$$(3.3) \quad \omega_\alpha^D(x, B) = \omega_\alpha^{D^*}(x, B)$$

for some $x \in S_+ \cup S_o \cup U$ if and only if $D \stackrel{C_\alpha}{\cong} D^*$.

(ii) Assume that $B \subset \mathbb{R}_-^n$. Then

$$(3.4) \quad \omega_\alpha^D(x, B) = \omega_\alpha^{D^*}(\hat{x}, \widehat{B})$$

for some $x \in S_- \cup S_o \cup V$ if and only if $D \stackrel{C_\alpha}{\cong} \widehat{D}^*$.

(iii) Assume that $B \subset \mathbb{R}_+^n$. Then

$$(3.5) \quad \omega_\alpha^D(x, B) < \omega_\alpha^{D^*}(\hat{x}, B)$$

for all $x \in S_- \cup V$.

(iv) Assume that $B \subset \mathbb{R}_-^n$. Then

$$(3.6) \quad \omega_\alpha^D(x, B) < \omega_\alpha^{D^*}(x, \widehat{B})$$

for all $x \in S_+ \cup U$.

(v) Assume that B is symmetric with respect to Π . Then

$$(3.7) \quad \omega_\alpha^D(x, B) = \omega_\alpha^{D^*}(x, B)$$

for some $x \in S_+ \cup U$ if and only if $D \stackrel{C_\alpha}{\cong} D^*$.

(vi) Assume that B is symmetric with respect to Π . Then

$$(3.8) \quad \omega_\alpha^D(x, B) = \omega_\alpha^{D^*}(\hat{x}, B)$$

for some $x \in S_- \cup V$ if and only if $D \stackrel{C_\alpha}{\cong} \widehat{D}^*$.

(vii) Assume that B is symmetric with respect to Π . Then

$$(3.9) \quad \omega_\alpha^D(s, B) = \omega_\alpha^{D^*}(s, B)$$

for some $s \in S_o$ if and only if either $D \stackrel{C_\alpha}{\cong} D^*$ or $D \stackrel{C_\alpha}{\cong} \widehat{D}^*$.

(viii) Assume that $B \subset \mathbb{R}_+^n$. Then

$$(3.10) \quad \omega_\alpha^D(s, B) + \omega_\alpha^D(\hat{s}, B) = \omega_\alpha^{D^*}(s, B) + \omega_\alpha^{D^*}(\hat{s}, B)$$

for some $s \in S$ if and only if $D \stackrel{C_\alpha}{\cong} D^*$.

(ix) Assume that $B \subset \mathbb{R}_-^n$. Then

$$(3.11) \quad \omega_\alpha^D(s, B) + \omega_\alpha^D(\hat{s}, B) = \omega_\alpha^{D^*}(s, \widehat{B}) + \omega_\alpha^{D^*}(\hat{s}, \widehat{B})$$

for some $s \in S$ if and only if $D \stackrel{C_\alpha}{\cong} \widehat{D}^*$.

(x) Assume that B is symmetric with respect to Π . Then

$$(3.12) \quad \omega_\alpha^D(s, B) + \omega_\alpha^D(\hat{s}, B) = \omega_\alpha^{D^*}(s, B) + \omega_\alpha^{D^*}(\hat{s}, B)$$

for some $s \in S$ if and only if either $D \stackrel{C_\alpha}{\cong} D^*$ or $D \stackrel{C_\alpha}{\cong} \widehat{D}^*$.

Theorems 4 and 5 lead to convex integral mean inequalities.

THEOREM 6. *Let D be an open set in \mathbb{R}^n and let B be a Borel set in D^c such that $B^* \subset (D^*)^c$. Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ be a nonconstant, convex, increasing function. Let Σ be a Borel set that lies on a k -dimensional plane, orthogonal to Π , and assume that Σ is symmetric with respect to Π . Then*

$$(3.13) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^D(x, B))m_k(dx) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^*}(x, B^*))m_k(dx).$$

NOTATION 4. Let S_B, U_B, V_B be the symmetric, upper nonsymmetric, and lower nonsymmetric parts of B , respectively. The notation $B \stackrel{D^*}{=} B^*$ means that \widehat{V}_B is D^* -null and the notation $B \stackrel{D^*}{=} \widehat{B}^*$ means that U_B is D^* -null.

THEOREM 7. *Let D, B, Φ, Σ be as in Theorem 6. Assume, in addition, that $\Sigma \subset S$, B^* is not D^* -null and*

$$\int_{\Sigma} \Phi(\omega_{\alpha}^D(p, B))m_1(dp) < \infty.$$

(i) *Suppose that $D \stackrel{C_{\alpha}}{=} \widehat{D}$ and that Φ is affine function. Then equality holds in (3.13).*

(ii) *Suppose that $D \stackrel{C_{\alpha}}{=} \widehat{D}$ and that Φ is not affine in any interval. Then equality holds in (3.13) if and only if $B \stackrel{D^*}{=} B^*$ or $B \stackrel{D^*}{=} \widehat{B}^*$.*

(iii) *Suppose that the condition $D \stackrel{C_{\alpha}}{=} \widehat{D}$ is not true. Then equality holds in (3.13) if and only if $(D \stackrel{C_{\alpha}}{=} D^*, B \stackrel{D^*}{=} B^*)$ or $(D \stackrel{C_{\alpha}}{=} \widehat{D}^*, B \stackrel{D^*}{=} \widehat{B}^*)$.*

The following lemma is necessary for the proof of Theorems 4 and 5. It is proved among with other related results in [8].

LEMMA 1. *Let D be an open set in \mathbb{R}^n . Suppose that D is polarized with respect to the plane Π , i.e., $D^* = D$. Let $B \subset \mathbb{R}_+^n \cap D^c$ be a Borel set. Then*

- (i) $\omega_{\alpha}^D(x, B) \geq \omega_{\alpha}^D(\hat{x}, B), x \in \mathbb{R}_+^n \cup \Pi;$
- (ii) $\omega_{\alpha}^D(x, B) \geq \omega_{\alpha}^D(x, \widehat{B}), x \in \mathbb{R}_+^n \cup \Pi;$
- (iii) $\omega_{\alpha}^D(x, B) + \omega_{\alpha}^D(\hat{x}, B) \geq \omega_{\alpha}^D(x, \widehat{B}) + \omega_{\alpha}^D(\hat{x}, \widehat{B}), x \in \mathbb{R}^n;$
- (iv) $\omega_{\alpha}^D(x, B) + \omega_{\alpha}^D(x, \widehat{B}) \geq \omega_{\alpha}^D(\hat{x}, B) + \omega_{\alpha}^D(\hat{x}, \widehat{B}), x \in \mathbb{R}^n.$

Assume, in addition, that B is not D -null. Then for $x \in D_+$, we have

$$(3.14) \quad \omega_{\alpha}^D(x, B) > \omega_{\alpha}^D(\hat{x}, B)$$

and

$$(3.15) \quad \omega_{\alpha}^D(x, B) > \omega_{\alpha}^D(x, \widehat{B}).$$

See Figure 5.

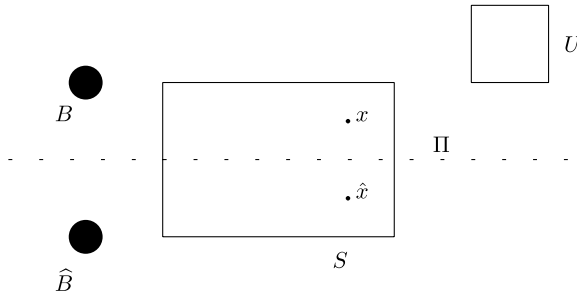


FIGURE 5. An illustration for Lemma 1.

4. Proof of Theorem 4

Let D be an open set in \mathbb{R}^n and B a Borel set in D^c . Let S, U, V denote the symmetric, upper nonsymmetric, and lower nonsymmetric part of D , respectively. If $x \in D^c$, the inequalities (3.1) and (3.2) are trivial. Also, if $x \in U$ or if $x \in V$, then (3.2) reduces to (3.1). We will prove (3.2) for $x = s \in S$; the proof of (3.1) is similar.

From now on $\{X_t, t \geq 0\}$ will denote a symmetric α -stable process ($0 < \alpha < 2$) starting from a point of D , with sample space Ω . We may and do assume that $\{X_t\}$ is the canonical version of the process [19, pp. 87–88]; so the elements ω of Ω are right-continuous functions defined on $[0, \infty)$ with values on \mathbb{R}^n (paths).

For the Borel sets U, V , we define inductively three sequences $\{\tau_k^U\}, \{\tau_k^V\}, \{\tau_k^{\widehat{V}}\}, k \geq 1$ of Markov times as follows:

$$(4.1) \quad \tau_1^U = \inf\{t > 0 : t \leq T^D, X_t \in U\},$$

$$(4.2) \quad \tau_1^V = \inf\{t > 0 : t \leq T^D, X_t \in V\},$$

$$(4.3) \quad \tau_1^{\widehat{V}} = \inf\{t > 0 : t \leq T^{D^*}, X_t \in \widehat{V}\}.$$

For $k \geq 2$, we set

$$(4.4) \quad \tau_k^U = \inf\{t > \tau_{k-1}^V : t \leq T^D, X_t \in U\},$$

$$(4.5) \quad \tau_k^V = \inf\{t > \tau_{k-1}^U : t \leq T^D, X_t \in V\},$$

$$(4.6) \quad \tau_k^{\widehat{V}} = \inf\{t > \tau_{k-1}^U : t \leq T^{D^*}, X_t \in \widehat{V}\}.$$

As usual, if the set $\{t > \tau_k^V : t \leq T^D, X_t \in U\}$ is empty, we set $\tau_{k+1}^U = \infty$; if the set $\{t > \tau_k^U : t \leq T^D, X_t \in V\}$ is empty, we set $\tau_{k+1}^V = \infty$; if the set $\{t > \tau_k^U : t \leq T^D, X_t \in \widehat{V}\}$ is empty, we set $\tau_{k+1}^{\widehat{V}} = \infty$. Thus, for example, $\tau_k^U(\omega)$ is the moment when the path ω makes the k th jump from $S \cup V$ to U (having stayed inside D until that moment). Note that if the process starts from a point in S and $X_{T^U} \in U$, then almost surely $0 < \tau_1^U < \tau_1^V < \tau_2^U < \tau_2^V < \dots$

provided that these Markov times are finite; if the process starts from a point in S and $\mathbf{X}_{T^S} \in V$, then almost surely $0 < \tau_1^V < \tau_1^U < \tau_2^V < \tau_2^U < \dots$ provided that these Markov times are finite.

We consider the sets of paths

$$\mathcal{B} = \{\omega \in \Omega : \mathbf{X}_{T^D} \in B\},$$

$$\mathcal{B}^* = \{\omega \in \Omega : \mathbf{X}_{T^{D^*}} \in B^*\}.$$

Then for $x \in D$, $\mathbf{P}^x(\mathcal{B}) = \omega_\alpha^D(x, B)$ and $\mathbf{P}^x(\mathcal{B}^*) = \omega_\alpha^{D^*}(x, B^*)$. Therefore, in order to prove (3.2) for $x = s \in S$, it suffices to prove that

$$(4.7) \quad \mathbf{P}^s(\mathcal{B}) + \mathbf{P}^{\hat{s}}(\mathcal{B}) \leq \mathbf{P}^s(\mathcal{B}^*) + \mathbf{P}^{\hat{s}}(\mathcal{B}^*), \quad s \in S.$$

We decompose the path set \mathcal{B} into *bracket sets* as follows:

$$(4.8) \quad \mathcal{B} = [S] \cup [U] \cup [V] \cup \mathcal{B}_\infty \cup \bigcup_{k=1}^\infty [UkU]$$

$$\cup \bigcup_{k=1}^\infty [UkV] \cup \bigcup_{k=1}^\infty [VkU] \cup \bigcup_{k=1}^\infty [VkV],$$

where

$$[S] = \{\omega \in \mathcal{B} : T^S = T^D\},$$

$$[U] = \{\omega \in \mathcal{B} : \mathbf{X}_{T^S} \in U, T^D = T^{S \cup U}\},$$

$$[V] = \{\omega \in \mathcal{B} : \mathbf{X}_{T^S} \in V, T^D = T^{S \cup V}\},$$

$$\mathcal{B}_\infty = \{\omega \in \mathcal{B} : \tau_k^U < \infty, \tau_k^V < \infty, \forall k \in \mathbb{N}\},$$

and for $k \in \mathbb{N}$,

$$[UkU] = \{\omega \in \mathcal{B} : \mathbf{X}_{T^S} \in U, \tau_{k+1}^U < T^D, \tau_{k+1}^V = \infty\},$$

$$[UkV] = \{\omega \in \mathcal{B} : \mathbf{X}_{T^S} \in U, \tau_k^V < T^D, \tau_{k+1}^U = \infty\},$$

$$[VkU] = \{\omega \in \mathcal{B} : \mathbf{X}_{T^S} \in V, \tau_k^U < T^D, \tau_{k+1}^V = \infty\},$$

$$[VkV] = \{\omega \in \mathcal{B} : \mathbf{X}_{T^S} \in V, \tau_{k+1}^V < T^D, \tau_{k+1}^U = \infty\}.$$

Intuitively, $[UkV]$ is the set of paths in Ω that visit successively U and V (staying in D) k times and then jump out of $S \cup V$ to the set B ; these paths are allowed to pass through S between the successive jumps from U to V or from V to U . Similarly, $[VkV]$ is the set of paths in Ω that visit successively V and U (staying in D) k times, then visit V once more and finally jump out of $S \cup V$ to the set B . The path set \mathcal{B}_∞ contains all paths in \mathcal{B} that visit successively U and V an infinite number of times.

The path set \mathcal{B}^* has a similar decomposition:

$$(4.9) \quad \mathcal{B}^* = [S] \cup [U] \cup [\widehat{V}] \cup \mathcal{B}_\infty^* \cup \bigcup_{k=1}^\infty [UkU]^* \cup \bigcup_{k=1}^\infty [Uk\widehat{V}]^* \cup \bigcup_{k=1}^\infty [\widehat{V}kU]^* \\ \cup \bigcup_{k=1}^\infty [\widehat{V}k\widehat{V}]^*,$$

where

$$[\widehat{V}] = \{\omega \in \mathcal{B}^* : \mathbf{x}_{TS} \in \widehat{V}, T^D = T^{S \cup \widehat{V}}\}, \\ \mathcal{B}_\infty^* = \{\omega \in \mathcal{B}^* : \tau_k^U < \infty, \tau_k^{\widehat{V}} < \infty, \forall k \in \mathbb{N}\},$$

and for $k \in \mathbb{N}$,

$$[UkU]^* = \{\omega \in \mathcal{B}^* : \mathbf{x}_{TS} \in U, \tau_{k+1}^U < T^D, \tau_{k+1}^{\widehat{V}} = \infty\}, \\ [Uk\widehat{V}] = \{\omega \in \mathcal{B}^* : \mathbf{x}_{TS} \in U, \tau_k^{\widehat{V}} < T^D, \tau_{k+1}^U = \infty\}, \\ [\widehat{V}kU] = \{\omega \in \mathcal{B}^* : \mathbf{x}_{TS} \in \widehat{V}, \tau_k^U < T^D, \tau_{k+1}^{\widehat{V}} = \infty\}, \\ [\widehat{V}k\widehat{V}] = \{\omega \in \mathcal{B}^* : \mathbf{x}_{TS} \in \widehat{V}, \tau_{k+1}^{\widehat{V}} < T^D, \tau_{k+1}^U = \infty\}.$$

By the strong Markov property, for every $k \in \mathbb{N}$ and every $s \in S$,

$$\mathbf{P}^s(\mathcal{B}_\infty) \leq \int_V \omega_\alpha^{S \cup U}(s, dv_1) \int_U \omega^{S \cup V}(v_1, du_1) \int_V \omega_\alpha^{S \cup U}(u_1, dv_2) \cdots \\ \times \int_V \omega_\alpha^{S \cup U}(u_{k-1}, dv_k) \int_U \omega_\alpha^{S \cup V}(v_k, du_k) \omega_\alpha^D(u_k, B).$$

Thus, by [8, Theorem 3],

$$\mathbf{P}^s(\mathcal{B}_\infty) \leq \left(\frac{1}{2}\right)^{2k-1}.$$

Hence, $\mathbf{P}^s(\mathcal{B}_\infty) = 0$.

Therefore, in order to prove (4.7), it suffices to prove the following inequalities for $s \in S$:

$$(4.10) \quad \mathbf{P}^s[V] + \mathbf{P}^{\hat{s}}[V] \leq \mathbf{P}^s[\widehat{V}] + \mathbf{P}^{\hat{s}}[\widehat{V}],$$

$$(4.11) \quad \mathbf{P}^s[UkU] + \mathbf{P}^{\hat{s}}[UkU] \leq \mathbf{P}^s[UkU]^* + \mathbf{P}^{\hat{s}}[UkU]^*, \quad k \in \mathbb{N},$$

$$(4.12) \quad \mathbf{P}^s[UkV] + \mathbf{P}^{\hat{s}}[UkV] \leq \mathbf{P}^s[Uk\widehat{V}]^* + \mathbf{P}^{\hat{s}}[Uk\widehat{V}]^*, \quad k \in \mathbb{N},$$

$$(4.13) \quad \mathbf{P}^s[VkU] + \mathbf{P}^{\hat{s}}[VkU] \leq \mathbf{P}^s[\widehat{V}kU]^* + \mathbf{P}^{\hat{s}}[\widehat{V}kU]^*, \quad k \in \mathbb{N},$$

$$(4.14) \quad \mathbf{P}^s[VkV] + \mathbf{P}^{\hat{s}}[VkV] \leq \mathbf{P}^s[\widehat{V}k\widehat{V}]^* + \mathbf{P}^{\hat{s}}[\widehat{V}k\widehat{V}]^*, \quad k \in \mathbb{N}.$$

We now prove the inequality (4.10). By the strong Markov property,

$$(4.15) \quad \mathbf{P}^s[V] + \mathbf{P}^{\hat{s}}[V] = \int_V [\omega_\alpha^S(s, dv) + \omega_\alpha^S(\hat{s}, dv)] \omega_\alpha^{S \cup V}(v, B)$$

and similarly,

$$(4.16) \quad \mathbf{P}^s[\widehat{V}] + \mathbf{P}^{\hat{s}}[\widehat{V}] = \int_V [\omega_\alpha^S(s, \widehat{dv}) + \omega_\alpha^S(\hat{s}, \widehat{dv})] \omega_\alpha^{S \cup \widehat{V}}(\hat{v}, B^*),$$

where $\omega_\alpha^S(s, \widehat{dv})$ is the measure μ on V defined by $\mu(A) = \omega_\alpha^S(s, \widehat{A})$ with a similar definition for the measure $\omega_\alpha^S(\hat{s}, \widehat{dv})$.

We denote by S_B, U_B, V_B the symmetric, upper nonsymmetric, and lower nonsymmetric part of B , respectively. Then, of course, we have

$$(4.17) \quad \omega_\alpha^{S \cup V}(v, B) = \omega_\alpha^{S \cup V}(v, S_B) + \omega_\alpha^{S \cup V}(v, U_B) + \omega_\alpha^{S \cup V}(v, V_B)$$

and

$$(4.18) \quad \omega_\alpha^{S \cup \widehat{V}}(\hat{v}, B^*) = \omega_\alpha^{S \cup \widehat{V}}(\hat{v}, S_B) + \omega_\alpha^{S \cup \widehat{V}}(\hat{v}, U_B) + \omega_\alpha^{S \cup \widehat{V}}(\hat{v}, \widehat{V}_B).$$

Because of symmetry,

$$(4.19) \quad \omega_\alpha^{S \cup V}(v, S_B) = \omega_\alpha^{S \cup \widehat{V}}(\hat{v}, S_B)$$

and

$$(4.20) \quad \omega_\alpha^{S \cup V}(v, V_B) = \omega_\alpha^{S \cup \widehat{V}}(\hat{v}, \widehat{V}_B).$$

Also, by Lemma 1,

$$(4.21) \quad \omega_\alpha^{S \cup V}(v, U_B) = \omega_\alpha^{S \cup \widehat{V}}(\hat{v}, \widehat{U}_B) \leq \omega_\alpha^{S \cup \widehat{V}}(\hat{v}, U_B).$$

The inequality (4.10) follows from (4.15)–(4.21).

Next, we prove the inequality (4.12). The inequalities (4.11), (4.13), and (4.14) are proved in a similar manner. Fix $k \in \mathbb{N}$. By the strong Markov property,

$$\begin{aligned} & \mathbf{P}^s[UkV] + \mathbf{P}^{\hat{s}}[UkV] \\ &= \int_U [\omega_\alpha^S(s, du_1) + \omega_\alpha^S(\hat{s}, du_1)] \int_V \omega_\alpha^{S \cup U}(u_1, dv_1) \int_U \omega_\alpha^{S \cup V}(v_1, du_2) \cdots \\ & \quad \times \int_U \omega_\alpha^{S \cup V}(v_{k-1}, du_k) \int_V \omega_\alpha^{S \cup U}(u_k, dv_k) \omega_\alpha^{S \cup V}(v_k, B) \end{aligned}$$

and similarly

$$\begin{aligned} & \mathbf{P}^s[Uk\widehat{V}]^* + \mathbf{P}^{\hat{s}}[Uk\widehat{V}]^* \\ &= \int_U [\omega_\alpha^S(s, du_1) + \omega_\alpha^S(\hat{s}, du_1)] \int_V \omega_\alpha^{S \cup U}(u_1, \widehat{dv}_1) \int_U \omega_\alpha^{S \cup \widehat{V}}(\widehat{v}_1, du_2) \cdots \\ & \quad \times \int_U \omega_\alpha^{S \cup \widehat{V}}(\widehat{v}_{k-1}, du_k) \int_V \omega_\alpha^{S \cup U}(u_k, \widehat{dv}_k) \omega_\alpha^{S \cup \widehat{V}}(\widehat{v}_k, B^*). \end{aligned}$$

Therefore, the inequality (4.12) follows from the following inequalities:

(i) For all $u \in U$ and all Borel sets $V_1 \subset V$,

$$(4.22) \quad \omega_\alpha^{S \cup U}(u, V_1) \leq \omega_\alpha^{S \cup U}(u, \widehat{V}_1).$$

(ii) For all $v \in V$ and all Borel sets $U_1 \subset U$,

$$(4.23) \quad \omega_\alpha^{S \cup V}(v, U_1) \leq \omega_\alpha^{S \cup \widehat{V}}(\widehat{v}, U_1).$$

(iii) For all $v \in V$,

$$(4.24) \quad \omega_\alpha^{S \cup V}(v, B) \leq \omega_\alpha^{S \cup \widehat{V}}(\widehat{v}, B^*).$$

The inequalities (i) and (ii) follow at once from Lemma 1. The inequality (iii) follows from symmetry, Lemma 1 and equations (4.17), (4.18).

The proof of Theorem 4 is now complete.

5. Proof of Theorem 5

In this section, we will prove Theorem 5. We recall the setting of the theorem: Let D be an open set in $\mathbb{R}^n, n \geq 2$ and let B be a Borel set in D^c . Denote by S, U , and V the symmetric, upper nonsymmetric, and lower nonsymmetric part of D , respectively. Suppose that B is not a D^* -null set and that $B^* \subset (D^*)^c$.

In the proof, we will make repeated use of the decomposition of the path sets \mathcal{B} and \mathcal{B}^* into bracket sets; see the proof of Theorem 4 in Section 4. We will also need the following lemma.

LEMMA 2. *Suppose that $B \subset \mathbb{R}_+^n$ is not a D^* -null set. If B is both $S \cup \widehat{V}$ -null and $S \cup U$ -null set, then for $x \in D_+$,*

$$(5.1) \quad 0 = \omega_\alpha^D(x, B) < \omega_\alpha^{D^*}(x, B).$$

Proof. By Lemma 1, since B is $S \cup \widehat{V}$ -null, it is also $S \cup V$ -null. It is also $S \cup U$ -null, and therefore all the bracket sets in the decomposition (4.8) have zero \mathbf{P}^x probability. Similarly, all the sets in the decomposition (4.9), except possibly \mathcal{B}_∞^* , have zero \mathbf{P}^x probability. Hence,

$$\omega_\alpha^D(x, B) = 0 \quad \text{and} \quad \omega_\alpha^{D^*}(x, B) = \mathbf{P}^x(\mathcal{B}_\infty^*).$$

Since B is not D^* -null, we have $\omega_\alpha^{D^*}(x, B) > 0$. □

Because of symmetry, part (i) of Theorem 5 is equivalent to part (ii). Also part (iii) is equivalent to (iv), part (v) is equivalent to (vi), and part (viii) is equivalent to (ix). We give here only the proofs of parts (i), (iii), and (viii). Parts (v), (vii), and (x) can be proved by similar arguments.

Proof of part (i) of Theorem 5. Suppose that $B \subset \mathbb{R}_+^n$ and that $C_\alpha(V) = 0$. Then \widehat{V} is $S \cup U$ -null. It follows then from the decompositions (4.8) and (4.9) that

$$\omega_\alpha^D(x, B) = \omega_\alpha^{S \cup U}(x, B) = \omega_\alpha^{D^*}(x, B).$$

for all $x \in S_+ \cup S_o \cup U$.

Conversely, suppose that for some $x \in S_+ \cup S_o \cup U$,

$$(5.2) \quad \omega_\alpha^D(x, B) = \omega_\alpha^{D^*}(x, B).$$

Seeking for a contradiction, assume also that

$$(5.3) \quad \widehat{V} \text{ is not } S \cup U\text{-null.}$$

This assumption implies that $\widehat{V} \neq \emptyset$. The assumption (5.2) and the proof of Theorem 4 imply that $\mathbf{P}^x[V] = \mathbf{P}^x[\widehat{V}]^*$ which means that

$$\int_V \omega_\alpha^{S \cup U}(x, dv) \omega_\alpha^{S \cup V}(v, B) = \int_V \omega_\alpha^{S \cup U}(x, \widehat{dv}) \omega_\alpha^{S \cup \widehat{V}}(\widehat{v}, B),$$

or, equivalently,

$$(5.4) \quad \int_V [\omega_\alpha^{S \cup U}(x, \widehat{dv}) - \omega_\alpha^{S \cup U}(x, dv)] \omega_\alpha^{S \cup V}(v, B) \\ + \int_V [\omega_\alpha^{S \cup \widehat{V}}(\widehat{v}, B) - \omega_\alpha^{S \cup V}(v, B)] \omega_\alpha^{S \cup U}(x, \widehat{dv}) = 0.$$

By Lemma 1, if B is not $S \cup \widehat{V}$ -null, we have $\omega_\alpha^{S \cup V}(v, B) < \omega_\alpha^{S \cup \widehat{V}}(\widehat{v}, B)$ and this together with (5.3) contradict (5.4). Therefore,

$$(5.5) \quad B \text{ is } S \cup \widehat{V}\text{-null.}$$

The equality (5.2) and the proof of Theorem 4 also imply that $\mathbf{P}^x[V1U] = \mathbf{P}^x[\widehat{V}1U]^*$ which means that

$$\int_V \omega_\alpha^{S \cup U}(x, dv) \int_U \omega_\alpha^{S \cup V}(v, du) \omega_\alpha^{S \cup U}(u, B) \\ = \int_V \omega_\alpha^{S \cup U}(x, \widehat{dv}) \int_U \omega_\alpha^{S \cup \widehat{V}}(\widehat{v}, du) \omega_\alpha^{S \cup U}(u, B),$$

or equivalently,

$$(5.6) \quad \int_V \omega_\alpha^{S \cup U}(x, \widehat{dv}) \int_U [\omega_\alpha^{S \cup \widehat{V}}(\widehat{v}, du) - \omega_\alpha^{S \cup V}(v, du)] \omega_\alpha^{S \cup U}(u, B) \\ + \int_V [\omega_\alpha^{S \cup U}(x, \widehat{dv}) - \omega_\alpha^{S \cup U}(x, dv)] \int_U \omega_\alpha^{S \cup V}(v, du) \omega_\alpha^{S \cup U}(u, B) = 0.$$

Because of (5.3) and (5.5), the equality (5.6) implies that for every $v \in V$, the measure

$$\omega_\alpha^{S \cup \widehat{V}}(\widehat{v}, du) - \omega_\alpha^{S \cup V}(v, du)$$

is the zero measure on U . In particular,

$$(5.7) \quad \omega_\alpha^{S \cup \widehat{V}}(\widehat{v}, U) = \omega_\alpha^{S \cup V}(v, U), \quad v \in V.$$

Because of Lemma 1, (5.7) implies that

$$(5.8) \quad U \text{ is } S \cup \widehat{V}\text{-null.}$$

If $S = \emptyset$, then U is a nonempty open set (it contains the point x) and V is a nonempty open set; this contradicts (5.8). Hence, $S \neq \emptyset$. Let $s \in S$. Then

$$(5.9) \quad \omega_\alpha^{S \cup U}(s, B) = \omega_\alpha^S(s, B) + \int_U \omega^S(s, du) \omega_\alpha^{S \cup U}(u, B).$$

But B is S -null (by (5.5)) and U is S -null (by (5.8)). Therefore, (5.9) implies that

$$(5.10) \quad B \text{ is } S \cup U\text{-null.}$$

Since B is both $S \cup \widehat{V}$ -null and $S \cup U$ -null but it is not D^* -null, Lemma 2 yields

$$(5.11) \quad 0 = \omega_\alpha^D(x, B) < \omega_\alpha^{D^*}(x, B), \quad x \in D_+.$$

This contradicts (5.2). Hence, \widehat{V} is $S \cup U$ -null. By [8, Lemma 3], $C_\alpha(V) = 0$. □

Proof of part (iii) of Theorem 5. Suppose that $B \subset \mathbb{R}_+^n$. Seeking for a contradiction, we suppose that

$$(5.12) \quad \omega_\alpha^D(x_o, B) = \omega_\alpha^{D^*}(\widehat{x}_o, B),$$

for some $x_o \in D_-$. By the proof of Theorem 4,

$$\mathbf{P}^{x_o}([S] \cup [V]) = \mathbf{P}^{\widehat{x}_o}([S] \cup [\widehat{V}]^*)$$

which means that

$$\omega_\alpha^{S \cup V}(x_o, B) = \omega_\alpha^{S \cup \widehat{V}}(\widehat{x}_o, B).$$

Lemma 1 implies that

$$(5.13) \quad B \text{ is } S \cup \widehat{V}\text{-null.}$$

Again by the proof of Theorem 4,

$$\mathbf{P}^{x_o}([U] \cup [V1U]) = \mathbf{P}^{\widehat{x}_o}([U] \cup [\widehat{V}1U]^*)$$

which means that

$$\int_U \omega_\alpha^{S \cup V}(x_o, du) \omega_\alpha^{S \cup U}(u, B) = \int_U \omega_\alpha^{S \cup \widehat{V}}(\widehat{x}_o, du) \omega_\alpha^{S \cup U}(u, B),$$

or, equivalently,

$$(5.14) \quad \int_U [\omega_\alpha^{S \cup \widehat{V}}(\widehat{x}_o, du) - \omega_\alpha^{S \cup V}(x_o, du)] \omega_\alpha^{S \cup U}(u, B) = 0.$$

Then either B is $S \cup U$ -null, or $\omega_\alpha^{S \cup \widehat{V}}(\widehat{x}_o, du) - \omega_\alpha^{S \cup V}(x_o, du)$ is the zero measure on U . In the first case, Lemma 2 implies that

$$\omega_\alpha^D(x_o, B) = 0 < \omega_\alpha^{D^*}(\widehat{x}_o, B)$$

contradicting (5.12). In the second case, we have

$$\omega_\alpha^{S \cup \widehat{V}}(\widehat{x}_o, U) = \omega_\alpha^{S \cup V}(x_o, U) = \omega_\alpha^{S \cup \widehat{V}}(\widehat{x}_o, \widehat{U}),$$

and thus Lemma 1 implies that

$$(5.15) \quad U \text{ is } S \cup \widehat{V}\text{-null.}$$

Then (5.13), (5.15), and the proof of Theorem 4 imply that

$$\omega_\alpha^{D^*}(\widehat{x}_o, B) = \omega_\alpha^{S \cup \widehat{V}}(\widehat{x}_o, B) = 0,$$

i.e., B is D^* -null; contradiction. □

Proof of part (viii) of Theorem 5. Suppose that $B \subset \mathbb{R}_+^n$, and that $C_\alpha(V) = 0$. Then \widehat{V} is $S \cup U$ -null. It follows from the proof of Theorem 4 that

$$\omega_\alpha^D(s, B) = \omega_\alpha^{D^*}(s, B) = \omega_\alpha^{S \cup U}(s, B), \quad s \in S.$$

Therefore,

$$(5.16) \quad \omega_\alpha^D(s, B) + \omega_\alpha^D(\widehat{s}, B) = \omega_\alpha^{D^*}(s, B) + \omega_\alpha^{D^*}(\widehat{s}, B)$$

for all $s \in S$.

Conversely, suppose that (5.16) holds for some $s \in S_+$. By the strong Markov property,

$$\begin{aligned} \omega_\alpha^D(s, B) &= \omega_\alpha^{S \cup U}(s, B) + \int_V \omega_\alpha^{S \cup U}(s, dv) \omega_\alpha^D(v, B), \\ \omega_\alpha^D(\widehat{s}, B) &= \omega_\alpha^{S \cup U}(\widehat{s}, B) + \int_V \omega_\alpha^{S \cup U}(\widehat{s}, dv) \omega_\alpha^D(v, B), \\ \omega_\alpha^{D^*}(s, B) &= \omega_\alpha^{S \cup U}(s, B) + \int_V \omega_\alpha^{S \cup U}(s, \widehat{dv}) \omega_\alpha^{D^*}(\widehat{v}, B), \\ \omega_\alpha^{D^*}(\widehat{s}, B) &= \omega_\alpha^{S \cup U}(\widehat{s}, B) + \int_V \omega_\alpha^{S \cup U}(\widehat{s}, \widehat{dv}) \omega_\alpha^{D^*}(\widehat{v}, B). \end{aligned}$$

Therefore, (5.16) implies that

$$\begin{aligned} 0 &= \int_V [\omega_\alpha^{S \cup U}(s, \widehat{dv}) + \omega_\alpha^{S \cup U}(\widehat{s}, \widehat{dv})] \omega_\alpha^{D^*}(\widehat{v}, B) \\ &\quad - \int_V [\omega_\alpha^{S \cup U}(s, dv) + \omega_\alpha^{S \cup U}(\widehat{s}, dv)] \omega_\alpha^D(v, B) \\ &= \int_V [\omega_\alpha^{S \cup U}(s, \widehat{dv}) + \omega_\alpha^{S \cup U}(\widehat{s}, \widehat{dv})] [\omega_\alpha^{D^*}(\widehat{v}, B) - \omega^D(v, B)] \\ &\quad + \int_V [\omega_\alpha^{S \cup U}(s, \widehat{dv}) + \omega_\alpha^{S \cup U}(\widehat{s}, \widehat{dv}) - \omega_\alpha^{S \cup U}(s, dv) - \omega_\alpha^{S \cup U}(\widehat{s}, dv)] \omega^D(v, B). \end{aligned}$$

It follows from the results in [8] that $\omega_\alpha^{S \cup U}(s, \widehat{dv})$ is the zero measure on \widehat{V} , and hence \widehat{V} is $S \cup U$ -null. By [8, Lemma 3], $C_\alpha(V) = 0$ which means $D \stackrel{C_\alpha}{=} D^*$. □

6. Proofs of Theorems 6 and 7

In this section, we prove Theorem 6 for the behavior of the convex integral means of α -harmonic measure under polarization. We also prove Theorem 7 which describes the corresponding equality cases. The proof of these results uses the following elementary lemma whose easy proof is omitted (see [25]).

LEMMA 3. Let $a_1, b_1, a_2, b_2 \in \mathbb{R}$ be such that

$$a_2 + b_2 \leq a_1 + b_1 \quad \text{and} \quad 0 \leq a_1 \leq a_2 \leq b_2 < b_1.$$

Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ be a nonconstant, convex, increasing function. Then

$$(6.1) \quad \Phi(a_2) + \Phi(b_2) \leq \Phi(a_1) + \Phi(b_1).$$

Equality holds in (6.1) if and only if Φ is affine on $[a_1, b_1]$ and $a_1 + b_1 = a_2 + b_2$.

Proof of Theorem 6. The inequality (3.13) is equivalent to

$$(6.2) \quad \begin{aligned} & \int_{\Sigma_+} [\Phi(\omega_\alpha^D(x, B)) + \Phi(\omega_\alpha^D(\hat{x}, B))] m_k(dx) \\ & \leq \int_{\Sigma_+} [\Phi(\omega_\alpha^{D^*}(x, B^*)) + \Phi(\omega_\alpha^{D^*}(\hat{x}, B^*))] m_k(dx) \end{aligned}$$

and this follows from Theorem 4 and Lemma 3. □

Proof of Theorem 7.

(i) Suppose that $D \stackrel{C_\alpha}{=} \widehat{D}$ and that Φ is affine function. It follows from the strong Markov property that for $s \in S$,

$$\omega_\alpha^D(s, B) = \omega_\alpha^S(s, B) \quad \text{and} \quad \omega_\alpha^{D^*}(s, B^*) = \omega_\alpha^S(s, B^*).$$

Also, because of symmetry, for $s \in S_+$,

$$\omega_\alpha^S(s, B) + \omega_\alpha^S(\hat{s}, B) = \omega_\alpha^S(s, B^*) + \omega_\alpha^S(\hat{s}, B^*).$$

Since Φ is affine, it follows that equality holds in (3.13).

(ii) Suppose that $D \stackrel{C_\alpha}{=} \widehat{D}$. Assume first that $B \stackrel{D^*}{=} B^*$. Then

$$\omega_\alpha^D(s, B) = \omega_\alpha^S(s, U_B \cup S_B) = \omega_\alpha^S(s, B^*), \quad s \in S,$$

and therefore, for $s \in \Sigma_+$,

$$\Phi(\omega_\alpha^S(s, B)) + \Phi(\omega_\alpha^S(\hat{s}, B)) = \Phi(\omega_\alpha^S(s, B^*)) + \Phi(\omega_\alpha^S(\hat{s}, B^*)).$$

Hence, (6.2) holds with equality. Similarly, if $B \stackrel{D^*}{=} \widehat{B^*}$, again (6.2) holds with equality. So (3.13) holds with equality.

Conversely, assume that (3.13) holds with equality. Then for $s \in \Sigma_+$,

$$\Phi(\omega_\alpha^S(s, B)) + \Phi(\omega_\alpha^S(\hat{s}, B)) = \Phi(\omega_\alpha^S(s, B^*)) + \Phi(\omega_\alpha^S(\hat{s}, B^*)).$$

Since Φ is not affine in any interval, it follows from Lemma 3 that for every $s \in \Sigma_+$,

$$\omega_\alpha^S(s, B^*) = \omega_\alpha^S(s, B) \quad \text{or} \quad \omega_\alpha^S(s, B^*) = \omega_\alpha^S(\hat{s}, B).$$

By Lemma 1, U_B is S -null or V_B is S -null which means $B \stackrel{D^*}{=} B^*$ or $B \stackrel{D^*}{=} \widehat{B^*}$.

(iii) Suppose that the condition $D \stackrel{C_\alpha}{\cong} \widehat{D}$ is not true. If $D \stackrel{C_\alpha}{\cong} D^*$, $B \stackrel{D^*}{\cong} B^*$ or if $D \stackrel{C_\alpha}{\cong} \widehat{D}^*$, $B \stackrel{D^*}{\cong} \widehat{B}^*$, then it is easy to show that equality holds in (3.13).

Conversely, assume that equality holds in (3.13), and hence in (6.2). Then for all $s \in \Sigma_+$,

$$\Phi(\omega_\alpha^D(s, B)) + \Phi(\omega_\alpha^D(\hat{s}, B)) = \Phi(\omega_\alpha^{D^*}(s, B^*)) + \Phi(\omega_\alpha^{D^*}(\hat{s}, B^*)).$$

By Lemma 3, for each $s \in \Sigma_+$, at least one of the following three equalities must be satisfied.

$$(6.3) \quad \omega_\alpha^D(s, B) + \omega_\alpha^D(\hat{s}, B) = \omega_\alpha^{D^*}(s, B^*) + \omega_\alpha^{D^*}(\hat{s}, B^*),$$

$$(6.4) \quad \omega_\alpha^{D^*}(s, B^*) = \omega_\alpha^D(s, B),$$

$$(6.5) \quad \omega_\alpha^{D^*}(s, B^*) = \omega_\alpha^D(\hat{s}, B).$$

By using the various parts of Theorem 5, we conclude that the equalities (6.3), (6.4), and (6.5) imply the following three conditions, respectively.

Condition 1: $[C_\alpha(V) = 0$ or U_B is D^* -null] and $[C_\alpha(U) = 0$ or V_B is D^* -null] and $[C_\alpha(U) = 0$ or $C_\alpha(V) = 0$ or S_B is D^* -null].

Condition 2: $[C_\alpha(V) = 0$ or U_B is D^* -null] and $[V_B$ is D^* -null] and $[C_\alpha(V) = 0$ or S_B is D^* -null].

Condition 3: $[C_\alpha(U) = 0$ or V_B is D^* -null] and $[U_B$ is D^* -null] and $[C_\alpha(U) = 0$ or S_B is D^* -null].

We perform the logical operations and using the assumptions that $D \stackrel{C_\alpha}{\cong} \widehat{D}$ is not true and that B^* is not D^* -null, we find that $[D \stackrel{C_\alpha}{\cong} D^*$ and $B \stackrel{D^*}{\cong} B^*]$ or $[D \stackrel{C_\alpha}{\cong} \widehat{D}^*$ and $B \stackrel{D^*}{\cong} \widehat{B}^*]$. □

7. Proofs of Theorems 1, 2 and 3

In the proof of Theorem 1, we will use the following two lemmas.

LEMMA 4. *Let $\{D_k\}$ be a sequence of open sets in \mathbb{R}^n such that for every $k \in \mathbb{N}$, $D_k \subset D_{k+1}$. Set $D := \bigcup_{k=1}^\infty D_k$. Let B be a compact set in \mathbb{R}^n . Assume that one of the following two conditions holds:*

(i) $B \subset (\overline{D})^c$.

(ii) $B \subset \partial D_k$ for all $k \in \mathbb{N}$ and there exists an open neighborhood O of B such that $D_1 \cap O = D \cap O$.

Then for $x \in D$,

$$(7.1) \quad \lim_{k \rightarrow \infty} \omega_\alpha^{D_k}(x, B) = \omega_\alpha^D(x, B).$$

Proof. By domain monotonicity, the sequence $\omega_\alpha^{D_k}(x, B)$ is increasing. We set

$$h(x) = \lim_{k \rightarrow \infty} \omega_\alpha^{D_k}(x, B), \quad x \in \mathbb{R}^n.$$

Then $0 \leq h(x) \leq \omega_\alpha^D(x, B)$. As in the classical case (by Harnack’s inequality), h is α -harmonic in D . Let

$$u(x) = h(x) - \omega_\alpha^D(x, B).$$

The function u is α -harmonic in D and $u = 0$ in $(\overline{D})^c$. Let E be the set of points in ∂D which are irregular for the Dirichlet problem for α -harmonic functions in D . Then $C_\alpha(E) = 0$; see e.g., [22, p. 296]. Let $\zeta \in \partial D \setminus E$. If $\zeta \in \partial D$ (in case (i)), or if $\zeta \in \partial D \setminus B$ (in case (ii)), then

$$\liminf_{D \ni x \rightarrow \zeta} u(x) = \liminf_{D \ni x \rightarrow \zeta} h(x) \geq 0.$$

If $\zeta \in \partial D \cap B$ (this can occur only in case (ii)), then by the strong Markov property and the fact that ζ is an interior point of $\partial D_1 \setminus (D \setminus D_1)$ (in the topology of ∂D_1),

$$\begin{aligned} \liminf_{D \ni x \rightarrow \zeta} u(x) &= \liminf_{D \ni x \rightarrow \zeta} [h(x) - \omega_\alpha^D(x, B)] \\ &\geq \liminf_{D \ni x \rightarrow \zeta} [\omega_\alpha^{D_1}(x, B) - \omega_\alpha^D(x, B)] \\ &= \liminf_{D \ni x \rightarrow \zeta} \left[- \int_{D \setminus D_1} \omega_\alpha^{D_1}(x, dy) \omega_\alpha^D(y, B) \right] \\ &\geq \liminf_{D \ni x \rightarrow \zeta} [-\omega_\alpha^{D_1}(x, D \setminus D_1)] = 0. \end{aligned}$$

By the minimum principle for α -harmonic functions [8, Lemma 5], $u = 0$ in \mathbb{R}^n . □

LEMMA 5. *Let D be an open set in \mathbb{R}^n lying in a striplike set G . Let Σ be a vertical line intersecting D . Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ be a nonconstant, convex, increasing function. Let B be a closed set in G^c . Suppose that $D = D^\sharp$. Then*

$$(7.2) \quad \int_\Sigma \Phi(\omega_\alpha^D(p, B)) m_1(dp) \leq \int_\Sigma \Phi(\omega_\alpha^D(p, B^\sharp)) m_1(dp).$$

Proof. We may assume that B is compact. By [14, Lemma 7.2], there exist horizontal, oriented planes H_j with corresponding polarizations P_j , $j \in \mathbb{N}$, such that for the sequence of sets $F_k := P_k \cdots P_2 P_1(B)$, we have

$$\lim_{k \rightarrow \infty} d_{\text{Haus}}(F_k, B^\sharp) = 0.$$

Here, d_{Haus} denotes the Hausdorff metric. Thus, for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$(7.3) \quad F_k \subset (B^\sharp)_\varepsilon,$$

where

$$(B^\sharp)_\varepsilon = \{x \in G^c : d(x, B^\sharp) \leq \varepsilon\}.$$

By Theorem 6 and (7.3),

$$(7.4) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^D(p, B))m_1(dp) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^D(p, F_k))m_1(dp) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^D(p, (B^{\sharp})_{\varepsilon}))m_1(dp).$$

But

$$\omega_{\alpha}^D(p, (B^{\sharp})_{\varepsilon}) - \omega_{\alpha}^D(p, B^{\sharp}) \leq \omega_{\alpha}^D(p, K_{\varepsilon}) \leq \omega_{\alpha}^G(p, K_{\varepsilon}),$$

where K_{ε} is the intersection of G^c with a finite union of balls of radius ε . Clearly,

$$\lim_{\varepsilon \rightarrow 0} \omega_{\alpha}^D(p, K_{\varepsilon}) = 0.$$

We take limits in (7.4) as $\varepsilon \rightarrow 0$ and obtain (7.2). □

Proof of Theorem 1. We may assume that B is compact. Let $B = B_1 \cup B_2$ with $B_1 = B \cap (\overline{G})^c$, $B_2 = B \cap \partial G \cap \partial D$. We may assume that B_1 and B_2 are both compact. We prove first that

$$(7.5) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^D(p, B_1))m_1(dp) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, B_1^{\sharp}))m_1(dp).$$

Let $\{D_k\}$ be sequence of bounded open sets in \mathbb{R}^n such that $\overline{D_k} \subset D_{k+1}$ and $\bigcup_{k=1}^{\infty} D_k = D$. By [14, Lemma 7.1], for each D_k , there exist horizontal, oriented planes H_j , $j = 1, 2, \dots, N_k$ such that

$$(7.6) \quad \Omega_k := P_{N_k} \cdots P_2 P_1(D_k) \subset D^{\sharp}.$$

Here, we denote by P_j the polarization with respect to H_j . We also set

$$(7.7) \quad F_k := P_{N_k} \cdots P_2 P_1(B_1).$$

By applying Theorem 6 N_k times, we obtain

$$(7.8) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^{D_k}(p, B_1))m_1(dp) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{\Omega_k}(p, F_k))m_1(dp).$$

By (7.6) and the domain monotonicity of α -harmonic measure,

$$(7.9) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^{\Omega_k}(p, F_k))m_1(dp) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, F_k))m_1(dp).$$

By Lemma 5,

$$(7.10) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, F_k))m_1(dp) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, B_1^{\sharp}))m_1(dp).$$

The inequalities (7.8)–(7.10) yield

$$(7.11) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^{D_k}(p, B_1))m_1(dp) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, B_1^{\sharp}))m_1(dp).$$

The inequality (7.5) follows from (7.11) by taking limits as $k \rightarrow \infty$ and using Lemma 4(i).

Next, we prove the inequality

$$(7.12) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^D(p, B_2)) m_1(dp) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, B_2^{\sharp})) m_1(dp).$$

Since B_2 is compact, we may assume that B_2 lies in the cube $[-1, 1]^n$. Recall that for $x \in \mathbb{R}^n$, we denote by x^{\sharp} the orthogonal projection of x on the horizontal plane Π . Let also B_2^{Π} denote the orthogonal projection of B_2 on Π . We first prove (7.12) under the additional assumption that for some $\varepsilon > 0$, the open set

$$\{x \in \mathbb{R}^n : \text{dist}(x^{\sharp}, B_2^{\Pi}) < \varepsilon\}$$

is subset of D . This assumption means that D contains an ε -neighborhood of the cylinder generated by B_2^{Π} .

We construct now a special cube-approximation $D_k \uparrow D$ with $B_2 \subset \partial D_k$ for all $k \in \mathbb{N}$. Let G^{Π} be the projection of G on Π . This is an open set in Π . For $k \in \mathbb{N}$, we consider the set \mathcal{Q}_1^k of all closed, nonoverlapping, $(n - 1)$ -dimensional cubes in Π , with sides parallel to the coordinate planes, side-length $\frac{1}{k}$, and all lying inside the cube $[-k, k]^{n-1}$. Let

$$\mathcal{Q}_2^k := \{Q \cap G^{\Pi} : Q \in \mathcal{Q}_1^k\}, \quad k \in \mathbb{N}.$$

Next, consider the set \mathcal{L}^k of all closed, nonoverlapping intervals in \mathbb{R} , with length $\frac{1}{k}$, lying inside the interval $[-k, k]$; that is \mathcal{L}^k contains the intervals

$$\left[0, \frac{1}{k}\right], \left[\frac{1}{k}, \frac{2}{k}\right], \dots, \left[k - \frac{1}{k}, k\right], \left[\frac{-1}{k}, 0\right], \dots, \left[-k, -k + \frac{1}{k}\right].$$

Let

$$\mathcal{Q}^k := \{Q \times l \subset D : Q \in \mathcal{Q}_2^k, l \in \mathcal{L}^k\}, \quad k \in \mathbb{N}.$$

Let D_k be the interior of the union of all sets in \mathcal{Q}^k :

$$D_k := \left(\bigcup_{Q \in \mathcal{Q}^k} Q\right)^{\circ}.$$

Then $\{D_k\}$ is an increasing sequence of open subsets of D and $\bigcup_k D_k = D$. Note that each D_k is a finite union of cube-like sets in \mathcal{Q}^k . Note also that an open set of the form

$$\{x \in \mathbb{R}^n : \text{dist}(x^{\sharp}, B_2^{\Pi}) < \varepsilon'\},$$

with $0 < \varepsilon' \leq \varepsilon$, is subset of D_k for all $k \in \mathbb{N}$; therefore $B_2 \subset \partial D_k$, for all $k \in \mathbb{N}$. By a standard technique (cf. [17], [18]), there exists a finite number of horizontal, oriented planes H_j , $j = 1, 2, \dots, N_k$ with corresponding polarizations P_j such that

$$D_k^{\sharp} = P_{N_k} \cdots P_2 P_1(D_k).$$

Moreover, it is clear that $D_k^\# \subset D^\#$. By Theorem 6 and Lemma 5,

$$\begin{aligned}
 (7.13) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^{D_k}(p, B_2))m_1(dp) &\leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D_k}(p, P_{N_k} \dots P_2 P_1(B_2)))m_1(dp) \\
 &\leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D_k}(p, B_2^\#))m_1(dp) \\
 &\leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^\#}(p, B_2^\#))m_1(dp).
 \end{aligned}$$

By Lemma 4(ii),

$$(7.14) \quad \lim_{k \rightarrow \infty} \omega_{\alpha}^{D_k}(p, B_2) = \omega_{\alpha}^D(p, B_2), \quad p \in D.$$

The inequality (7.12) follows from (7.13) and (7.14). So it remains to remove the additional assumption we made after the statement of (7.12).

We will use another approximation technique (see [1]). For $j \in \mathbb{N}$, let

$$D_j := D \cup \left\{ x \in \mathbb{R}^n : \text{dist}(x^\#, B_2^\Pi) < \frac{1}{j} \right\}$$

and

$$u_j(x) := \omega_{\alpha}^{D_j}(x, B_2), \quad x \in \mathbb{R}^n.$$

We also set

$$v_j(x) := \omega_{\alpha}^{D_j^\#}(x, B_2^\#), \quad x \in \mathbb{R}^n.$$

Then each of the sets D_j contains D and satisfies the additional assumption, and therefore, by what we have proved so far,

$$(7.15) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^D(p, B_2))m_1(dp) \leq \int_{\Sigma} \Phi(u_j(p))m_1(dp) \leq \int_{\Sigma} \Phi(v_j(p))m_1(dp).$$

The sequence $\{v_j\}$ is decreasing and therefore it converges to a function v_{∞} . As in the classical case, the convergence $v_j \rightarrow v_{\infty}$ is uniform on compact subsets of $D^\#$ and the function v_{∞} is α -harmonic in $D^\#$. Because of (7.15), in order to prove (7.12), it suffices to prove that

$$(7.16) \quad v_{\infty}(x) = \omega_{\alpha}^{D^\#}(x, B_2^\#), \quad x \in D^\#.$$

Since $v_j(x) \geq \omega_{\alpha}^{D^\#}(x, B_2^\#)$ for all $x \in D^\#$ and all $j \in \mathbb{N}$, we have $v_{\infty}(x) \geq \omega_{\alpha}^{D^\#}(x, B_2^\#)$. To prove the converse inequality we consider a function h in the upper Perron family for the function $\chi_{B_2^\#}$. Thus, h is α -superharmonic and bounded below in $D^\#$ and satisfies the following inequalities:

$$\liminf_{D^\# \ni x \rightarrow b} h(x) \geq 1 \quad \forall b \in B_2^\#$$

and

$$\liminf_{D^\# \ni x \rightarrow \zeta} h(x) \geq 0 \quad \forall \zeta \in \partial D^\# \setminus B_2^\#.$$

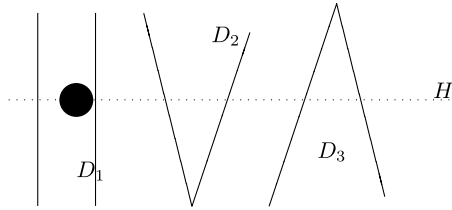


FIGURE 6. The set D_j is of type \mathcal{G}_j , $j = 1, 2, 3$.

Consider the function $h - v_\infty$ which is α -superharmonic in D^\sharp . Since $0 \leq v_\infty \leq 1$, we have

$$(7.17) \quad \liminf_{D^\sharp \ni x \rightarrow b} (h(x) - v_\infty(x)) \geq 0 \quad \forall b \in B_2^\sharp.$$

Next, let $\zeta \in \partial D^\sharp \setminus B_2^\sharp$ be regular point for the α -Dirichlet problem. Since $\partial D^\sharp \setminus B_2^\sharp$ is an open subset of ∂D^\sharp , we have

$$\lim_{D^\sharp \ni x \rightarrow \zeta} v_j(x) = 0 \quad \forall j \in \mathbb{N}.$$

Using the fact that the sequence v_j decreases to v_∞ , we conclude that

$$\lim_{D^\sharp \ni x \rightarrow \zeta} v_\infty(x) = 0$$

and therefore

$$(7.18) \quad \liminf_{D^\sharp \ni x \rightarrow \zeta} (h(x) - v_\infty(x)) \geq 0.$$

By (7.17), (7.18) and the minimum principle for α -superharmonic functions [8, Lemma 5], $h(x) \geq v_\infty(x)$, $x \in D^\sharp$. Taking infimum for all h , we conclude that $\omega_\alpha^{D^\sharp}(x, B_2^\sharp) \geq v_\infty(x)$. Therefore, (7.12) is proved. This completes the proof of the inequality (1.1). The inequality (1.2) is proved similarly. \square

For the proof of Theorem 2, we need some definitions; see Figure 6.

DEFINITION 1. (i) Let Ω be a Borel set in \mathbb{R}^n . We say that $\Omega \in \mathcal{A}_1$ if there exists a horizontal plane H such that for every vertical line Σ that intersects Ω , the set $\Sigma \cap (\mathbb{R}^n \setminus \Omega)$ is either empty or a nonempty, bounded, vertical segment, symmetric with respect to H . We say that $\Omega \in \mathcal{A}_2$ if for every vertical line Σ that intersects Ω , the set $\Sigma \cap \Omega$ is either the whole line Σ or an upward half-line. We say that $\Omega \in \mathcal{A}_3$ if for every vertical line Σ that intersects Ω , the set $\Sigma \cap \Omega$ is either the whole line Σ or a downward half-line.

(ii) Let D be an open set such that $D \stackrel{C_\alpha}{\neq} C$ for any striplike set C . We say that $D \in \mathcal{G}_j$ if $D \stackrel{C_\alpha}{=} \Omega$, for some $\Omega \in \mathcal{A}_j$, $j = 1, 2, 3$.

(iii) Let B be a closed set such that $B \stackrel{m_n}{\neq} C$ for any striplike set C . We say that $B \in \mathcal{F}_j$ if $B \stackrel{m_n}{=} \Omega$, for some $\Omega \in \mathcal{A}_j$, $j = 1, 2, 3$.

We will also need the following two lemmas taken from [9].

LEMMA 6. *Let D be an open set in \mathbb{R}^n . Assume that $D \notin \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. There exists a horizontal plane H_o such that $S_{H_o}D \stackrel{C_\alpha}{=} D$ if and only if for every horizontal plane H , either $D \stackrel{C_\alpha}{=} P_H D$ or $R_H D \stackrel{C_\alpha}{=} P_H D$.*

LEMMA 7. *Let B be a closed set in \mathbb{R}^n . Assume that $B \notin \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. There exists a horizontal plane H_o such that $S_{H_o}B \stackrel{m_n}{=} B$ if and only if for every horizontal plane H , either $B \stackrel{m_n}{=} P_H B$ or $R_H B \stackrel{m_n}{=} P_H B$.*

We now prove Theorem 2. The proof of Theorem 3 is similar, and so we omit it.

Proof of Theorem 2.

(a) By [14, Lemma 7.2], there exist horizontal, oriented planes H_j with corresponding polarizations $P_j, j \in \mathbb{N}$, such that for the sequence of sets $F_k := P_k \cdots P_2 P_1(B)$, we have

$$\lim_{k \rightarrow \infty} d_{\text{Haus}}(F_k, B^\sharp) = 0.$$

Thus, for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$B^\sharp \subset (F_k)_\varepsilon := \{x \in G^c : d(x, F_k) \leq \varepsilon\}.$$

By Theorems 7(i) and 1, the domain monotonicity of α -harmonic measure, and the boundedness of B , we obtain:

$$\begin{aligned} (7.19) \quad \int_\Sigma \Phi(\omega_\alpha^D(p, B))m_1(dp) &= \int_\Sigma \Phi(\omega_\alpha^D(p, F_k))m_1(dp) \\ &\leq \int_\Sigma \Phi(\omega_\alpha^D(p, B^\sharp))m_1(dp) \\ &\leq \int_\Sigma \Phi(\omega_\alpha^D(p, (F_k)_\varepsilon))m_1(dp) \\ &\leq \int_\Sigma \Phi(\omega_\alpha^D(p, F_k))m_1(dp) + o(\varepsilon) \\ &= \int_\Sigma \Phi(\omega_\alpha^D(p, B))m_1(dp) + o(\varepsilon). \end{aligned}$$

We take limits as $\varepsilon \rightarrow 0$ and obtain (1.3).

(b) The sufficiency of the conditions $S_H B_1 \stackrel{m_n}{=} B_1$ and $S_H B_2 \stackrel{D}{=} B_2$ for the equality

$$(7.20) \quad \int_\Sigma \Phi(\omega_\alpha^D(p, B))m_1(dp) = \int_\Sigma \Phi(\omega_\alpha^{D^\sharp}(p, B^\sharp))m_1(dp)$$

is easy to prove. Conversely, assume that (7.20) holds. If $m_n(B_1) = 0$, then trivially $S_H B_1 \stackrel{m_n}{=} B_1$. Suppose that $m_n(B_1) > 0$. Note that if $B_1 \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, then $B_1 \subsetneq B_1^\sharp$ and $m_n(B_1^\sharp \setminus B_1) > 0$; therefore (7.20) cannot

hold. Hence, $B_1 \notin \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Assume that the condition $S_H B_1 \stackrel{m_n}{=} B_1$ does not hold for any horizontal plane H . Then Lemma 7 implies that there exists a plane H for which neither of the conditions $B_1 \stackrel{m_n}{=} P_H B_1$ and $R_H B_1 \stackrel{m_n}{=} P_H B_1$ hold. By Theorem 7(ii) and Theorem 1, we obtain

$$(7.21) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^D(p, B_1))m_1(dp) < \int_{\Sigma} \Phi(\omega_{\alpha}^{P_H D}(p, P_H B_1))m_1(dp) \\ \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, B_1^{\sharp}))m_1(dp).$$

This contradicts (7.20). Hence, $S_H B_1 \stackrel{m_n}{=} B_1$. Finally, applying Theorem 7 for a suitable horizontal plane, we see that $S_H B_2 \stackrel{D}{=} B_2$. Note that Lemma 7 implies that the condition $B_1 \stackrel{m_n}{=} P_H(B_1)$ is false, while Theorem 7 involves the condition $B_1 \stackrel{D}{=} P_H(B_1)$; this is not a problem since the harmonic measure is absolutely continuous with respect to m_n in $(\overline{D})^c$.

(c) If $S_H D \stackrel{C_{\alpha}}{=} D$, $S_H B_1 \stackrel{m_n}{=} B_1$ and $S_H B_2 \stackrel{D}{=} B_2$ for some horizontal plane H , then we trivially have

$$(7.22) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^D(p, B))m_1(dp) = \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, B^{\sharp}))m_1(dp).$$

Conversely, assume that (7.22) holds. If $D \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, then D^{\sharp} is a striplike set with $D \subsetneq D^{\sharp}$ and $D \stackrel{C_{\alpha}}{\neq} D^{\sharp}$; therefore

$$\int_{\Sigma} \Phi(\omega_{\alpha}^D(p, B))m_1(dp) < \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, B))m_1(dp) \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, B^{\sharp}))m_1(dp)$$

contradicting (7.22). Hence, $D \notin \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

Suppose that the condition $S_H D \stackrel{C_{\alpha}}{=} D$ does not hold for any plane H . By Lemma 6, there exists a horizontal plane H such that neither of the conditions $D \stackrel{C_{\alpha}}{=} P_H D$ or $R_H D \stackrel{C_{\alpha}}{=} P_H D$ hold. By Theorem 7(iii) and Theorem 1, we obtain

$$(7.23) \quad \int_{\Sigma} \Phi(\omega_{\alpha}^D(p, B))m_1(dp) < \int_{\Sigma} \Phi(\omega_{\alpha}^{P_H D}(p, P_H B))m_1(dp) \\ \leq \int_{\Sigma} \Phi(\omega_{\alpha}^{D^{\sharp}}(p, B^{\sharp}))m_1(dp).$$

This contradicts (7.22). Therefore, there exists a horizontal plane H such that $S_H D \stackrel{C_{\alpha}}{=} D$. We may assume that $H = \Pi$; so it remains to prove that $B_1^{\sharp} \stackrel{m_n}{=} B_1$ and $B_2^{\sharp} \stackrel{D}{=} B_2$.

Suppose that the condition $B_1^{\sharp} \stackrel{m_n}{=} B_1$ is not true. Then by Lemma 7, there exists a horizontal plane H such that neither of the conditions $B_1 \stackrel{m_n}{=} P_H B_1$ or $R_H B_1 \stackrel{m_n}{=} P_H B_1$ hold. We continue as above and using Theorems 7 and 1, we

arrive at a strict inequality that contradicts (7.22). The proof of the condition $B_2^\# \stackrel{D}{=} B_2$ is similar. □

8. Special cases

8.1. Classical harmonic measure ($\alpha = 2$). In this subsection, we assume that $\alpha = 2$. So, we deal with classical harmonic measure and Brownian motion on a domain D . The various results of the previous sections hold for $\alpha = 2$ with some modifications.

The corresponding equality cases (the analogs of Theorems 2 and 3) are the following below.

THEOREM 8. *Let D be a domain in \mathbb{R}^n lying in a striplike set G . Let Σ be a vertical line intersecting D . Let $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ be a nonconstant, convex, increasing function. Let B be a closed set in $\partial G \cap \partial D$. Assume that B is not D -null and that*

$$\int_{\Sigma} \Phi(\omega_2^D(p, B))m_1(dp) < \infty.$$

(a) *Suppose that D is an essentially striplike set, B is bounded, and Φ is affine function. Then*

$$(8.1) \quad \int_{\Sigma} \Phi(\omega_2^D(p, B))m_1(dp) = \int_{\Sigma} \Phi(\omega_2^{D^\#}(p, B^\#))m_1(dp).$$

(b) *Suppose that D is an essentially striplike set and Φ is not affine in any interval. Then*

$$(8.2) \quad \int_{\Sigma} \Phi(\omega_2^D(p, B))m_1(dp) = \int_{\Sigma} \Phi(\omega_2^{D^\#}(p, B^\#))m_1(dp)$$

if and only if there exists a horizontal plane H such that $S_H B \stackrel{D}{=} B$.

(c) *Suppose that D is not an essentially striplike set. Then*

$$(8.3) \quad \int_{\Sigma} \Phi(\omega_2^D(p, B))m_1(dp) = \int_{\Sigma} \Phi(\omega_2^{D^\#}(p, B^\#))m_1(dp)$$

if and only if there exists a horizontal plane H such that $S_H D \stackrel{C_2}{=} D$ and $S_H B \stackrel{D}{=} B$.

An illustration for Theorem 8 appears in Figure 7.

THEOREM 9. *Let D, G, B be as in Theorem 8. Let $p \in D$. Assume that B is not D -null. Then*

$$(8.4) \quad \omega_2^D(p, B) = \omega_2^{D^\#}(p^\#, B^\#)$$

if and only if there exists a horizontal plane H , such that $p \in H$, $S_H D \stackrel{C_2}{=} D$, and $S_H B \stackrel{D}{=} B$.

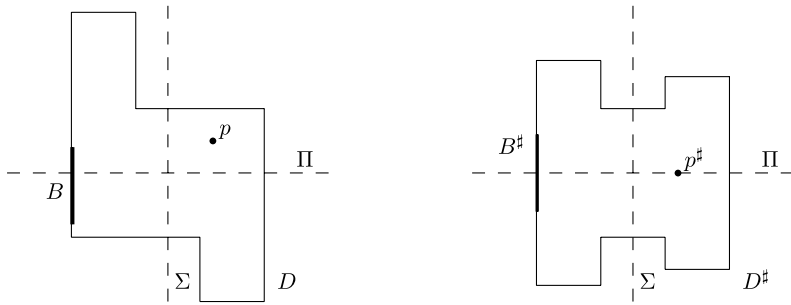


FIGURE 7. A domain D and its symmetrization D^\sharp . The set B is on the boundary of D and its symmetrization B^\sharp is on the boundary of D^\sharp . Then $\int_\Sigma \Phi(\omega_2^D(p, B))m_1(dp) \leq \int_\Sigma \Phi(\omega_2^{D^\sharp}(p, B^\sharp))m_1(dp), \omega_2^D(p, B) \leq \omega_2^{D^\sharp}(p, B^\sharp)$.

The proof of Theorems 8 and 9 parallels the proofs of Theorems 2 and 3. So, one must first prove polarization inequalities with the corresponding equality cases (the analogs of Theorems 4, 5, 6, 7). These polarization results are also proved in a similar way; in fact their proof follows closely the probabilistic method in Sections 4–6.

8.2. Regular sets. We examine in this subsection what happens if we assume that D is an open set, regular for the Dirichlet problem (for α -harmonic functions; $0 < \alpha \leq 2$).

LEMMA 8. *Let D be an open set in \mathbb{R}^n , regular for the Dirichlet problem. Let U, V be the upper and lower nonsymmetric parts of D with respect to the plane Π . If $C_\alpha(V) = 0$, then $V = \emptyset$. If $C_\alpha(U) = 0$, then $U = \emptyset$.*

Proof. Suppose that $C_\alpha(V) = 0$. Then V has empty interior. Assume that $V \neq \emptyset$. Let $x \in V$. It is easy to see that $\hat{x} \in \partial D$. Moreover, if $B(\hat{x}, r)$ is any ball centered at \hat{x} with sufficient small radius r , then $B(\hat{x}, r) \cap D^c \subset \hat{V}$. By Wiener’s regularity criterion [22, Chapter V, Section 1], \hat{x} is an irregular point of ∂D . This proves to be a contradiction. \square

REMARKS 1. Using Lemma 8, we see that in Theorems 8 and 9, with the additional assumption that D is regular, we can replace the condition $S_H D \stackrel{C_2}{\cong} D$ by the condition $S_H D = D$ (of course, the condition $S_H B \stackrel{D}{=} B$ cannot be replaced by $S_H B = B$).

2. The condition $S_H B_2 \stackrel{D}{=} B_2$ which appears in Theorems 5 and 6 is not geometric because there is no known geometric characterization of D -null sets. The condition $S_H D \stackrel{C_\alpha}{\cong} D$ is geometric because the α -capacity is connected to

the α -transfinite diameter which is a geometric quantity; see [22, Chapter II, Section 3].

3. If in Theorems 2 and 3, we make the assumption that G is a half-space (or some other similar assumption), then B_2 is a D -null set and therefore we can remove the condition $S_H B_2 \stackrel{D}{=} B_2$. Also, if B_2 lies on a vertical straight line (or any other lower dimensional vertical plane), isolated from the rest of the boundary of D , then we can use [27, Theorem 3] and replace the condition $S_H B_2 \stackrel{D}{=} B_2$ by the geometric condition $S_H B_2 \stackrel{C_\alpha}{=} B_2$.

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