Illinois Journal of Mathematics
Volume 52, Number 3, Fall 2008, Pages 745–756
S 0019-2082

SHARP LLOGL INEQUALITIES FOR DIFFERENTIALLY SUBORDINATED MARTINGALES AND HARMONIC FUNCTIONS

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Abstract. Let (x_n), (y_n) be two martingales adapted to the same filtration $(\mathcal{F}_n)$ satisfying, with probability 1,

$$|dx_n| \le |dy_n|, \quad n = 0, 1, 2, \dots.$$

For every $K > 0$, we determine the best constant $L = L(K)$ for which the inequality

$$\mathbb{E}|x_n| \le K\mathbb{E}|y_n| \log|y_n| + L, \quad n = 0, 1, 2, \dots$$

holds true. We also prove a similar inequality for harmonic functions.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a discrete filtration $(\mathcal{F}_n)_{n\ge0}$. Throughout the paper, $(x_n)_{n\ge0}$, $(y_n)_{n\ge0}$ will denote $(\mathcal{F}_n)$-martingales taking values in a certain separable Hilbert space $\mathcal{H}$. The norm in this Hilbert space will be denoted by $|\cdot|$ and $x \cdot y$ will stand for the scalar product of the vectors x, $y \in \mathcal{H}$. The difference sequences of the martingales (x_n), (y_n) will be denoted by $(dx_n)_{n\ge0}$, $(dy_n)_{n\ge0}$, respectively; that is, we set

$$\begin{aligned} dx_0 &= x_0, \qquad dx_n = x_n - x_{n-1}, \\ dy_0 &= y_0, \qquad dy_n = y_n - y_{n-1}, \quad n = 1, 2, \dots. \end{aligned}$$

Given a sequence (v_n) of $(\mathcal{F}_n)$-predictable random variables, (x_n) is said to be a transform of the martingale (y_n) by the sequence (v_n), if for any n we have $dx_n = v_n dy_n$. In particular, if for any n, v_n is constant almost surely and equal to ±1, we will say that (x_n) is a ±1 transform of (y_n).

Received May 3, 2007; received in final form September 11, 2007.
Partially supported by MEiN Grant 1 PO3A 012 29.
2000 *Mathematics Subject Classification.* 60G42, 31B05.

In [2], Burkholder introduced a notion of differential subordination (though it appears in earlier papers of Burkholder, see for example [1]). A martingale (x_n) is said to be differentially subordinate to a martingale (y_n), if for any $n=0,1,2,\dots$, with probability 1, we have

$$|dx_n| \le |dy_n|. \tag{1.1}$$

This generalizes the notion of martingale transforms; if (x_n) is the transform of (y_n) by a sequence (v_n) which is bounded by 1, then (x_n) is differentially subordinate to (y_n).

The famous results of Burkholder establish sharp weak and strong type inequalities for differentially subordinated martingales.

THEOREM 1.1. *Let (x_n), (y_n) be two $(\mathcal{F}_n)$-martingales such that (x_n) is differentially subordinate to (y_n) and n be a fixed nonnegative integer.*

(i) *(The weak type $(1,1)$ inequality.) For any positive λ,*

$$\lambda \mathbb{P}(|x_n| \ge \lambda) \le 2\mathbb{E}|y_n|. \tag{1.2}$$

(ii) *(The strong type (p,p) inequality.) For $1<p<\infty$,*

$$(\mathbb{E}|x_n|^p)^{1/p} \le (p^*-1)(\mathbb{E}|y_n|^p)^{1/p}, \tag{1.3}$$

where $p^* = \max\{p, p/(p-1)\}$.

Both constants 2 and p^*-1 are best possible.

Since the paper [2], many interesting sharp inequalities for differentially subordinated martingales were established. These include the further results of Burkholder [3], [4], and Suh [6]. Moreover, the differential subordination (1.1) was successfully transferred to the case of continuous-time martingales by Wang [7] and similar sharp inequalities were proved in this setting.

In the paper, we study LlogL inequalities for the differential subordinated martingales. It is well known, that strong type $(1,1)$ inequalities fail to hold and only the weak-type estimates are true. However, the first moment of (x_n) can be bounded in terms of (y_n) as follows: since the strong (p,p) estimates hold true for $1<p<\infty$, classical extrapolation arguments yield the existence of an absolute constants K, L such that

$$\mathbb{E}|x_n| \le K\mathbb{E}|y_n|\log|y_n| + L, \quad n=0,1,2,\dots. \tag{1.4}$$

This inequality is of our main interest. The contribution of this paper is to determine, for any $K>0$, the optimal value of the constant L. Precisely, our main results are contained in the following theorem.

THEOREM 1.2. *Let (x_n), (y_n) be two $(\mathcal{F}_n)$-martingales taking values in $\mathcal{H}$ such that (x_n) is differentially subordinate to (y_n). Fix a nonnegative integer n and a positive number K. Then*

(i) *If $K \le 1$, then the inequality (1.4) does not hold in general for any $L>0$ as it does not even for ± 1 transforms.*

(ii) *If* $1 < K < 2$, *then the inequality* (1.4) *holds with*

$$L = L(K) = \frac{K^2}{2(K-1)} \exp(-K^{-1}). \tag{1.5}$$

The constant L *is best possible, it is already best possible for* $\mathcal{H} = \mathbb{R}$ *and* x *being a* ± 1 *transform of* y. *Furthermore, the inequality is strict in all nontrivial cases.*

(iii) *If* $K \geq 2$, *then the inequality* (1.4) *holds with*

$$L = L(K) = K \exp(K^{-1} - 1). \tag{1.6}$$

The constant is best possible, it is already best possible for $\mathcal{H} = \mathbb{R}$ *and* x *being a* ± 1 *transform of* y. *Furthermore, in general, the inequality is* not *strict.*

The optimality of the constants L is understood in the sense that for any $L' < L$ there exists a pair (x_n), (y_n) of differentially subordinated martingales, for which (1.4) is not true with L replaced by L'.

Note that quite unexpectedly, the inequality (1.4) behaves quite differently for $K < 2$ and $K \geq 2$. We have different expressions for the constant L, and which is more important, for $K < 2$ the inequality is strict, while for $K \geq 2$, we may have equality in (1.4) for some nontrivial martingales.

Our second result concerns the LlogL inequality for harmonic functions. Let Ω be an open subset of $\mathbb{R}^n$, n being a positive integer. Let D be a subdomain of Ω with $0 \in D$ and $\partial D \subset \Omega$. Denote by μ the harmonic measure on ∂D with respect to 0. Consider two harmonic functions v, w on Ω, taking values in a Hilbert space $\mathcal{H}$. Following [3], we say that v is differentially subordinate to w if

$$|\nabla v(x)| \leq |\nabla w(x)| \quad \text{for } x \in \Omega.$$

Burkholder proved the following result.

THEOREM 1.3. *Suppose* v *is differentially subordinate to* w *and* $v(0) \leq w(0)$.

(i) *(The weak type* $(1,1)$ *inequality.) For any positive* λ,

$$\lambda \mu\big(\{x \in \partial D : |v(x)| \geq \lambda\}\big) \leq 2 \int_{\partial D} |w(x)|\, d\mu(x). \tag{1.7}$$

(ii) *(The strong type* (p,p) *inequality.) For* $1 < p < \infty$,

$$\left[\int_{\partial D} |v(x)|^p \, d\mu(x)\right]^{1/p} \leq (p^* - 1) \left[\int_{\partial D} |w(x)|^p \, d\mu(x)\right]^{1/p}. \tag{1.8}$$

Our result can be stated as follows.

THEOREM 1.4. *Suppose* v *is differentially subordinate to* w *and* $K > 1$. *Then*

$$\int_{\partial D} |v(x)|\, d\mu(x) \leq K \int_{\partial D} |w(x)| \log |w(x)|\, d\mu(x) + L, \tag{1.9}$$

where $L = L(K)$ is defined by (1.5) if $1 < K < 2$ and (1.6) in the case $K \geq 2$. The constant $L(K)$ is best possible for $K \geq 2$.

We do not know the best constant L for $1 < K < 2$. We also do not know if (1.9) fails to hold for $K \leq 1$.

The paper is organized as follows. In the next section, we describe the method of proving certain martingale inequalities as well as inequalities for harmonic functions, invented by Burkholder. Section 3 contains the proofs of the inequalities (1.4) and (1.9). The sharpness of these estimates is investigated in Section 4. In the last section, we show that (1.4) is strict for $1 < K < 2$, and that it fails to hold for $K \leq 1$.

2. Burkholder's method

Let us briefly describe the method Burkholder invented for proving inequalities for differentially subordinated martingales/differentially subordinated harmonic functions. Let us first deal with the martingale setting. Given a Borel function $r : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, suppose we want to show that

$$\mathbb{E}r(x_n, y_n) \geq 0, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

for any martingales (x_n), (y_n) with (x_n) differentially subordinate to (y_n). The main idea is to construct a special function b, which satisfies the following properties.

1° For $x, y \in \mathcal{H}$ with $|x| \leq |y|$, we have $b(x, y) \geq 0$.

2° For any $x, y \in \mathcal{H}$, $b(x, y) \leq r(x, y)$.

3° For any $x, y \in \mathcal{H}$ there exist $A = A(x, y)$, $B = B(x, y) \in \mathcal{H}$ such that for any $h, k \in \mathcal{H}$ with $|h| \leq |k|$,

$$b(x + h, y + k) \geq b(x, y) + A(x, y) \cdot h + B(x, y) \cdot k$$

(if b is differentiable in (x, y) then one is forced to take $A(x, y) = \frac{\partial b}{\partial x}(x, y)$, $B(x, y) = \frac{\partial b}{\partial y}(x, y)$).

These conditions immediately yield (2.1): note that by 3°, we have, for any $n \geq 0$,

$$\mathbb{E}b(x_n, y_n) \leq \mathbb{E}b(x_{n+1}, y_{n+1}).$$

Combining this inequality with 1° and 2°, we may write

$$\mathbb{E}r(x_n, y_n) \geq \mathbb{E}b(x_n, y_n) \geq \mathbb{E}b(x_{n-1}, y_{n-1}) \geq \cdots \geq \mathbb{E}b(x_0, y_0) \geq 0, \tag{2.2}$$

thus completing the proof.

The inequalities for harmonic functions can be proved in a similar manner. Assume we want to establish an inequality

$$\int_{\partial D} r(v(x), w(x))\, d\mu(x) \geq 0$$

for differentially subordinated harmonic functions v, w. Again, the key tool is the special function b, satisfying 1°, 2°, and the following harmonic analogue of the condition 3°.

3°′ If v is differentially subordinate to w, then $b(v,w)$ is superharmonic on Ω.

Therefore, as in (2.2), using 1°, 2°, and 3°′,

$$\int_{\partial D} r(v(x),w(x))\,d\mu(x) \geq \int_{\partial D} b(v(x),w(x))\,d\mu(x) \geq b(v(0),w(0)) \geq 0. \tag{2.3}$$

For example, let us consider the weak type estimates (1.2) and (1.7). Clearly, by homogeneity, it suffices to prove them for $\lambda = 1$; then we have $r(x,y) = 2|y| - \chi_{\{|x|\geq 1\}}$. Following Burkholder [3], consider $b : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ defined by

$$b(x,y) = \begin{cases} |y|^2 - |x|^2 & \text{if } |x|+|y| \leq 1, \\ 2|y| - 1 & \text{if } |x|+|y| > 1. \end{cases} \tag{2.4}$$

Then b satisfies 1°, 2°, 3°, and 3°′, which yields (1.2) and (1.7).

In the proofs of Theorems 1.2 and 1.4, we follow the same pattern and construct the special function u with respect to $r(x,y) = K|y|\log|y| - |x| + L$.

3. Proofs of (1.4) and (1.9)

We start from defining the special function $u : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$. Our approach is based on the integration method, introduced by the author in [5]. The special function is obtained by integration of scaled function b from the preceding section against a positive kernel. Then most of its properties follow by the ones of the function b. Precisely, let

$$\begin{aligned} u(x,y) &= \int_1^\infty b(x/t, y/t)\,dt \\ &= \begin{cases} |y|^2 - |x|^2 & \text{if } |x|+|y| \leq 1, \\ 2|y|\log(|x|+|y|) - 2|x| + 1 & \text{if } |x|+|y| > 1. \end{cases} \end{aligned} \tag{3.1}$$

We start from some technical result to be needed later.

LEMMA 3.1. *Fix $y \in \mathcal{H}$. For $1 < K \leq 2$, let*

$$\psi_K(s) = 2|y|\log(s+|y|) + \left(\frac{2}{K} - 2\right)s, \quad s \geq (1-|y|)^+.$$

Then ψ_K attains its maximum only in one point $\frac{|y|}{K-1} \vee (1-|y|)$.

Proof. Straightforward analysis of the derivative. □

The two lemmas below present the most important properties of the function u.

LEMMA 3.2. (i) *If* (x_n) *is a martingale which is differentially subordinate to a martingale* (y_n)*, then for any nonnegative integer* n*,*

$$\mathbb{E}u(x_n, y_n) \ge 0. \tag{3.2}$$

(ii) *If* $v,\ w : \Omega \to \mathcal{H}$ *are harmonic functions and* v *is differentially subordinate to* w*, then*

$$\int_{\partial D} u(v(x), w(x))\, d\mu(x) \ge 0.$$

Proof. (i) For any positive t, (x_n/t) is differentially subordinated to (y_n/t). Therefore, by (2.2), for any nonnegative integer n we have

$$\mathbb{E}b(x_n/t, y_n/t) \ge 0$$

and Fubini's theorem yields the claim.

(ii) We use the same scaling argument and (2.3). □

LEMMA 3.3. *Fix* $x, y \in \mathcal{H}$*. For* $K > 1$*, recall* $L = L(K)$ *given by* (1.5) *and* (1.6)*.*

(i) *If* $1 < K < 2$*, then*

$$u(x,y) \le 2|y| \log \frac{2L|y|}{K} - \frac{2}{K}|x| + 1. \tag{3.3}$$

Furthermore, if $|x|^2 + |y|^2 > 0$*,*

$$\text{we have equality in (3.3) iff } |x| + |y| \ge 1 \text{ and } |x| = \frac{1}{K-1}|y|. \tag{3.4}$$

(ii) *If* $K \ge 2$*, then*

$$u(x,y) \le K|y| \log \frac{2L|y|}{K} - |x| + \frac{K}{2}. \tag{3.5}$$

Proof. (i) If $|x| + |y| \le 1$, then the inequality (3.3) is equivalent to

$$|y|^2 - 2|y| \log \frac{2L|y|}{K} - 1 + \frac{1}{K^2} \le |x|^2 - \frac{2}{K}|x| + \frac{1}{K^2} = \left(|x| - \frac{1}{K}\right)^2.$$

Obviously, the right-hand side is nonnegative. The left-hand side, as a function of $|y| \in [0,1]$, is strictly concave and vanishes along with its derivative at $|y| = 1 - K^{-1}$. Hence, we are done. Note that (3.4) holds true in this case.

If $|x| + |y| > 1$, then (3.3) takes form

$$2|y| \log(|x| + |y|) + \left(\frac{2}{K} - 2\right)|x| \le 2|y| \log \frac{2L|y|}{K}. \tag{3.6}$$

By Lemma 3.1, if $|y|/(K-1) \ge 1 - |y|$ (equivalently, $\frac{|y|}{K-1} + |y| \ge 1$), then the left-hand side does not exceed

$$\psi_K\left(\frac{|y|}{K-1}\right) = 2|y| \log \frac{K|y|}{K-1} - \frac{2}{K}|y| = 2|y| \log \frac{2L|y|}{K},$$

with equality only if $|x| = |y|/(K-1)$. This shows (3.3) and (3.4). Finally, if $|y|/(K-1) < 1 - |y|$, then Lemma 3.1 reduces (3.6) to the case $|x| + |y| = 1$, which we have already considered.

(ii) This is proved exactly in the same manner: for $|x|+|y| \le 1$, one rewrites (3.5) in the form

$$|y|^2 - K|y|\log\frac{2L|y|}{K} + \frac{1}{4} - \frac{K}{2} \le |x|^2 - |x| + \frac{1}{4} = \left(|x| - \frac{1}{2}\right)^2.$$

It is clear that right-hand side is nonnegative, while the left-hand side, considered as a function of $|y|$, is concave on $[0,1]$ and vanishes along with its derivative for $|y| = \frac{1}{2}$.

If $|x| + |y| > 1$, then (3.5) is equivalent to

$$2|y|\log(|x|+|y|) - |x| + 1 \le K|y|\log\frac{2L|y|}{K} + \frac{K}{2}. \tag{3.7}$$

Lemma 3.1 reduces the case $|y| < \frac{1}{2}$ (or $|y| < 1 - |y|$) to the case $|x| + |y| = 1$, which we have just considered. If $|y| \ge \frac{1}{2}$, then, by Lemma 3.1, the left-hand side of (3.7) does not exceed $\psi_2(|y|) + 1$ and we are left to show that

$$\psi_2(|y|) + 1 = 2|y|\log 2|y| - |y| + 1 \le K|y|\log\frac{2L|y|}{K} + \frac{K}{2},$$

or, equivalently, $(K-2)[2|y|\log(2|y|) - 2|y| + 1] \ge 0$. However, $K \ge 2$ and the expression in the square bracket is nonnegative. The proof is complete. □

Now, we are ready for the following proof.

Proof of the inequality (1.4). Fix a nonnegative integer n and let (x_n), (y_n) be two martingales, with (x_n) being differentially subordinate to (y_n). Then $(x'_n) = (x_n \cdot K/2L)$ is differentially subordinated to $(y'_n) = (y_n \cdot K/2L)$. Therefore, we may apply Lemmas 3.2 and 3.3 to these new martingales; for $1 < K < 2$, we obtain

$$\begin{aligned} K\mathbb{E}|y_n|\log|y_n| - \mathbb{E}|x_n| + L &= L\left[2\mathbb{E}|y'_n|\log\frac{2L|y'_n|}{K} - \frac{2}{K}\mathbb{E}|x'_n| + 1\right] \\ &\ge L\mathbb{E}u(x'_n, y'_n) \ge 0, \end{aligned} \tag{3.8}$$

while for $K \ge 2$,

$$\begin{aligned} K\mathbb{E}|y_n|\log|y_n| - \mathbb{E}|x_n| + L &= \frac{2L}{K}\left[K\mathbb{E}|y'_n|\log\frac{2L|y'_n|}{K} - \mathbb{E}|x'_n| + \frac{K}{2}\right] \\ &\ge \frac{2L}{K}\mathbb{E}u(x'_n, y'_n) \ge 0. \end{aligned}$$

□

Proof of the inequality (1.9). We repeat all the arguments from the proof above, replacing the martingales x_n, y_n by the functions v, w, and expectations by the integrals over ∂D. □

4. Optimality of $L = L(K)$

Throughout this section, we assume $\mathcal{H} = \mathbb{R}$.

Let us start with two simple properties of the function u. The conditional versions of the identities below will be needed later.

LEMMA 4.1. *Let d be a centered random variable and x, y be two positive numbers.*

(i) *If $-x \le d \le y$ almost surely, then*

$$\mathbb{E}u(x+d, y+d) = u(x,y). \tag{4.1}$$

(ii) *If $y \ge 1$ and $d \ge -y$ almost surely, then*

$$\mathbb{E}u(d, y+d) = u(0,y) + \mathbb{E}\chi_{\{d\ge 0\}}\left(2(y+d)\log\frac{y+2d}{y} - 4d\right). \tag{4.2}$$

Proof. (i) From (3.1), we infer that the function $\phi_{x,y} : [-x, y] \to \mathbb{R}$ given by $\phi_{x,y}(r) = u(x+r, y-r)$ is linear. This yields (4.1).

(ii) We have

$$\begin{aligned}\mathbb{E}\big(u(d, y+d) - u(0,y)\big) &= \mathbb{E}[u(d, y+d) - u(0,y) - 2d(\log y + 1)]\\ &= \mathbb{E}\chi_{\{d\ge 0\}}[u(d, y+d) - u(0,y) - 2d(\log y+1)]\\ &= \mathbb{E}\chi_{\{d\ge 0\}}\left[2(y+d)\log\frac{y+2d}{y} - 4d\right]. \qquad \square\end{aligned}$$

Now, we will construct a crucial pair of martingales. Let the underlying probability space be the interval $[0,1]$ with the Lebesgue measure. Fix numbers $\beta > 0$ and $\gamma > 1$. Let, for $k = 1, 2, \ldots,$

$$\begin{aligned}T_k &= (1+\beta)^{k-1} \ge 1,\\ m_k &= \frac{\gamma - 1}{2\gamma}\cdot(1+\beta)^{1-k}\left(1 - \frac{2\beta}{2\gamma + \beta(\gamma+1)}\right)^{k-1}.\end{aligned}$$

Now, define (for simplicity, we identify a set with its indicator function)

$$\begin{aligned}x_0 = y_0 &\equiv \frac{1}{2},\\ dx_1 = -dy_1 &= -\frac{1}{2}\cdot\left[0, \frac{\gamma-1}{2\gamma}\right] + \frac{1}{2}\cdot\frac{\gamma-1}{\gamma+1}\cdot\left[\frac{\gamma-1}{2\gamma}, 1\right].\end{aligned}$$

Furthermore, for $k = 1, 2, \ldots,$

$$\begin{aligned}dx_{2k} = dy_{2k} &= \frac{\beta T_k}{2}\cdot\left[0, \frac{2\gamma m_k}{2\gamma+\beta(\gamma+1)}\right] - \frac{\gamma T_k}{\gamma+1}\cdot\left(\frac{2\gamma m_k}{2\gamma+\beta(\gamma+1)}, m_k\right],\\ dx_{2k+1} &= -dy_{2k+1}\\ &= -\frac{\beta T_k}{2}\cdot[0, m_{k+1}] + \frac{T_k(2\gamma+\beta(\gamma-1))}{2(\gamma+1)}\cdot\left(m_{k+1}, \frac{2\gamma m_k}{2\gamma+\beta(\gamma+1)}\right].\end{aligned}$$

Note that (x_n) is a ± 1 transform of (y_n). Some of the properties of these martingales are described in the following lemma.

LEMMA 4.2. *For $k=1,2,\dots,$ we have*

$$|x_{k-1}|+|y_{k-1}|\geq 1 \quad \text{on } [0,1], \tag{4.3}$$

$$(x_{2k},y_{2k})=\left(\frac{\beta T_k}{2},T_k+\frac{\beta T_k}{2}\right) \quad \text{on } \left[0,\frac{2\gamma m_k}{2\gamma+\beta(\gamma+1)}\right], \tag{4.4}$$

$$(x_{2k-1},y_{2k-1})=(0,T_k) \quad \text{on } [0,m_k], \tag{4.5}$$

$$|x_{2k-1}|=\gamma|y_{2k-1}| \quad \text{on } (m_k,1]. \tag{4.6}$$

Proof. Straightforward induction. □

LEMMA 4.3. *For $k=1,2,\dots$ we have*

$$\mathbb{E}u(x_{2k+1},y_{2k+1})=\mathbb{E}u(x_{2k},y_{2k}) \tag{4.7}$$

and

$$\begin{aligned}\mathbb{E}u(x_{2k},y_{2k})&=\mathbb{E}u(x_{2k-1},y_{2k-1})\\ &\quad+\frac{2\gamma m_k T_k}{2\gamma+\beta(\gamma+1)}[(2+\beta)\log(1+\beta)-2\beta].\end{aligned} \tag{4.8}$$

Proof. The property (4.4), combined with the inequality

$$-\frac{\beta T_k}{2}\leq dx_{2k+1}\leq T_k+\frac{\beta T_k}{2} \quad \text{on } \left[0,\frac{2\gamma m_k}{2\gamma+\beta(\gamma+1)}\right]$$

and Lemma 4.1(i), gives (4.7). Similarly, the property (4.5), together with $dx_{2k}\geq -T_k$ on $[0,m_k]$, in view of Lemma 4.1(ii), yield

$$\begin{aligned}&\mathbb{E}u(x_{2k},y_{2k})-\mathbb{E}u(x_{2k-1},y_{2k-1})\\ &\quad=\mathbb{P}(dx_{2k}\geq 0)\cdot\left[2\left(T_n+\frac{\beta T_n}{2}\right)\log\frac{T_n+\beta T_n}{T_n}-2\beta T_n\right]\\ &\quad=\frac{2\gamma m_k T_k}{2\gamma+\beta(\gamma+1)}[(2+\beta)\log(1+\beta)-2\beta],\end{aligned}$$

which is (4.8). □

Proof of the sharpness of (1.4) *with $L=L(K)$ and* (1.9) *for $K\geq 2$.* We consider the cases $K\geq 2$ and $1<K<2$ separately.

The case $K\geq 2$. We simply set $x_n=y_n\equiv\exp(K^{-1}-1)$ almost surely, $n=0,1,2,\dots$ and obtain equality in (1.4). Exactly in the same manner, the choice $v=w\equiv\exp(K^{-1}-1)$ on Ω gives equality in (1.9).

The case $1<K<2$. This is more involved. For $\gamma=1/(K-1)>1$, let (x_n), (y_n) be the martingales constructed above and set $x_n'=x_n\cdot 2L/K$,

$y_n' = y_n \cdot 2L/K$. For positive integer k, let

$$z_{2k-1} = \frac{1}{L}[K|y_{2k-1}'|\log|y_{2k-1}'| - |x_{2k-1}'| + L] = 2y_{2k-1}\log\left(\frac{2L}{K}y_{2k-1}\right) - \frac{2}{K}|x_{2k-1}| + 1.$$

Combining (3.4), (4.3), and (4.6), we may write

$$\begin{aligned}\mathbb{E}z_{2k-1} &= \mathbb{E}z_{2k-1}\chi_{[0,m_k]} + \mathbb{E}z_{2k-1}\chi_{(m_k,1]}\\ &= \mathbb{E}z_{2k-1}\chi_{[0,m_k]} + \frac{K}{2}\mathbb{E}u(x_{2k-1},y_{2k-1})\chi_{(m_k,1]}\\ &= \mathbb{E}z_{2k-1}\chi_{[0,m_k]} - \frac{K}{2}\mathbb{E}u(x_{2k-1},y_{2k-1})\chi_{[0,m_k]}\\ &\quad + \frac{K}{2}\mathbb{E}u(x_{2k-1},y_{2k-1}).\end{aligned} \tag{4.9}$$

Now, fix $\varepsilon > 0$. By (4.7) and (4.8),

$$\begin{aligned}\frac{K}{2}\mathbb{E}u(x_{2k-1},y_{2k-1}) &= \frac{K}{2}[\mathbb{E}u(x_{2k-1},y_{2k-1}) - \mathbb{E}u(x_0,y_0)]\\ &= \frac{K}{2}\sum_{n=0}^{2k-2}[\mathbb{E}u(x_{n+1},y_{n+1}) - \mathbb{E}u(x_n,y_n)]\\ &= \frac{K}{2}\sum_{n=1}^{k-1}[\mathbb{E}u(x_{2n},y_{2n}) - \mathbb{E}u(x_{2n-1},y_{2n-1})]\\ &= \frac{K}{2}\sum_{n=1}^{k-1}\frac{2\gamma m_n T_n}{2\gamma+\beta(\gamma+1)}[(2+\beta)\log(1+\beta) - 2\beta]\\ &\le \frac{K\gamma((2+\beta)\log(1+\beta)-2\beta)}{2\gamma+\beta(\gamma+1)}\sum_{n=1}^{\infty}m_nT_n\\ &= \frac{K\gamma((2+\beta)\log(1+\beta)-2\beta)}{2\gamma+\beta(\gamma+1)}\cdot\frac{\gamma-1}{2\gamma}\cdot\frac{2\gamma+\beta(\gamma+1)}{2\beta}\\ &< \varepsilon\end{aligned}$$

for β sufficiently close to 0. Now, in virtue of (4.5), we have

$$\mathbb{E}z_{2k-1}\chi_{[0,m_k]} = m_k\left(2T_k\log\frac{2LT_k}{K} + 1\right) < \varepsilon$$

for k large enough. Finally, by (4.5),

$$\frac{K}{2}\mathbb{E}u(x_{2k-1},y_{2k-1})\chi_{[0,m_k]} = \frac{K}{2}m_k(2T_k\log T_k + 1) < \varepsilon$$

for k large enough. Therefore, by (4.9), we have shown that by a proper choice of β and k, $\mathbb{E}z_{2k-1}$ can be arbitrarily close to 0, which clearly implies the optimality of $L = L(K)$. $\square$

5. Strictness and the case $K \le 1$

Proof of the strictness of (1.4) *for* $1 < K < 2$. Assume (x_n) is differentially subordinate to (y_n). Fix an integer n and suppose $\mathbb{P}(|x_n|^2 + |y_n|^2 > 0) > 0$. Consider the martingales $(x_n') = (x_n \cdot K/2L)$, $(y_n') = (y_n \cdot K/2L)$ as in the proof of (1.4). By (1.3), we have

$$\mathbb{E}|x_n'|^2 \le \mathbb{E}|y_n'|^2,$$

which implies $\mathbb{P}(|x_n'|^2 + |y_n'|^2 > 0, |x_n'| \le |y_n'|) > 0$. Therefore, by (3.4), the first inequality in (3.8) is strict, and hence so is the inequality (1.4). □

The inequality (1.4) *fails to hold for* $K \le 1$. Suppose (1.4) holds for some $K \le 1$ and $L < \infty$. Fix $K' \in (1, 2)$, let (y_n) be any martingale and (x_n) be its ± 1 transform. Since $t \log t \ge -e^{-1}$ for nonnegative t, we may write

$$\begin{aligned}
\mathbb{E}|x_n| &\le K\mathbb{E}|y_n| \log |y_n| + L \\
&= K\mathbb{E}(|y_n| \log |y_n| + e^{-1}) + L - Ke^{-1} \\
&\le K'\mathbb{E}(|y_n| \log |y_n| + e^{-1}) + L - Ke^{-1} \\
&= K'\mathbb{E}|y_n| \log |y_n| + L + (K' - K)e^{-1},
\end{aligned}$$

which, by Theorem 1.2(ii), implies $L(K') \le L + (K' - K)e^{-1}$ and contradicts (1.5) if K' is taken sufficiently close to 1. □

Acknowledgments. The author obtained the results above during his visit to Université de Franche–Comte in Besançon, France. The author would also like to thank the referee for the careful reading of the first version of the paper.

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