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EXTENSION THEOREMS FOR THE FOURIER TRANSFORM ASSOCIATED WITH NONDEGENERATE QUADRATIC SURFACES IN VECTOR SPACES OVER FINITE FIELDS

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Abstract. We study the restriction of the Fourier transform to quadratic surfaces in vector spaces over finite fields. In two dimensions, we obtain the sharp result by considering the sums of arbitrary two elements in the subset of quadratic surfaces on two dimensional vector spaces over finite fields. For higher dimensions, we estimate the decay of the Fourier transform of the characteristic functions on quadratic surfaces so that we obtain the Tomas–Stein exponent. Using incidence theorems, we also study the extension theorems in the restricted settings to sizes of sets in quadratic surfaces. Estimates for Gauss and Kloosterman sums and their variants play an important role.

1. Introduction

Let S be a subset of $\mathbb{R}^d$ and $d\sigma$ a positive measure supported on S. Then one may ask that for which values of p and r does the estimate

$$\|\widehat{f\,d\sigma}\|_{L^r(\mathbb{R}^d)} \leq C_{p,r}\|f\|_{L^p(S,\,d\sigma)} \quad \text{for all } f \in L^p(S,d\sigma) \tag{1.1}$$

hold? This problem is known as the extension theorems. See, for example, [1], [5], [7], [9], and the references contained therein on recent progress related to this problem and its analogs. In the case of $p = 2$ in (1.1), Strichartz [6] gave a complete solution when S is a quadratic surface given by $S = \{x \in \mathbb{R}^d : Q(x) = j\}$, where $Q(x)$ is a polynomial of degree of two with real coefficients and j is a real constant. In this paper, we study the analogous extension operators given by quadratic forms in the finite field setting, building upon earlier work of Mockenhaupt and Tao [3] for the paraboloid in vector spaces over finite fields. We begin with some notation and definitions to describe our main results. Let

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$\mathbb{F}_q$ be a finite field of characteristic $char(\mathbb{F}_q) > 2$ with q elements, and let $\mathbb{F}_q^d$ be a d-dimensional vector space over $\mathbb{F}_q$. Given a function $f : \mathbb{F}_q^d \to \mathbb{C}$, $d \geq 1$, define the Fourier transform of f by the formula

$$\widehat{f}(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(-x \cdot m) f(x),$$

where χ is a nontrivial additive character on $\mathbb{F}_q$. When $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$ for some prime q, we could take $\chi(t) = e^{2\pi i t/q}$, and the calculations in the paper are independent of the exact choice of the character. Recall that the Fourier inversion theorem is given by

$$f(x) = \sum_{m \in \mathbb{F}_q^d} \chi(x \cdot m) \widehat{f}(m).$$

Also, recall that the Plancherel theorem says in this context that

$$\sum_{m \in \mathbb{F}_q^d} |\widehat{f}(m)|^2 = q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2.$$

Let $S \subset \mathbb{F}_q^d$ be an algebraic variety in $\mathbb{F}_q^d$. We denote by $d\sigma$ normalized surface measure on S defined by the relation

$$\widehat{f\, d\sigma}(m) = \frac{1}{\#S} \sum_{x \in S} \chi(-x \cdot m) f(x),$$

where $\#S$ denotes the number of elements in S. In other words,

$$q^{-d} \cdot \sigma(x) = (\#S)^{-1} \cdot S(x).$$

Here, and throughout the paper, $E(x)$ denotes the characteristic function, χ_E, of the subset E of $\mathbb{F}_q^d$. We therefore, denote by $E\, d\sigma$ the measure $\chi_E\, d\sigma$.

For $1 \leq p, r < \infty$, define

$$\|f\|^p_{L^p(\mathbb{F}_q^d, dx)} = q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^p,$$

$$\|\widehat{f}\|^r_{L^r(\mathbb{F}_q^d, dm)} = \sum_{m \in \mathbb{F}_q^d} |\widehat{f}(m)|^r$$

and

$$\|f\|^p_{L^p(S, d\sigma)} = \frac{1}{\#S} \sum_{x \in S} |f(x)|^p.$$

Similarly, denote by $\|f\|_{L^\infty}$ the maximum value of $|f|$.

Observe that the measure on the "space" variables, dx, is the normalized measure obtained by dividing the counting measure by q^d, whereas the measure on the "phase" variables, dm, is just the usual counting measure. These

normalizations are chosen in such a way that the Plancherel inequality takes the familiar form

$$\|\widehat{f}\|_{L^2(\mathbb{F}_q^d, dm)} = \|f\|_{L^2(\mathbb{F}_q^d, dx)}.$$

We now define the nondegenerate quadratic surfaces in $\mathbb{F}_q^d$ in the usual way. Let $x = (x_1, x_2, \dots, x_d) \in \mathbb{F}_q^d$. Denote by $Q(x)$ a homogeneous polynomial in $\mathbb{F}_q[x_1, \dots, x_d]$ of degree 2. Since $char(\mathbb{F}_q) > 2$, throughout this paper, we can express $Q(x)$ in the form

$$Q(x_1, x_2, \dots, x_d) = \sum_{i,j=1}^{d} a_{ij} x_i x_j \quad \text{with } a_{ij} = a_{ji}.$$

If the $d \times d$ matrix $\{a_{ij}\}$ is invertible, we say that the polynomial $Q(x)$ is a nondegenerate quadratic form over $\mathbb{F}_q$. For each $j \in \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, the multiplicative group of $\mathbb{F}_q$, consider a set S_j in $\mathbb{F}_q^d$ given by

$$S_j = \{x \in \mathbb{F}_q^d : Q(x_1, \dots, x_d) = j\}, \tag{1.2}$$

where $Q(x)$ is a nondegenerate quadratic form. We call such a set S_j a nondegenerate quadratic surface in $\mathbb{F}_q^d$. For example, the sphere

$$S^{d-1} = \{x \in \mathbb{F}_q^d : x_1^2 + x_2^2 + \cdots + x_d^2 = 1\}$$

is a nondegenerate quadratic surface in $\mathbb{F}_q^d$.

1.1. Extension theorems and main results of this paper. Let $1 \le p, r \le \infty$. We define $R^*(p \to r)$ to be the best constant such that the extension estimate

$$\|\widehat{f\, d\sigma}\|_{L^r(\mathbb{F}_q^d, dm)} \le R^*(p \to r) \|f\|_{L^p(S_j, d\sigma)}$$

holds for all functions f on S_j. The main goal of this paper is to determine the set of exponents p and r such that

$$R^*(p \to r) \le C_{p,r} < \infty,$$

where $C_{p,r}$ is independent of the size of $\mathbb{F}_q$. We note that

$$R^*(p_1 \to r) \le R^*(p_2 \to r) \quad \text{for } p_1 \ge p_2, \tag{1.3}$$

and

$$R^*(p \to r_1) \le R^*(p \to r_2) \quad \text{for } r_1 \ge r_2,$$

which will allow us to reduce the analysis below to certain endpoint estimates.

Let S be an algebraic variety in $\mathbb{F}_q^d$ with $\#S \approx q^k$ for some $0 < k < d$. Here, and throughout the paper, $X \lesssim Y$ means that there exists $C > 0$, independent of q such that $X \le CY$, and $X \approx Y$ means both $X \lesssim Y$ and $Y \lesssim X$. Mockenhaupt and Tao [3] proved that $R^*(p \to r)$ is uniformly bounded ($O(1)$ with constants independent of the size of $\mathbb{F}_q$) only if

$$r \ge \frac{2d}{k} \quad \text{and} \quad r \ge \frac{dp}{k(p-1)}. \tag{1.4}$$

For the detailed proofs of these assertions, see [3], pages 41–42.

Mockenhaupt and Tao also showed that $R^*(2 \to r)$ is uniformly bounded whenever

$$r \geq \frac{2d+2}{d-1} \tag{1.5}$$

if

$$S = \{(x, x \cdot x) : x \in F_q^{d-1}\},$$

an analog of the Euclidean paraboloid. Moreover, when $d = 3$ and -1 is not a square in $\mathbb{F}_q$, they improved the result in (1.5) by showing that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$, such that

$$R^*\left(\frac{8}{5} \to 4\right) \lesssim 1 \quad \text{and} \quad R^*\left(2 \to \frac{18}{5}\right) \leq C_\varepsilon q^\varepsilon. \tag{1.6}$$

If we replaced the paraboloid by a general nondegenerate quadratic surface, the extension problem becomes more complicated, in part because the Fourier transform of quadratic surfaces cannot be computed by simply considering the Gauss sums, as was pointed out by the authors in [3]. Using generalized Kloosterman sums, we estimate the decay of the Fourier transform of nondegenerate quadratic surfaces. As a result, we obtain Theorem 1 below which gives the same exponents as (1.5) (see Figure 1).

THEOREM 1. *Let S_j be a nondegenerate quadratic surface in $\mathbb{F}_q^d$ defined as in (1.2). If $d \geq 2$ and $r \geq \frac{2d+2}{d-1}$, then*

$$R^*(2 \to r) \lesssim 1.$$

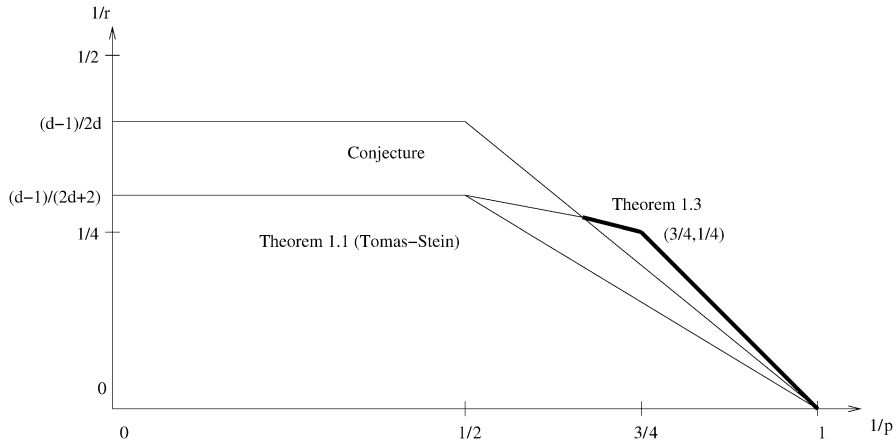

FIGURE 1. Tomas–Stein exponent and extension estimates in a restricted setting to big sets ($q^{\frac{d+1}{2}} \lesssim \#E \lesssim q^{d-1}$).

In the case $d = 2$, Mochenhaupt and Tao [3] showed that the necessary conditions for the boundedness of $R^*(p \to r)$ in (1.4) are also sufficient when S is the parabola. Theorem 2 below implies that this also holds in the case when S is a nondegenerate quadratic curve. To see this, observe from Corollary 10 that $\#S \approx q$ for $d = 2$. Thus, the necessary conditions in (1.4) take the form

$$r \geq 4 \quad \text{and} \quad r \geq \frac{2p}{p-1}. \tag{1.7}$$

Combining (1.3) with Theorem 2 below, we see that

$$R^*(p \to 4) \lesssim 1 \quad \text{for } 2 \leq p \leq \infty. \tag{1.8}$$

By direct estimation, we have

$$R^*(p \to \infty) \lesssim 1 \quad \text{for } 1 \leq p \leq \infty. \tag{1.9}$$

Interpolating (1.8) and (1.9), we see that the necessary conditions given by (1.7) are in fact sufficient as we claim once we establish the following result.

THEOREM 2. *Let $d \geq 2$. Let S_j be the nondegenerate quadratic surface in $\mathbb{F}_q^d$ defined as in (1.2). Then we have*

$$R^*(2 \to 4) \lesssim 1.$$

Observe that Theorem 2 is stronger in two dimensions. Theorem 1 and Theorem 2 are the same in three dimensions, and Theorem 1 is stronger in dimensions four and higher. Using incidence theory, we are able to improve the exponents above in a restricted setting. See Theorem 3 and Figure 1 for extension theorems restricted to big sets, and also see Theorem 4, Figures 2, 3,

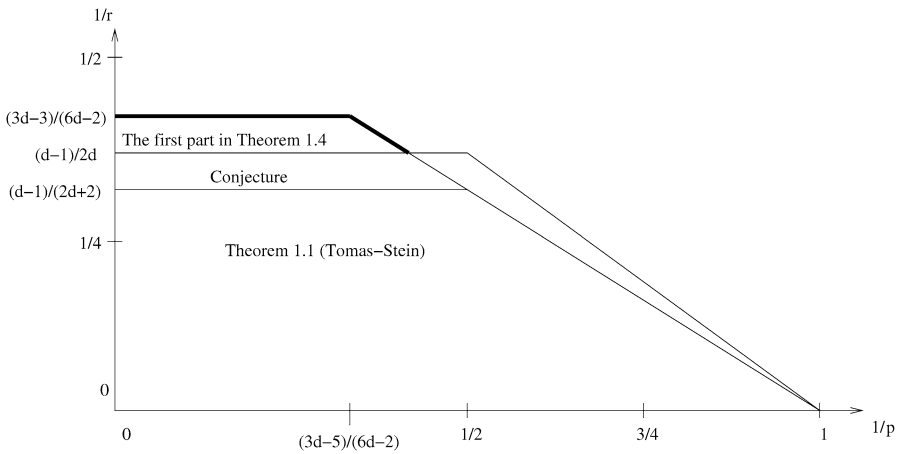

FIGURE 2. Extension estimates in a restricted setting ($1 \lesssim \#E \lesssim q^{\frac{d-1}{2}}$).

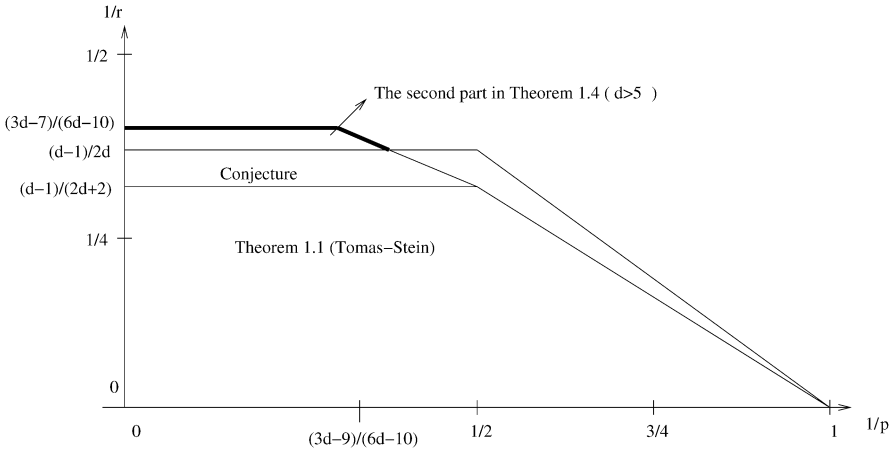

FIGURE 3. Extension estimates in a restricted setting ($1 \lesssim \#E \lesssim q^{\frac{d+1}{2}}, d > 5$).

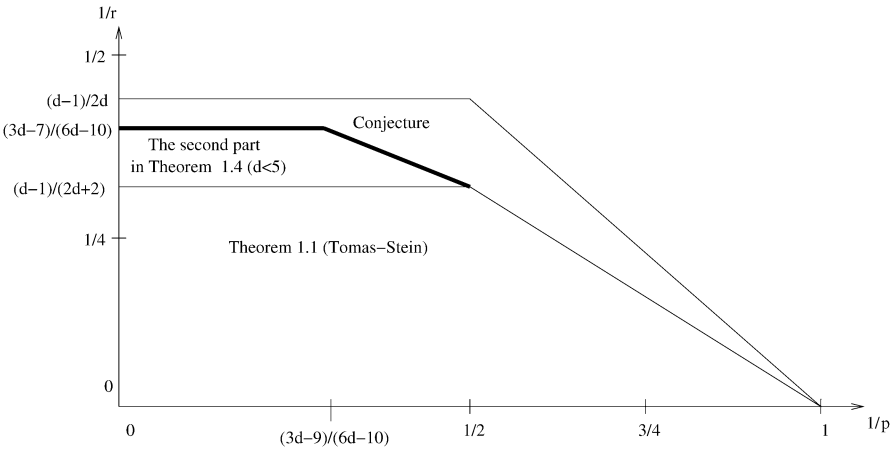

FIGURE 4. Extension estimates in a restricted setting ($1 \lesssim \#E \lesssim q^{\frac{d+1}{2}}, d < 5$).

and 4 for extension theorems restricted to small sets. These results are analogous to those obtained by Mockenhaupt and Tao as described in (1.6) above. While the aforementioned authors use combinatorial methods to prove their incidence theorems, we use Fourier analytic methods which eventually reduce proofs to the estimates for Kloosterman and related sums.

THEOREM 3. *Let S_j be a nondegenerate quadratic surface in $\mathbb{F}_q^d$ and E be a subset of S_j. Then we have the following estimate:*

$$\|\widehat{E\,d\sigma}\|_{L^4(\mathbb{F}_q^d,dm)} \lesssim \|E\|_{L^{4/3}(S_j,d\sigma)} \quad \text{for } q^{\frac{d+1}{2}} \lesssim \#E \lesssim q^{d-1}. \tag{1.10}$$

THEOREM 4. *Let S_j be a nondegenerate quadratic surface in $\mathbb{F}_q^d$ and E be a subset of S_j. Then for every $p_0 \geq 2$, we have the following estimates:*

$$\|\widehat{E\,d\sigma}\|_{L^r(\mathbb{F}_q^d,dm)} \lesssim \|E\|_{L^p(S_j,d\sigma)} \quad \text{for } 1 \lesssim \#E \lesssim q^{\frac{d-1}{2}} \tag{1.11}$$

where the exponents p and r are given by

$$p \geq \frac{(6d-2)p_0 - 8d + 8}{(3d-5)p_0 - 4d + 12} \quad \text{and} \quad r \geq \frac{(6d-2)p_0 - 8d + 8}{(3d-3)p_0 - 4d + 4},$$

and

$$\|\widehat{E\,d\sigma}\|_{L^r(\mathbb{F}_q^d,dm)} \lesssim \|E\|_{L^p(S_j,d\sigma)} \quad \text{for } 1 \lesssim \#E \lesssim q^{\frac{d+1}{2}}, \tag{1.12}$$

where the exponents p and r are given by

$$p \geq \frac{(6d-10)p_0 - 8d + 24}{(3d-9)p_0 - 4d + 20} \quad \text{and} \quad r \geq \frac{(6d-10)p_0 - 8d + 24}{(3d-7)p_0 - 4d + 12}.$$

1.2. Outline of this paper. In Section 2, we shall introduce few theorems related to bounds on exponential sums. As an application, we get the decay of the Fourier transform of the characteristic functions on the nondegenerate quadratic surfaces in vector spaces over finite fields (see Lemma 9 below). In Section 3, we shall prove Theorem 1 which can be obtained from the results of Lemma 9. In Section 4, the proof of Theorem 2 will be given. In the final section, we prove Theorem 3 and Theorem 4.

2. Classical bounds on exponential sums and consequences

In this section, we shall estimate the decay of Fourier transform of the characteristic functions on nondegenerate quadratic surfaces in $\mathbb{F}_q^d$ using the classical bounds on exponential sums. To do this, we first introduce the well known theorems for exponential sums. The following theorem is a well-known estimate for Gauss sums.

THEOREM 5. *Let χ be a nontrivial additive character of $\mathbb{F}_q$, and ψ a multiplicative character of $\mathbb{F}_q^*$. It follows that*

$$G_a(\chi,\psi) = \sum_{t\in\mathbb{F}_q^*} \chi(at)\psi(t) = O\big(q^{\frac{1}{2}}\big), \quad a \in \mathbb{F}_q^*.$$

Proof. If $\psi = 1$, the result is obvious, so we may assume that ψ is a nontrivial multiplicative character of $\mathbb{F}_q^*$. We have

$$\begin{aligned}|G_a(\chi,\psi)|^2 &= \sum_{t\in\mathbb{F}_q^*}\sum_{s\in\mathbb{F}_q^*}\chi(at-as)\psi(ts^{-1})\\ &= \sum_{t\in\mathbb{F}_q^*}\psi(t)\sum_{s\in\mathbb{F}_q^*}\chi(ast-as)\\ &= \sum_{t\in\mathbb{F}_q^*}\psi(t)\Big(-1+\sum_{s\in\mathbb{F}_q}\chi(st-s)\Big)\\ &= -\sum_{t\in\mathbb{F}_q^*}\psi(t)+\sum_{t\in\mathbb{F}_q^*}\sum_{s\in\mathbb{F}_q}\chi\big((t-1)s\big)\\ &= 0+q=q.\end{aligned}$$

Thus,

$$|G_a(\chi,\psi)| = q^{\frac{1}{2}}$$

and the proof is complete. □

The following theorem gives us the relation between more general exponential sums and Gauss sums in Theorem 5. For a nice proof, see [2].

THEOREM 6. *Let χ be a nontrivial additive character of $\mathbb{F}_q$, $n\in\mathbb{N}$, and ψ a multiplicative character of $\mathbb{F}_q^*$ of order $h = gcd(n, q-1)$. Then*

$$\sum_{s\in\mathbb{F}_q}\chi(ts^n) = \sum_{k=1}^{h-1}\psi^{-k}(t)G(\psi^k,\chi)$$

for any $t\in\mathbb{F}_q^*$, where $G(\psi^k,\chi) = \sum_{s\in\mathbb{F}_q^*}\psi^k(s)\chi(s)$.

The following theorem is well known as the estimation of the Salié sum, often referred to as the twisted Kloosterman sum (see [4]).

THEOREM 7. *Let ψ be a multiplicative character of order two of $\mathbb{F}_q^*$, q odd, and $a, b\in\mathbb{F}_q$. Then for any additive character χ of $\mathbb{F}_q$,*

$$\left|\sum_{t\in\mathbb{F}_q^*}\psi(t)\chi(at+bt^{-1})\right| \lesssim q^{\frac{1}{2}}.$$

The following is a classical estimate for Kloosterman sums due to Wey [8] (see also [2]).

THEOREM 8. *If χ is a nontrivial additive character of $\mathbb{F}_q$, and $a, b\in\mathbb{F}_q$ are not both 0, then we have*

$$\left|\sum_{t\in\mathbb{F}_q^*}\chi(at+bt^{-1})\right| \lesssim q^{\frac{1}{2}}.$$

With the same notation as above, we have the following estimate on the Fourier transform of the characteristic function of a nondegenerate quadratic surface.

LEMMA 9. *Let $\mathbb{F}_q, q$ odd, be a finite field. Then*

$$|\widehat{S_j}(m)| = \left| q^{-d} \sum_{x \in S_j} \chi(-x \cdot m) \right| \lesssim q^{-\frac{d+1}{2}}$$

if $m \neq (0, \dots, 0)$, and

$$\widehat{S_j}(0, \dots, 0) \approx q^{-1}.$$

From Lemma 9, we obtain the following corollary.

COROLLARY 10.

$$\#S_j \approx q^{d-1}.$$

Proof. Using the second part of Lemma 9, we have

$$\widehat{S_j}(0, \dots, 0) = q^{-d} \sum_{x \in \mathbb{F}_q^d} S_j(x) \approx q^{-1},$$

and the result follows. □

2.1. Proof of Lemma 9. We first observe that

$$\begin{aligned}
\widehat{S_j}(m) &= q^{-d} \sum_{x \in S_j} \chi(-x \cdot m) \\
&= q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(-x \cdot m) q^{-1} \sum_{t \in \mathbb{F}_q} \chi\big(t\big(Q(x) - j\big)\big) \\
&= q^{-1} \delta_0(m) + q^{-d-1} \sum_{t \in \mathbb{F}_q^*} \chi(-jt) \sum_{x \in \mathbb{F}_q^d} \chi\big(tQ(x) - x \cdot m\big),
\end{aligned}$$

where $\delta_0(m) = 1$ if $m = (0, \dots, 0)$ and $\delta_0(m) = 0$ otherwise. To complete the proof of Lemma 9, it suffices to show that for $j \neq 0$, $m \in \mathbb{F}_q^d$,

$$D(j, m) \lesssim q^{\frac{d+1}{2}} \tag{2.1}$$

where

$$D(j, m) = \sum_{t \in \mathbb{F}_q^*} \chi(-jt) \sum_{x \in \mathbb{F}_q^d} \chi\big(tQ(x) - x \cdot m\big). \tag{2.2}$$

Let

$$W_t(m) = \sum_{x \in \mathbb{F}_q^d} \chi\big(tQ(x) - x \cdot m\big)$$

for $m \in \mathbb{F}_q^d, t \in \mathbb{F}_q^*$. We shall need the following theorem (see [2]).

THEOREM 11. *Every quadratic form $Q(x) = \sum_{i,k=1}^{d} a_{ik}x_ix_k$ over $\mathbb{F}_q, q$ odd, can be transformed into a diagonal form $a_1x_1^2 + \cdots + a_dx_d^2$ over $\mathbb{F}_q$ by means of a nonsingular linear substitution of indeterminates. Moreover, if $Q(x)$ is a nondegenerate quadratic form, then $a_i \neq 0$ for all $i = 1, 2, \ldots, d$.*

Using Theorem 11, we may write that for some $m' = (m'_1, \ldots, m'_d) \in \mathbb{F}_q^d$, and $a_i \in \mathbb{F}_q^*$ for all $i = 1, 2, \ldots, d$,

$$W_t(m) = \sum_{x \in \mathbb{F}_q^d} \chi(t\|x\|_a + x \cdot m'),$$

where $m' \in \mathbb{F}_q^d$ is determined by $m \in \mathbb{F}_q^d$ and $\|x\|_a$ is given by

$$\|x\|_a = a_1x_1^2 + \cdots + a_dx_d^2.$$

Since χ is an additive character of $\mathbb{F}_q$, we have

$$\begin{aligned} W_t(m) &= \prod_{k=1}^{d} \sum_{x_k \in \mathbb{F}_q} \chi(ta_kx_k^2 + m'_kx_k) \\ &= \prod_{k=1}^{d} \sum_{x_k \in \mathbb{F}_q} \chi\big(ta_k\big(x_k + (2ta_k)^{-1}m'_k\big)^2 - (4ta_k)^{-1}{m'_k}^2\big) \\ &= \prod_{k=1}^{d} \chi(-(4ta_k)^{-1}{m'_k}^2) \sum_{x_k \in \mathbb{F}_q} \chi(ta_kx_k^2). \end{aligned}$$

Using Theorem 6, we see that

$$\sum_{x_k \in \mathbb{F}_q} \chi(ta_kx_k^2) = \psi^{-1}(ta_k)G(\chi, \psi),$$

where ψ is a multiplicative character of $\mathbb{F}_q^*$ of order two and $G(\chi, \psi) = \sum_{s \in \mathbb{F}_q^*} \chi(s)\psi(s)$. Thus, we obtain that

$$\begin{aligned} W_t(m) &= \psi^{-d}(t)\psi^{-1}(a_1 \cdots a_d)(G(\chi, \psi))^d \prod_{k=1}^{d} \chi(-(4ta_k)^{-1}{m'_k}^2) \\ &= \psi^{-d}(t)\psi^{-1}(a_1 \cdots a_d)(G(\chi, \psi))^d \chi\left(t^{-1} \sum_{k=1}^{d} -(4a_k)^{-1}{m'_k}^2\right). \end{aligned} \tag{2.3}$$

Combining above fact in (2.3) with (2.2), we obtain that

$$D(j, m) = \psi^{-1}(a_1 \cdots a_d)(G(\chi, \psi))^d \sum_{t \in \mathbb{F}_q^*} \chi(-jt + t^{-1}M)\psi^{-d}(t), \tag{2.4}$$

where M is given by

$$M = \sum_{k=1}^{d} -(4a_k)^{-1}{m'_k}^2.$$

Since ψ is a multiplicative character of order two, we see that $\psi^{-d}=1$ for d even, and $\psi^{-d}=\psi$ for d odd. Therefore, in order to get the inequality in (2.1), we can apply Theorems 5 and 7 to (2.4) for d odd. On the other hand, if d is even, we can apply Theorems 5 and 8 to (2.4) because $j\neq 0$. This completes the proof of Lemma 9.

3. Proof of the Tomas–Stein exponent (Theorem 1)

Theorem 1 is a result from Lemma 9 in this paper and Lemma 6.1 in [3]. We first introduce Lemma 6.1 in [3]. Let S be an algebraic variety in $\mathbb{F}_q^d$ with a normalized surface measure $d\sigma$. We introduce the Bochner–Riesz kernel

$$K(m):=\widehat{d\sigma}(m)-\delta_0(m),$$

where $\delta_0(m)=1$ if $m=(0,\dots,0)$ and $\delta_0(m)=0$ otherwise. We need the following theorem. For a nice proof, see Lemma 6.1 in [3].

THEOREM 12. *Let $p,r\geq 2$, and $\mathbb{F}_q^d$ be a d-dimensional vector space over $\mathbb{F}_q$. Suppose that*

$$\|K\|_{L^\infty(\mathbb{F}_q^d,dm)}=\|\widehat{d\sigma}-\delta_0\|_{L^\infty(\mathbb{F}_q^d,dm)}\lesssim q^{-\frac{\tilde{d}}{2}}$$

for some $0<\tilde{d}<d$. Then for any $0<\theta<1$, we have

$$R^*\left(p\to\frac{r}{\theta}\right)\lesssim 1+R^*(p\to r)^\theta q^{-\frac{\tilde{d}(1-\theta)}{4}}.$$

We are now ready to prove Theorem 1. Recall that we are working with a nondegenerate quadratic surface S_j in $\mathbb{F}_q^d$. We now check that

$$\|K\|_{L^\infty(\mathbb{F}_q^d,dm)}\lesssim q^{-\frac{(d-1)}{2}}. \tag{3.1}$$

In fact, if $m=(m_1,m_2,\dots,m_d)\neq(0,\dots,0)$ then we have

$$\begin{aligned}K(m)=\widehat{d\sigma}(m)&=(\#S_j)^{-1}\sum_{x\in S_j}\chi(-x\cdot m)\\&=(\#S_j)^{-1}\sum_{x\in\mathbb{F}_q^d}\chi(-x\cdot m)S_j(x)\\&=(\#S_j)^{-1}q^d\widehat{S_j}(m).\end{aligned}$$

From Corollary 10 and Lemma 9, we have

$$(\#S_j)\approx q^{d-1}\quad\text{and}\quad|\widehat{S_j}(m)|\lesssim q^{-\frac{d+1}{2}}\quad\text{for } m\neq(0,\dots,0).$$

We therefore, obtain that

$$|K(m)|\lesssim q^{-\frac{d-1}{2}}\quad\text{for } m\neq(0,\dots,0).$$

On the other hand, we have

$$K(0,\dots,0)=\widehat{d\sigma}(0,\dots,0)-1=0.$$

Thus, the inequality in (3.1) holds. We now claim that

$$R^*(2 \to 2) \approx q^{\frac{1}{2}}. \tag{3.2}$$

To justify above claim, we shall show that

$$\|\widehat{f\,d\sigma}\|_{L^2(\mathbb{F}_q^d, dm)} \approx q^{\frac{1}{2}} \|f\|_{L^2(S_j, d\sigma)} \tag{3.3}$$

for all functions f on S_j. We first note that

$$\begin{aligned} |\widehat{f\,d\sigma}(m)|^2 &= (\#S_j)^{-2} \sum_{x \in S_j} \chi(-x \cdot m) f(x) \sum_{y \in S_j} \chi(y \cdot m) \overline{f(y)} \\ &= (\#S_j)^{-2} \sum_{x,y \in S_j} \chi\big((y-x) \cdot m\big) f(x) \overline{f(y)}. \end{aligned}$$

We have

$$\begin{aligned} \|\widehat{f\,d\sigma}\|_{L^2(\mathbb{F}_q^d, dm)} &= \bigg(\sum_{m \in \mathbb{F}_q^d} |\widehat{f\,d\sigma}(m)|^2 \bigg)^{\frac{1}{2}} \\ &= (\#S_j)^{-1} \bigg(\sum_{x,y \in S_j} \sum_{m \in \mathbb{F}_q^d} \chi\big((y-x) \cdot m\big) f(x) \overline{f(y)} \bigg)^{\frac{1}{2}} \\ &= (\#S_j)^{-1} q^{\frac{d}{2}} \bigg(\sum_{x \in S_j} |f(x)|^2 \bigg)^{\frac{1}{2}} \\ &= (\#S_j)^{-1} q^{\frac{d}{2}} (\#S_j)^{\frac{1}{2}} \|f\|_{L^2(S_j, d\sigma)} \approx q^{\frac{1}{2}} \|f\|_{L^2(S_j, d\sigma)}. \end{aligned}$$

In the last equality, we used the fact that $\#S_j \approx q^{d-1}$. Thus, our claim in (3.2) is proved. Using Theorem 12 with (3.1) and (3.2), we obtain that for any $0 < \theta < 1$,

$$\begin{aligned} R^*\bigg(2 \to \frac{2}{\theta}\bigg) &\lesssim 1 + R^*(2 \to 2)^{\theta} q^{-\frac{(d-1)(1-\theta)}{4}} \\ &\lesssim 1 + q^{\frac{\theta}{2}} q^{-\frac{(d-1)(1-\theta)}{4}}. \end{aligned}$$

Taking $0 < \theta \leq \frac{d-1}{d+1}$, we have

$$R^*\bigg(2 \to \frac{2}{\theta}\bigg) \lesssim 1.$$

Thus, Theorem 1 is proved with $r = \frac{2}{\theta}$.

4. Proof of the $L^2 \to L^4$ estimate (Theorem 2)

To prove Theorem 2, we make the following reduction.

LEMMA 13. *Let S_j be a nondegenerate quadratic surface in $\mathbb{F}_q^d$ defined as in (1.2). Suppose that for any $x \in (\mathbb{F}_q^d)^* = \mathbb{F}_q^d \setminus (0,\dots,0)$, we have*

$$\sum_{\{(\alpha,\beta)\in S_j\times S_j : \alpha+\beta=x\}} 1 \lesssim q^{d-2}.$$

Then for $d \geq 2$,

$$R^*(2\to 4) \lesssim 1.$$

Proof. We have to show that

$$\|\widehat{f\,d\sigma}\|_{L^4(\mathbb{F}_q^d,dm)} \lesssim \|f\|_{L^2(S_j,d\sigma)}$$

for all functions f on S_j. Using Plancherel, we have

$$\begin{aligned}\|\widehat{f\,d\sigma}\|_{L^4(\mathbb{F}_q^d,dm)} &= \|\widehat{f\,d\sigma}\,\widehat{f\,d\sigma}\|_{L^2(\mathbb{F}_q^d,dm)}^{\frac12}\\ &= \|f\,d\sigma * f\,d\sigma\|_{L^2(\mathbb{F}_q^d,dx)}^{\frac12}\end{aligned}$$

and so it suffices to show that

$$\|f\,d\sigma * f\,d\sigma\|_{L^2(\mathbb{F}_q^d,dx)}^2 \lesssim \|f\|_{L^2(S_j,d\sigma)}^4.$$

It follows that

$$\begin{aligned}&\|f\,d\sigma * f\,d\sigma\|_{L^2(\mathbb{F}_q^d,dx)}^2\\ &\quad= q^{-d}|f\,d\sigma * f\,d\sigma(0,\dots,0)|^2 + \|f\,d\sigma * f\,d\sigma\|_{L^2((\mathbb{F}_q^d)^*,dx)}^2.\end{aligned}$$

Thus, it will suffice to show that

$$q^{-d}|f\,d\sigma * f\,d\sigma(0,\dots,0)|^2 \lesssim \|f\|_{L^2(S_j,d\sigma)}^4 \tag{4.1}$$

and

$$\|f\,d\sigma * f\,d\sigma\|_{L^2((\mathbb{F}_q^d)^*,dx)}^2 \lesssim \|f\|_{L^2(S_j,d\sigma)}^4. \tag{4.2}$$

We first show that the inequality in (4.1) holds. We have

$$\begin{aligned}|f\,d\sigma * f\,d\sigma(0,\dots,0)| &\leq \sum_{m\in\mathbb{F}_q^d} |\widehat{f\,d\sigma}(m)|^2\\ &= (\#S_j)^{-2}q^d \sum_{x\in S_j} |f(x)|^2\\ &= (\#S_j)^{-1}q^d\|f\|_{L^2(S_j,d\sigma)}^2 \approx q\|f\|_{L^2(S_j,d\sigma)}^2.\end{aligned}$$

Thus, the inequality in (4.1) holds because $d \geq 2$. It remains to show that the inequality in (4.2) holds. Without loss of generality, we may assume that f is positive. Using the Cauchy–Schwarz inequality, we see that

$$\begin{aligned}&f\,d\sigma * f\,d\sigma(x)\\ &\quad= (\#S_j)^{-2}q^d \sum_{\{(\alpha,\beta)\in S_j\times S_j : \alpha+\beta=x\}} f(\alpha)f(\beta)\end{aligned} \tag{4.3}$$

$$\leq (\#S_j)^{-2} q^d \Bigg(\sum_{\{(\alpha,\beta)\in S_j\times S_j : \alpha+\beta=x\}} 1 \Bigg)^{\frac{1}{2}}$$
$$\times \Bigg(\sum_{\{(\alpha,\beta)\in S_j\times S_j : \alpha+\beta=x\}} f^2(\alpha) f^2(\beta) \Bigg)^{\frac{1}{2}}$$
$$= (d\sigma * d\sigma)^{\frac{1}{2}}(x) (f^2\, d\sigma * f^2\, d\sigma)^{\frac{1}{2}}(x).$$

From our hypothesis, and the fact that $\#S_j \approx q^{d-1}$, we obtain that for $x \neq (0,\dots,0)$,

$$d\sigma * d\sigma(x) \approx q^{-d+2} \sum_{\{(\alpha,\beta)\in S_j\times S_j : \alpha+\beta=x\}} 1 \lesssim 1. \tag{4.4}$$

From Fubini's theorem, we also have

$$\|f^2\, d\sigma * f^2\, d\sigma\|_{L^1(\mathbb{F}_q^d, dx)} = \|f\|^4_{L^2(S_j, d\sigma)}. \tag{4.5}$$

Using Hölder inequality, and estimates (4.3), (4.4), and (4.5), we obtain that

$$\begin{aligned}
\|f\, d\sigma * f\, d\sigma\|^2_{L^2((\mathbb{F}_q^d)^*, dx)} &= \|(f\, d\sigma * f\, d\sigma)^2\|_{L^1((\mathbb{F}_q^d)^*, dx)} \\
&\leq \|(d\sigma * d\sigma)\cdot (f^2\, d\sigma * f^2\, d\sigma)\|_{L^1((\mathbb{F}_q^d)^*, dx)} \\
&\leq \|d\sigma * d\sigma\|_{L^\infty((\mathbb{F}_q^d)^*, dx)} \|f^2\, d\sigma * f^2\, d\sigma\|_{L^1((\mathbb{F}_q^d)^*, dx)} \\
&\lesssim \|f\|^4_{L^2(S_j, d\sigma)}.
\end{aligned}$$

Thus, the inequality in (4.2) holds and so the proof of Lemma 13 is complete. □

We now prove Theorem 2. By Lemma 13, it is enough to show that for any $x \in (\mathbb{F}_q^d)^*, d \geq 2$,

$$\sum_{\{(\alpha,\beta)\in S_j\times S_j : \alpha+\beta=x\}} 1 \lesssim q^{d-2}, \tag{4.6}$$

where S_j is the nondegenerate quadratic surface in $\mathbb{F}_q^d$. Using Theorem 11, we may assume that the nondegenerate quadratic surface in $\mathbb{F}_q^d$ is given by

$$S_j = \{y \in \mathbb{F}_q^d : a_1 y_1^2 + \cdots + a_d y_d^2 = j \neq 0\}$$

for all $a_k \neq 0, k = 1, 2, \dots, d$. Therefore, the left-hand side of the equation in (4.6) can be estimated by the number of common solutions $\alpha = (\alpha_1, \dots, \alpha_d)$ in $\mathbb{F}_q^d$ of the equations

$$\begin{aligned}
a_1\alpha_1^2 + \cdots + a_d\alpha_d^2 &= j, \\
2a_1x_1\alpha_1 + \cdots + 2a_dx_d\alpha_d &= \sum_{k=1}^d a_k x_k^2
\end{aligned} \tag{4.7}$$

for $x=(x_1,\dots,x_d)\neq(0,\dots,0)$ and $a_k\neq 0$ for all $k=1,2,\dots,d$. Note that $2a_kx_k\neq 0$ for some $k=1,2,\dots,d$ because $x\neq(0,\dots,0)$ and $a_k\neq 0$. Thus, a routine algebraic computation shows that the number of common solutions of equations in (4.7) is less than equal to $2q^{d-2}$. This means that the inequality in (4.6) holds and so we complete the proof of Theorem 2.

5. Incidence theorems and the proofs of Theorem 3 and Theorem 4

The purpose of this section is to develop the incidence theory needed to prove both Theorem 3 and Theorem 4.

THEOREM 14. *Let S_j be a nondegenerate quadratic surface in $\mathbb{F}_q^d$ defined as before. If E is any subset of S_j, then we have*

$$\sum_{\{(x,y)\in E\times E: x-y+z\in S_j\}} 1 \lesssim (\#E)^2q^{-1}+(\#E)q^{\frac{d-1}{2}}$$

for all $z\in\mathbb{F}_q^d$ where the bound is independent of $z\in\mathbb{F}_q^d$.

Proof. Fix $E\subset S_j$. For each $z\in\mathbb{F}_q^d$, consider

$$\begin{aligned}
&\sum_{\{(x,y)\in E\times E: x-y+z\in S_j\}} 1\\
&\quad=\sum_{(x,y)\in\mathbb{F}_q^d\times\mathbb{F}_q^d} E(x)E(y)S_j(x-y+z)\\
&\quad=\sum_{(x,y)\in\mathbb{F}_q^d\times\mathbb{F}_q^d} E(x)E(y)\sum_{m\in\mathbb{F}_q^d}\chi\big(m\cdot(x-y+z)\big)\widehat{S_j}(m)\\
&\quad=q^{2d}\sum_{m\in\mathbb{F}_q^d}|\widehat{E}(m)|^2\chi(m\cdot z)\widehat{S_j}(m)=I+II,
\end{aligned}$$

where

$$I=q^{2d}|\widehat{E}(0,\dots,0)|^2\widehat{S_j}(0,\dots,0)$$

and

$$II=q^{2d}\sum_{m\neq(0,\dots,0)}|\widehat{E}(m)|^2\chi(m\cdot z)\widehat{S_j}(m).$$

Using Lemma 9 and Plancherel, we obtain that

$$I\approx(\#E)^2q^{-1},$$

and

$$\begin{aligned}
|II|&\lesssim q^{2d}q^{-\frac{d+1}{2}}\sum_{m\neq(0,\dots,0)}|\widehat{E}(m)|^2\\
&\leq q^{2d}q^{-\frac{d+1}{2}}q^{-d}\sum_{x\in\mathbb{F}_q^d}|E(x)|^2=q^{\frac{d-1}{2}}(\#E).
\end{aligned}$$

This completes the proof. □

COROLLARY 15. *Let S_j be a nondegenerate quadratic surface in $\mathbb{F}_q^d$ and E be any subset of S_j. Then we have*

$$\sum_{\{(x,y,z,s)\in E^4 : x+z=y+s\}} 1 \lesssim \min\big\{(\#E)^3, (\#E)^3 q^{-1} + (\#E)^2 q^{\frac{d-1}{2}}\big\}.$$

Proof. Since E is a subset of S_j, we have

$$\sum_{\{(x,y,z,s)\in E^4 : x+z=y+s\}} 1 \le \sum_{z\in E} \sum_{\{(x,y)\in E^2 : x-y+z\in S_j\}} 1.$$

Thus, Corollary 15 is the immediate result from Theorem 14 and the obvious fact that

$$\sum_{\{(x,y,z,s)\in E^4 : x+z=y+s\}} 1 \le (\#E)^3. \qquad \square$$

5.1. Proof of Theorem 3. In order to prove Theorem 3, we first expand the left-hand side of the inequality in (1.10). It follows that

$$\begin{aligned}
(5.1)\quad \|\widehat{E\,d\sigma}\|_{L^4(\mathbb{F}_q^d, dm)} &= \bigg(\sum_{m\in\mathbb{F}_q^d} |\widehat{E\,d\sigma}(m)|^4\bigg)^{\frac{1}{4}} \\
&= \bigg(\sum_{m\in\mathbb{F}_q^d} \bigg|\frac{1}{\#S_j}\sum_{x\in S_j} \chi(-x\cdot m)E(x)\bigg|^4\bigg)^{\frac{1}{4}} \\
&= \frac{1}{\#S_j}\bigg(\sum_{x,y,z,s\in E\subset S_j} \sum_{m\in\mathbb{F}_q^d} \chi\big((x-y+z-s)\cdot m\big)\bigg)^{\frac{1}{4}} \\
&= \frac{q^{\frac{d}{4}}}{\#S_j}\bigg(\sum_{\{(x,y,z,s)\in E^4 : x+z=y+s\}} 1\bigg)^{\frac{1}{4}}.
\end{aligned}$$

Since $q^{\frac{d+1}{2}} \lesssim \#E \lesssim q^{d-1}$ from the hypothesis, we use Corollary 15 to obtain that

$$(5.2)\qquad \sum_{\{(x,y,z,s)\in E^4 : x+z=y+s\}} 1 \lesssim (\#E)^3 q^{-1}.$$

Combining (5.1) and (5.2), we have

$$(5.3)\qquad \|\widehat{E\,d\sigma}\|_{L^4(\mathbb{F}_q^d, dm)} \lesssim \frac{q^{\frac{d-1}{4}}(\#E)^{\frac{3}{4}}}{\#S_j}.$$

On the other hand, by expanding the right-hand side of the inequality in (1.10), we see that

$$(5.4)\qquad \|E\|_{L^{\frac{4}{3}}(S_j, d\sigma)} = \bigg(\frac{\#E}{\#S_j}\bigg)^{\frac{3}{4}}.$$

Since $\#S_j \approx q^{d-1}$ by Corollary 10, comparing (5.3) with (5.4) yields the inequality in (1.10) and completes the proof.

5.2. Proof of Theorem 4. In order to prove Theorem 4, we need the following lemma.

LEMMA 16. *Let S_j be a nondegenerate quadratic surface in $\mathbb{F}_q^d$ and E be a subset of S_j. For $p_0 \geq 2$, we have the following estimates:*

$$\|\widehat{E\,d\sigma}\|_{L^4(\mathbb{F}_q^d,dm)} \lesssim q^{\frac{-3d+5}{8}+\frac{d-1}{2p_0}} \|E\|_{L^{p_0}(S_j,d\sigma)} \quad \text{for } 1 \lesssim \#E \lesssim q^{\frac{d-1}{2}}, \tag{5.5}$$

$$\|\widehat{E\,d\sigma}\|_{L^4(\mathbb{F}_q^d,dm)} \lesssim q^{\frac{-3d+9}{8}+\frac{d-3}{2p_0}} \|E\|_{L^{p_0}(S_j,d\sigma)} \quad \text{for } q^{\frac{d-1}{2}} \lesssim \#E \lesssim q^{\frac{d+1}{2}}, \tag{5.6}$$

and

$$\|\widehat{E\,d\sigma}\|_{L^4(\mathbb{F}_q^d,dm)} \lesssim q^{\frac{-3d+9}{8}+\frac{d-3}{2p_0}} \|E\|_{L^{p_0}(S_j,d\sigma)} \quad \text{for } 1 \lesssim \#E \lesssim q^{\frac{d+1}{2}}. \tag{5.7}$$

Proof. We first prove the inequality in (5.5). Since $1 \lesssim \#E \lesssim q^{\frac{d-1}{2}}$, using Corollary 15 we have

$$\sum_{\{(x,y,z,s)\in E^4 : x+z=y+s\}} 1 \lesssim (\#E)^3.$$

Combining this with the fact in (5.1), we obtain that

$$\|\widehat{E\,d\sigma}\|_{L^4(\mathbb{F}_q^d,dm)}) \lesssim \frac{q^{\frac{d}{4}}(\#E)^{\frac{3}{4}}}{\#S_j}. \tag{5.8}$$

As before, we note that

$$\|E\|_{L^{p_0}(S_j,d\sigma)} \approx \left(\frac{\#E}{\#S_j}\right)^{\frac{1}{p_0}}. \tag{5.9}$$

From (5.8) and (5.9), it suffices to show that for every $1 \lesssim \#E \lesssim q^{\frac{d-1}{2}}$,

$$\frac{q^{\frac{d}{4}}(\#E)^{\frac{3}{4}-\frac{1}{p_0}}}{(\#S_j)^{1-\frac{1}{p_0}}} \lesssim q^{\frac{-3d+5}{8}+\frac{d-1}{2p_0}} \tag{5.10}$$

Since $p_0 \geq 2$ and $\#S_j \approx q^{d-1}$, the inequality in (5.10) follows by a direct calculation. Thus, the inequality in (5.5) holds. In order to prove the inequality in (5.6), just note from Corollary 15 that since $q^{\frac{d-1}{2}} \lesssim \#E \lesssim q^{\frac{d+1}{2}}$, we have

$$\sum_{\{(x,y,z,s)\in E^4 : x+z=y+s\}} 1 \lesssim (\#E)^2 q^{\frac{d-1}{2}},$$

and then follow the same argument as in the proof of the inequality (5.5). The inequality in (5.7) follows from the inequalities in (5.5) and (5.6) because

$$q^{\frac{-3d+5}{8}+\frac{d-1}{2p_0}} \lesssim q^{\frac{-3d+9}{8}+\frac{d-3}{2p_0}}, \quad \text{for } p_0 \geq 2.$$

Thus, the proof of Lemma 16 is complete. □

We now return to proof of Theorem 4. From (3.3), recall that we have

$$\|\widehat{E\,d\sigma}\|_{L^2(\mathbb{F}_q^d,dm)} \approx q^{\frac{1}{2}}\|E\|_{L^2(S_j,d\sigma)} \tag{5.11}$$

for all characteristic functions $E(x)$ on S_j. Therefore, Theorem 4 can be obtained by interpolating (5.11) and the inequalities in Lemma 16.

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