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SPECTRAL PROPERTIES OF THE LAYER POTENTIALS ON LIPSCHITZ DOMAINS

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ABSTRACT. We study the invertibility of the operator $\beta I - K^*$ in $H^{-\alpha}(\partial\Omega)$, $0 \le \alpha \le 1$ for $\beta \in \mathbf{C} \setminus (-\frac{1}{2}, \frac{1}{2}]$ where K^* is a adjoint operator of the double layer potential K related to the Laplace equation and Ω is a bounded Lipschitz domain in $\mathbf{R}^n$. Consequently, the spectrum on the real line lies in $(-\frac{1}{2}, \frac{1}{2}]$.

1. Introduction

In this paper, we study the resolvent sets of K^*, the adjoint operator of the double layer potential K related to the Laplace equation on a bounded Lipschitz domain $\Omega \subset \mathbf{R}^n$, $n \ge 2$.

If the boundary of Ω is smooth, then K^* is a compact operator and $\beta I - K^*$ is one-to-one in $L^2(\partial\Omega)$ for all $\beta \in \mathbf{C} \setminus (-\frac{1}{2}, \frac{1}{2}]$ (see [4], [5]). Hence, by Fredholm Alternative, $\beta I - K^*$ is invertible for all $\beta \in \mathbf{C} \setminus (-\frac{1}{2}, \frac{1}{2}]$. On the contrary, if the boundary of Ω is not smooth, the operator K^* may not be compact, and hence we can not apply Fredholm theory. But, when $\beta \in \mathbf{R} \setminus (-\frac{1}{2}, \frac{1}{2}]$, authors in [4] showed that $\beta I - K^*$ is invertible on $L^2(\partial\Omega)$ (see [4]).

Careful consideration on geometric property of domain allows us to obtain certain spectral property of layer potential operator for some limited cases. For example, when Ω is a convex bounded Lipschitz domain, authors in [6] showed that the spectral radius of K^* over $L^2(\partial\Omega)$ is $\frac{1}{2}$ and the spectral radius of K^* over $L^2_0(\partial\Omega)$ is strictly less than $\frac{1}{2}$ (see [6]).

Several authors were interested in the resolvent sets of double layer potentials related to other equations ([1], [2], [3], [7], [8]).

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In this paper, we will improve the result in [6] for more general domain than convex Lipschitz domains. For example, we will consider a certain domain which may not be a convex domain. Also, we will show that resolvent sets of K^* over $H^{-\alpha}(\partial\Omega), 0 \le \alpha \le 1$ are contained in $\{z \in \mathbf{C} : |z| > \frac{1}{2}\}$. In particular, the resolvent set of K^* over $H^{-\frac{1}{2}}(\partial\Omega)$ is contained in $\mathbf{C} \setminus (-\frac{1}{2}, \frac{1}{2}]$.

In Section 2, we state main results and in Sections 3 and 4 we present the proofs of the main theorems.

2. Statement of main results

For a given domain Ω, the letters P, Q denote points on the boundary of the domain. Also, we denote points in $\mathbf{R}^n$ by X.

We introduce the fundamental solution of the Laplace equation

$$\Gamma(X) = \frac{1}{\omega_n(n-2)} \frac{1}{|X|^{n-2}} \quad \text{if } n \ge 3,$$

$$\Gamma(X) = \frac{1}{2\pi} \log|X| \quad \text{if } n = 2,$$

where ω_n is the measure of the unit sphere in $\mathbf{R}^n$.

For $0 < \alpha < 1$, we introduce the Besov space

$$H^\alpha(\partial\Omega) = \left\{ f \in L^2(\partial\Omega) \Big| \int\int_{\partial\Omega\times\partial\Omega} \frac{|f(P)-f(Q)|^2}{|P-Q|^{n-1+2\alpha}} \, dP\, dQ < \infty \right\}$$

with the norm

$$\|f\|_{H^\alpha(\partial\Omega)} := \|f\|_{L^2(\partial\Omega)} + \left(\int\int_{\partial\Omega\times\partial\Omega} \frac{|f(P)-f(Q)|^2}{|P-Q|^{n-1+2\alpha}} \, dP\, dQ \right)^{\frac{1}{2}}.$$

We denote $H^0(\partial\Omega) := L^2(\partial\Omega), H^1(\partial\Omega) := L^2_1(\partial\Omega)$. $H^\alpha(\partial\Omega), 0 < \alpha < 1$ are real interpolation spaces, i.e.,

$$(L^2(\partial\Omega), L^2_1(\partial\Omega))_{\alpha,2} = H^\alpha(\partial\Omega).$$

Let us denote the dual space of $H^\alpha(\partial\Omega)$ by $H^{-\alpha}(\partial\Omega)$.

We define the single layer potential of $f \in L^2(\partial\Omega)$ by

$$u(X) = \mathcal{S}f(X) = \int_{\partial\Omega} \Gamma(X-Q) f(Q)\, dQ, \quad X \in \mathbf{R}^n \setminus \partial\Omega. \tag{2.1}$$

Then we have

$$\Delta u = 0 \quad \text{in } \mathbf{R}^n \setminus \partial\Omega$$

and for $P \in \partial\Omega$, we have

$$Sf(P) = \lim_{X\to P, X\in\Gamma_\pm(P)} \mathcal{S}f(X) = \int_{\partial\Omega} \Gamma(P-Q) f(Q)\, dQ.$$

Let

$$K^* f(P) = p.v \frac{1}{\omega_n} \int_{\partial\Omega} \frac{\langle P-Q, \mathbf{n}(P)\rangle}{|P-Q|^n} f(Q)\, dQ,$$

where $\mathbf{n}(P)$ is the outer normal vector at $P \in \partial\Omega$. Then

$$\frac{\partial u}{\partial \mathbf{n}^{\pm}} = -\frac{1}{2}I \pm K^*,$$

where $\frac{\partial u}{\partial \mathbf{n}^+}$ is outer normal derivative from Ω and $\frac{\partial u}{\partial \mathbf{n}^-}$ is outer normal derivative from $\mathbf{R}^n \setminus \bar{\Omega}$.

Also, we define the double layer potential $\mathcal{K}$. Let $f \in L^2(\partial\Omega)$. Then the double layer potential is defined by

$$\mathcal{K}f(X) = \frac{1}{\omega_n} \int_{\partial\Omega} \frac{\langle Q - X, \mathbf{n}(Q)\rangle}{|X - Q|^n} f(Q)\, dQ, \quad X \in \mathbf{R}^n \setminus \partial\Omega.$$

It is known that for $P \in \partial\Omega$

$$\lim_{X \to P, X \in \Gamma_{\pm}} \mathcal{K}f(X) = \left(\pm\frac{1}{2}I + K\right) f(P),$$

where $Kf(P) = p.v. \frac{1}{\omega_n} \int_{\partial\Omega} \frac{\langle Q-P, \mathbf{n}(Q)\rangle}{|P-Q|^n} f(Q)\, dQ$.

$K : L^2(\partial\Omega) \to L^2(\partial\Omega), \quad H^1(\partial\Omega) \to H^1(\partial\Omega)$ are bounded operators (see [9]). By interpolation theorem, it follows that $K : H^{\alpha}(\partial\Omega) \to H^{\alpha}(\partial\Omega)$, $0 < \alpha < 1$ is a bounded operator, and hence the dual operator K^* of K is also a bounded operator from $H^{-\alpha}(\partial\Omega)$ to $H^{-\alpha}(\partial\Omega)$.

Next, we define single layer potential in $H^{-\frac{1}{2}}(\partial\Omega)$. Given $f \in H^{-\frac{1}{2}}(\partial\Omega)$, we define single layer potential as

$$u(X) = \mathcal{S}f(X) = \langle f, \Gamma(X - \cdot)\rangle, \quad X \in \mathbf{R}^n \setminus \partial\Omega,$$

and

$$Sf(P) = \lim_{X \to P} \mathcal{S}f(X).$$

Then we have $u \in H^1(\Omega), \nabla u \in L^2(\mathbf{R}^n \setminus \bar{\Omega})$ and $S : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is bounded operator. Define $\frac{\partial u}{\partial \mathbf{n}^+}, \frac{\partial u}{\partial \mathbf{n}^-} \in H^{-\frac{1}{2}}(\partial\Omega)$ as

$$\left\langle \frac{\partial u}{\partial \mathbf{n}}, v \right\rangle = \int_{\Omega} \nabla u \cdot \nabla \overline{V^+}, \qquad \left\langle \frac{\partial u}{\partial \mathbf{n}^-}, v \right\rangle = \int_{\mathbf{R}^n \setminus \Omega} \nabla u \cdot \nabla \overline{V^-},$$

where $v \in H^{\frac{1}{2}}(\partial\Omega)$ and $V^+ \in H^1(\Omega), V^- \in H^1(\mathbf{R}^n \setminus \bar{\Omega})$ with $V^+|_{\partial\Omega} = v = V^-|_{\partial\Omega}$ and $\|V\|_{H^1(\Omega)} \le c\|v\|_{H^{1/2}(\partial\Omega)}, \|V\|_{H^1(\mathbf{R}^n \setminus \bar{\Omega})} \le c\|v\|_{H^{1/2}(\partial\Omega)}$. Then

$$\begin{aligned} \left\| \frac{\partial u}{\partial \mathbf{n}^+} \right\|_{H^{-1/2}(\partial\Omega)} &\le c \int_{\Omega} |\nabla u|^2, \\ \left\| \frac{\partial u}{\partial \mathbf{n}^-} \right\|_{H^{-1/2}(\partial\Omega)} &\le c \int_{\mathbf{R}^n \setminus \bar{\Omega}} |\nabla u|^2. \end{aligned} \tag{2.2}$$

Moreover, $\langle \frac{\partial u}{\partial \mathbf{n}^+}, 1\rangle = 0$ and

$$\frac{\partial u}{\partial \mathbf{n}^+} = \left(-\frac{1}{2}I + K^*\right) f, \qquad \frac{\partial u}{\partial \mathbf{n}^-} = -\left(\frac{1}{2}I + K^*\right) f. \tag{2.3}$$

Hence, $-\frac{1}{2}I+K^*$ is a bounded operator from $H^{-\frac{1}{2}}(\partial\Omega)$ to $H_0^{-\frac{1}{2}}(\partial\Omega) := \{f \in H^{-\frac{1}{2}}(\partial\Omega) : \langle f, 1\rangle = 0\}$.

The following proposition is available (see [9]).

PROPOSITION 2.1. *Let Ω is bounded Lipschitz domain in $\mathbf{R}^n, n \geq 2$. Then*

(1) *$\frac{1}{2}I + K$ is invertible in $L^2(\partial\Omega)$,*

(2) *$\frac{1}{2}I + K$ is invertible in $H^1(\partial\Omega)$,*

(3) *S is invertible from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$ for $n \geq 3$,*

(4) *S is invertible from $H^{-1}(\partial\Omega)$ to $L^2(\partial\Omega)$ for $n \geq 3$,*

(5) *when $n = 2$, for any $f_0 \neq 0$ satisfying $(-\frac{1}{2}I + K^*)f_0 = 0$, if $Sf_0 \neq 0$, then S is invertible from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$ and if $Sf_0 = 0$, then the range of S is $H_0^1(\partial\Omega) = \{f \in H^1(\partial\Omega) | \int f = 0\}$.*

REMARK 2.2. By Proposition 2.1 and the interpolation theorem, the operator $\frac{1}{2}I + K$ is invertible from $H^\alpha(\partial\Omega)$ to $H^\alpha(\partial\Omega)$ and S is invertible from $H^{-\alpha}(\partial\Omega)$ to $H^{1-\alpha}(\partial\Omega)$ for $0 \leq \alpha \leq 1, n \geq 3$.

DEFINITION 2.3. We call $\Omega \subset \mathbf{R}^n$ a locally convex bounded Lipschitz domain if there are $r_0 > 0$ and $P_i \in \partial\Omega, 1 \leq i \leq N$, such that $\partial\Omega \subset \bigcup_{i=1}^N B_{r_0}(P_i)$ and for each i there is a Lipschitz function ψ_i on $\mathbf{R}^{n-1}$ which is either convex or concave satisfying

$$\Omega \cap B_{r_0}(P_i) = \{(x, x_n) \in \mathbf{R}^n : x_n > \psi_i(x)\} \cap B_{r_0}(P_i).$$

For example, when $n \geq 2$ the domain $(-2,2)^n \setminus \overline{B_1(0)}$ is a locally convex bounded Lipschitz domain. When $n = 2$, the domains with boundary consisting of finite number of edges are also locally convex ones.

Now, we state our main results.

THEOREM 2.4. *Let Ω be a locally convex bounded Lipschitz domain in $\mathbf{R}^n$. Then for all complex numbers β satisfying $|\beta| > \frac{1}{2}$, $\beta I - K^*$ is invertible in $H^{-\alpha}(\partial\Omega), 0 \leq \alpha \leq 1$.*

THEOREM 2.5. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$. Then for any $\beta \in \mathbf{C} \setminus (-\frac{1}{2}, \frac{1}{2}]$, $\beta I - K^*$ is invertible in $H^{-\frac{1}{2}}(\partial\Omega)$.*

3. Proof of Theorem 2.4

For a given $\beta \in \mathbf{C}$, we denote the operator $\beta I - K^*$ by T_β. We prepare the following lemmas for the proof of Theorem 2.4.

LEMMA 3.1. *Let Ω be a bounded Lipschitz domain, then T_β is one-to-one in $H^{-\frac{1}{2}}(\partial\Omega)$ for $\beta \in \mathbf{C} \setminus (-\frac{1}{2}, \frac{1}{2}]$.*

Proof. Suppose that T_β is not one-to-one for some $\beta \in \mathbf{C} \setminus (-\frac{1}{2}, \frac{1}{2}]$. Then there is $f \in H^{-\frac{1}{2}}(\partial\Omega)$, such that $T_\beta f = 0$ and f is not identically zero. Since $T_\beta f = (\beta - \frac{1}{2})f + (\frac{1}{2}I - K^*)f$ and $(\frac{1}{2}I - K^*)f \in H_0^{-\frac{1}{2}}(\partial\Omega)$, we have $\langle f, 1\rangle = 0$.

Set $u = \mathcal{S}f$. Then u satisfies $|u(X)| = O(|X|^{1-n})$ and $|\nabla u(X)| = O(|X|^{-n})$ at infinity for $n \geq 2$. Since f is not identically zero, the following numbers A and B cannot be zero:

$$A = \int_\Omega |\nabla u|^2 \, dX \quad \text{and} \quad B = \int_{\mathbf{R}^n \setminus \Omega} |\nabla u|^2 \, dX.$$

By Green's formula, we have

$$A = \left\langle \left(-\frac{1}{2} I + K^* \right) f, Sf \right\rangle \quad \text{and} \quad B = \left\langle \left(\frac{1}{2} I + K^* \right) f, Sf \right\rangle.$$

Since $T_\beta f = 0$, we have that $\beta = \frac{1}{2} \frac{B-A}{B+A}$. Note that β is real and $|\beta| \leq \frac{1}{2}$ since $A, B \geq 0$.

Now, we have a contradiction for $\beta \in \mathbf{C} \setminus [-\frac{1}{2}, \frac{1}{2}]$. If $\beta = -\frac{1}{2}$, we have $B = 0$. By the decay of u at infinity, we have $u \equiv 0$ in $\mathbf{R}^n \setminus \Omega$. Since u is continuous up to the boundary of Ω, $u \equiv 0$ in $\mathbf{R}^n$ by maximum principle. Hence, $0 = \frac{\partial u}{\partial \mathbf{n}} + \frac{\partial u}{\partial \mathbf{n}^-} = -f$ by (2.3). We also have a contradiction for $\beta = -\frac{1}{2}$. Therefore, T_β is one-to-one in $H^{-\frac{1}{2}}(\partial\Omega)$ for $\beta \in \mathbf{C} \setminus (-\frac{1}{2}, \frac{1}{2}]$. □

LEMMA 3.2. *Let $n \geq 2$ and $D = \{X = (x, x_n) \in \mathbf{R}^n | x_n > \phi(x)\}$ be a convex Lipschitz graph domain. Then the spectral radius $\rho(K^*)$ of K^* over $L^2(\partial D)$ is strictly less than $\frac{1}{2}$.*

Proof. Let $f \in L^2(\partial D)$ be a Lipschitz function, compact support, and $u(X) = \mathcal{S}f(X)$ for $X \in \mathbf{R}^n \setminus \partial D$. By Rellich-identity, we have

$$\int_{\partial D} \langle e_n, \mathbf{n} \rangle |\nabla u|^2 = 2 \int_{\partial D} \langle e_n, \nabla u \rangle \frac{\partial u}{\partial \mathbf{n}},$$

where $e_n = (0, \ldots, -1)$. Since $\langle e_n, \mathbf{n} \rangle \geq c > 0$ on ∂D and $\nabla u = \frac{\partial u}{\partial \mathbf{n}} \mathbf{n} + \sum_{i=1}^{i=n-1} \frac{\partial u}{\partial T_i} T_i$ where T_i are unit tangential vectors on ∂D, we have

$$c_1 \int_{\partial D} \left| \frac{\partial u}{\partial \mathbf{n}} \right|^2 \leq \int_{\partial D} \sum_{i=1}^{i=n-1} \left| \frac{\partial u}{\partial T_i} \right|^2 \leq c_2 \int_{\partial D} \left| \frac{\partial u}{\partial \mathbf{n}} \right|^2$$

with the positive constants c_1, c_2 depending only on the Lipschitz constant of the domain. Hence, we get

$$\begin{aligned} (3.1) \qquad c_1 \int_{\partial D} \left| -\frac{1}{2} f + K^* f \right|^2 &\leq \int_{\partial D} \sum_{i=1}^{i=n-1} \left| \frac{\partial u}{\partial T_i} \right|^2 \\ &\leq c_2 \int_{\partial D} \left| -\frac{1}{2} f + K^* f \right|^2. \end{aligned}$$

For the domain $\mathbf{R}^n \setminus \bar{D}$, we have similar inequalities:

$$(3.2) \quad c_1 \int_{\partial D} \left| \frac{1}{2} f + K^* f \right|^2 \leq \int_{\partial D} \sum_{i=1}^{i=n-1} \left| \frac{\partial u}{\partial T_i} \right|^2 \leq c_2 \int_{\partial D} \left| \frac{1}{2} f + K^* f \right|^2.$$

Combining (3.1) and (3.2), we obtain

$$\|f\|_{L^2(\partial D)} \le \left\| -\frac{1}{2}f - K^*f \right\|_{L^2(\partial D)} + \left\| \frac{1}{2}f - K^*f \right\|_{L^2(\partial D)} \le c\left\| \pm\frac{1}{2}f - K^*f \right\|_{L^2(\partial D)}$$

which holds not only for Lipschitz functions with compact support, but also for functions in $L^2(\partial D)$ by approximation.

For real β satisfying $|\beta| > \frac{1}{2}$, we already have (see [6])

$$\|f\|_{L^2(\partial D)} \le c_\beta \|\beta f - K^* f\|_{L^2(\partial D)}.$$

In other words, T_β is one to one and has closed range for any real $|\beta| \ge \frac{1}{2}$.

Let's assume that the spectral radius $\rho(K^*)$ of K^* is $\beta_0 \ge \frac{1}{2}$. Then we have

$$\|f\|_{L^2(\partial D)} \le c_{\beta_0} \|T_{\beta_0} f\|_{L^2(\partial D)} \tag{3.3}$$

for all $f \in L^2(\partial D)$. Since K^* is a positive preserving operator in $L^2(\partial D)$ by the convexity of the domain, β_0 belongs to the spectrum of T_{β_0}. This implies that T_{β_0} cannot be onto. Meanwhile, T_β is invertible for $|\beta| > \beta_0$. Hence, we can take a sequence $\{\beta_i\}$ such that $\beta_i \to \beta_0$ and T_{β_i} are invertible. Let $g \in L^2(\partial D)$. Then there is $f_i \in L^2(\partial D)$ such that $T_{\beta_i} f_i = g$ for all i. If $\{f_i\}$ is bounded in $L^2(\partial D)$, then we are complete since there are a subsequence (we say $\{f_i\}$) and $f \in L^2(\partial D)$ such that f_i weakly converges to f and we can observe

$$\left| \int (T_{\beta_0} f - g)\bar{h} \right| = \left| \int (T_{\beta_0} f - T_{\beta_0} f_i + T_{\beta_0} f_i - T_{\beta_i} f_i)\bar{h} \right| \le \left| \int (f - f_i)\overline{T^*_{\beta_0} h} \right| + |\beta_0 - \beta_i| \|f_i\|_{L^2(\partial D)} \|h\|_{L^2(\partial D)}$$

for any $h \in L^2(\partial D)$. Now, suppose that $\{f_i\}$ is unbounded in $L^2(\partial D)$. Setting $F_i = \frac{f_i}{\|f_i\|_{L^2(\partial D)}}$, we have $T_{\beta_i} F_i \to 0$ in $L^2(\partial D)$ and $\|F_i\|_{L^2(\partial D)} = 1$. By weak compactness of Hilbert spaces, there is a subsequence (we again say $\{F_i\}$) such that F_i weakly converges to F for some $F \in L^2(\partial D)$. Then by (3.3) we get

$$1 = \|F_i\|_{L^2(\partial D)} \le c_{\beta_0} \|T_{\beta_0} F_i\|_{L^2(\partial D)} \le c_{\beta_0} \big(|\beta_0 - \beta_i| + \|T_{\beta_i} F_i\|_{L^2(\partial D)} \big) \to 0$$

and we have a contradiction. Hence, $\beta_0 = \rho(K^*) < \frac{1}{2}$. □

We can derive the following lemma from Lemma 2.3 in [6].

LEMMA 3.3 (Localization lemma). *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n, n \ge 2$. Fix a complex number β and assume that there are a finite number of points $P_i \in \partial\Omega, 1 \le i \le N$, and a positive number $r_0 > 0$ with*

$\partial\Omega \subset \bigcup_{i=1}^{i=N} B_{r_0}(P_i)$ and positive constants $C_i, 1 \le i \le N$, such that for each boundary ball $\Delta_{i,r} := \partial\Omega \cap B_r(P_i), 0 < r \le r_0$, we have

$$\|f\|_{L^2(\Delta_{i,r})} \le C\|(\beta I - K^*)(f\chi_{\Delta_{i,r}})\|_{L^2(\Delta_{i,r})}$$

for all $f \in L^2(\Delta_{i,r})$ where χ_E denotes the characteristic function of the set E. If β is not an eigenvalue of K^* on $L^2(\partial\Omega)$, then $\beta I - K^*$ has closed range on $L^2(\partial\Omega)$.

Fix $|\beta| > \frac{1}{2}$. We prove Theorem 2.4 with $\alpha = 0$ first.

We take r_0, P_i, ψ_i from Definition 2.3, and for each i, we define $\Omega_i := \{(x, x_n) \in \mathbf{R}^n | x_n > \psi_i(x)\}$. Let K_i^* be the double layer potential on $\partial\Omega_i$. Since Ω_i is a convex domain or $\mathbf{R}^n \setminus \bar{\Omega}_i$ is a convex domain, the spectral radius $\rho(K_i^*) = \rho(-K_i^*)$ over $L^2(\partial\Omega_i)$ is strictly less than $\frac{1}{2}$ by Lemma 3.2. Then the spectral radius $\rho(K_{i,r}^*)$ of $K_{i,r}^* := \chi_{\Delta_{i,r}} K^* \chi_{\Delta_{i,r}}$ over $L^2(\Delta_{i,r})$ is strictly less than $\frac{1}{2}$ since

$$\lim_{k\to\infty} \|(K_{i,r}^*)^k\|^{1/k} \le \lim_{k\to\infty} \|(K_i^*)^k\|^{1/k} < \frac{1}{2}$$

(see [6]). Hence, $\beta I - K_{i,r}^*$ is invertible in $L^2(\Delta_{i,r})$ and we have

$$\|f\|_{L^2(\Delta_{i,r})} \le \|(\beta I - K_{i,r}^*)^{-1}\|\|(\beta I - K_{i,r}^*)f\|_{L^2(\Delta_{i,r})}$$

with an observation

$$\begin{aligned}\|(\beta I - K_{i,r}^*)^{-1}\| &\le \frac{1}{|\beta|}\sum_{k=0}^{\infty} \frac{1}{|\beta|^k}\|(K_{i,r}^*)^k\| \\ &\le \frac{1}{|\beta|}\sum_{k=0}^{\infty} \frac{1}{|\beta|^k}\|(K_i^*)^k\| \\ &\le \frac{1}{|\beta|}\sum_{k=0}^{N} \frac{1}{|\beta|^k}\|(K_i^*)^k\| + \frac{1}{|\beta|}\sum_{k=N+1}^{\infty} \frac{1}{|\beta|^k}\left(\frac{1}{2}\right)^k =: C_i\end{aligned}$$

for some N. Note that C_i only depends on K_i^*. Since β is not an eigenvalue by Lemma 3.1, we can use Lemma 3.3 and $\beta I - K^*$ as closed range.

Now, we will show that $\beta I - K^*$ is onto for $|\beta| > \frac{1}{2}$. Suppose that $\beta I - K^*$ is not onto for some $|\beta| > \frac{1}{2}$. Since the resolvent set is open in $\mathbf{C}$, we assume that β is in boundary of the resolvent set. Hence, we can take a sequence $\{\beta_i\}, |\beta_i| > \frac{1}{2}$ such that $\beta_i \to \beta$ and $\beta_i I - K^*$ is invertible in $L^2(\partial\Omega)$. By closed graph theorem, there is a positive constant C such that

$$\|f\|_{L^2(\partial\Omega)} \le C\|(\beta I - K^*)f\|_{L^2(\partial\Omega)} \tag{3.4}$$

for all $f \in L^2(\partial\Omega)$. The rest follows as in the proof of Lemma 3.2 and the invertibility follows.

Next, we consider the case $0 < \alpha \le 1$. It is known that

$$KS = SK^* \tag{3.5}$$

in $H^1(\partial\Omega)$ (see [9]). When $n \geq 3$, $S : L^2(\partial\Omega) \to H^1(\partial\Omega)$ is invertible, and hence we can have $-K = -SK^*S^{-1}$. Adding βI on both sides, we have

$$\beta I - K = S(\beta I - K^*)S^{-1} \tag{3.6}$$

in $H^1(\partial\Omega)$. Since $\beta I - K^*$ is invertible in $L^2(\partial\Omega)$, $\beta I - K$ is invertible in $H^1(\partial\Omega)$, and hence, by duality, $\beta I - K^*$ is invertible in $H^{-1}(\partial\Omega)$ for $|\beta| > \frac{1}{2}$. Using the real interpolation theorem, we have that $\beta I - K^*$ is invertible in $H^{-\alpha}(\partial\Omega), 0 \leq \alpha \leq 1$ for $|\beta| > \frac{1}{2}$.

Now, let $n = 2$. By the above argument and duality, it suffices to show that $\beta I - K$ is invertible in $H^1(\partial\Omega)$. Since $\beta I - K$ is invertible in $L^2(\partial\Omega)$, $\beta I - K$ is one-to-one in $H^1(\partial\Omega)$. So, we only need to show that $\beta I - K$ is onto. We use Proposition 2.1. If $Sf_0 \neq 0$, then $S : L^2(\partial\Omega) \to H^1(\partial\Omega)$ is invertible. Then $\beta I - K$ is invertible in $H^1(\partial\Omega)$ as in the case of $n \geq 3$. Let's assume $Sf_0 = 0$ and choose $f \in H^1_0(\partial\Omega)$. Then again by Proposition 2.1, there is a function $\phi \in L^2(\partial\Omega)$ such that $S\phi = f$. Then we can get $(\beta I - K)S(\beta I - K^*)^{-1}\phi = f$, using (3.5) and invertibility of $\beta I - K^*$ in $L^2(\partial\Omega)$. Hence, $H^1_0(\partial\Omega)$ is a subspace of the range of $\beta I - K$. On the other hand, we observe $(\beta I - K)1 = (\beta - \frac{1}{2})1$ which implies that constants are also contained in the range of $\beta I - K$. By considering decomposition of functions in $H^1(\partial\Omega)$, we conclude that $\beta I - K$ is onto in $H^1(\partial\Omega)$. Theorem 2.4 is proved.

REMARK 3.4. The proof of Theorem 2.4 says more than the statement of the theorem. In fact, the resolvent set $\rho(K^*)$ of K^* over $H^{-\alpha}(\partial\Omega)$ is contained in $\mathbf{C} \setminus (B_{\frac{1}{2}-\epsilon}(0) \cup [-\frac{1}{2}, \frac{1}{2}))$ for some $\epsilon > 0$.

4. Proof of Theorem 2.5

We will use the following simple lemma.

LEMMA 4.1. *Let H_1, H_2 be Hilbert spaces and $H_1 = H_{11} \oplus H_{12}$ where $\dim H_{12} = N$ is finite. Let $T : H_1 \to H_2$ be a bounded operator and one-to-one. If $T(H_{11})$ is a closed subspace of H_2, then T has closed range.*

Proof. Assume that Tg_k converges to $f \in H_2$ for some $\{g_k\} \subset H_1$. If $\{g_k\}$ is bounded sequence in H_1, then it is trivial. Suppose that $\{g_k\}$ is unbounded in H_1. We let $G_k = \frac{g_k}{\|g_k\|_{H_1}}$. Then TG_k converges to zero in H_2 and $\|G_k\|_{H_1} = 1$. Let $\{e_i\}_{1\leq i\leq N}$ be an orthonormal basis of H_{12}. We decompose G_k to $G_k = G_{k1} + \sum_{i=1}^N a_{ki}e_i$ where $G_{k1} \in H_{11}$ and $a_{ki} \in \mathbf{C}$. Since $\{G_k\}$ is bounded, by weakly compactness of Hilbert space there is subsequence (we say $\{G_k\}$) such that G_k weakly converges to zero since T is one-to-one. Since H_{11} and H_{12} are orthonormal, G_{k1}, G_{k2} also weakly converge to zero. Hence, $\{a_{ki}\}$ converge to zero for $1 \leq i \leq N$. Hence, $\|G_{k1}\|_{H_1} \to 1$ and TG_{k1} converges to zero. By the injectivity of T and closedness of $T(H_{11})$, we have G_{k1} converges to zero. It contradicts for $\|G_{k1}\|_{H_1}$ converges to 1. Hence, T has closed range. □

Take $\beta \in \mathbf{C} \setminus (-\frac{1}{2}, \frac{1}{2}]$. By Lemma 3.1 and (3.6), T_β is one-to-one in $H^{-\frac{1}{2}}(\partial\Omega)$. We will show that T_β has closed range in $H^{-\frac{1}{2}}(\partial\Omega)$. By the help of Lemma 4.1 it is enough to show that $T_\beta(H_0^{-\frac{1}{2}}(\partial\Omega))$ is closed in $H^{-\frac{1}{2}}(\partial\Omega)$. Assume $T_\beta g_k$ converges to $f \in H^{-\frac{1}{2}}(\partial\Omega)$ for some sequence $\{g_k\} \subset H_0^{-\frac{1}{2}}(\partial\Omega)$. If $\{g_k\}$ is bounded, then we are done as in Lemma 3.2. Suppose that $\{g_k\}$ is unbounded in $H^{-\frac{1}{2}}(\partial\Omega)$. We let $G_k = \frac{g_k}{\|g_k\|_{H^{-1/2}(\partial\Omega)}}$. Then $\|G_k\|_{H^{-1/2}(\partial\Omega)} = 1$ for all k and $T_\beta G_k$ converges to zero in $H^{-\frac{1}{2}}(\partial\Omega)$. Set $u_k = \mathcal{S}G_k$. Since $\{G_k\} \subset H_0^{-\frac{1}{2}}(\partial\Omega)$, $u_k \in H^1(\Omega)$ and $\nabla u_k \in L^2(\mathbf{R}^n \setminus \bar{\Omega})$ (in particular, when $n=2$). Let

$$A_k = \int_\Omega |\nabla u_k|^2 \, dX \quad \text{and} \quad B_k = \int_{\mathbf{R}^n \setminus \Omega} |\nabla u_k|^2 \, dX.$$

By Green's formula, we have

$$A_k = \langle T_\beta G_k, SG_k \rangle + \left\langle \left(\frac{1}{2} + \beta\right) G_k, SG_k \right\rangle,$$

$$B_k = \langle T_\beta G_k, SG_k \rangle - \left\langle \left(\frac{1}{2} - \beta\right) G_k, SG_k \right\rangle.$$

Hence, we have $\beta = \frac{1}{2} \frac{B_k - A_k - 2\epsilon_k}{A_k + B_k}$ for all k with $\epsilon_k = \langle T_\beta G_k, SG_k \rangle$. Suppose that $A_k + B_k$ goes to zero as $k \to \infty$. Then by (2.2), we have

$$\left\| \frac{\partial u_k}{\partial \mathbf{n}} \right\|_{H^{-1/2}(\partial\Omega)} \le cA_k, \qquad \left\| \frac{\partial u_k}{\partial \mathbf{n}^-} \right\|_{H^{-1/2}(\partial\Omega)} \le cB_k$$

and $\frac{\partial u_k}{\partial \mathbf{n}^+} + \frac{\partial u_k}{\partial \mathbf{n}^-} = -G_k$ goes to zero in $H^{-\frac{1}{2}}(\partial\Omega)$. But, it contradicts $\|G_k\|_{H^{-1/2}(\partial\Omega)} = 1$. Hence, $A_k + B_k$ has a lower bound which is bigger than zero. Since, ϵ_k go to zero, β has to be real and $|\beta| \le \frac{1}{2}$. We have a contradiction. Hence, T_β has closed range in $H^{-\frac{1}{2}}(\partial\Omega)$.

The surjectivity of T_β in $H^{-\frac{1}{2}}(\partial\Omega)$ follows as in the proof of Theorem 2.4 and we finish the proof.

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