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FINITE DIMENSIONAL POINT DERIVATIONS FOR GRAPH ALGEBRAS

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ABSTRACT. This paper focuses on finite dimensional point derivations for the non-self-adjoint operator algebras corresponding to directed graphs. We begin by analyzing the derivations corresponding to full matrix representations of the tensor algebra of a directed graph. We determine when such a derivation is inner, and describe situations that give rise to noninner derivations. We also analyze the situation when the derivation corresponds to a multiplicative linear functional.

1. Introduction

The non-self-adjoint operator algebras associated to directed graphs have undergone significant scrutiny of late. Specifically, the algebra $\mathcal{T}^+(Q)$, the norm closed algebra generated by the left regular representation of a directed graph Q, and $\mathcal{L}_Q$, the WOT-closure of $\mathcal{T}^+(Q)$, are studied abstractly as special cases of tensor algebras over C^*-correspondences in [17]. The case of tensor algebras corresponding to directed graphs were first studied in [18] in the case of the graph with a single vertex and n-edges. The weakly closed version was studied around the same time in [6], [7]. General directed graphs were taken up in [15]. Since that time the study has expanded significantly to various facets of these non-self-adjoint algebras; see, for example, [5], [10], [13], [14], [16].

Important in the analysis of these algebras is a recent paper [5] which analyzed finite dimensional representations and faithful irreducible representations for strongly transitive directed graphs. Building on their work, and combining with a recent paper [8], we undertake a description of noncommutative point derivations of directed graph algebras. This work was motivated

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by attempts to describe the automorphism group of the non-self-adjoint operator algebras associated to directed graphs, further described in [9]. Often times the group of derivations is more tractable than the automorphism group. However, we do not have a complete description for either the group of derivations or the group of automorphisms. We would suggest that one approach to either of these groups is to look at how derivations or automorphisms factor through subalgebras. For derivations, this is the approach of this paper. For automorphisms, we refer the reader to [9].

When one views the classical situation of uniform algebras a strong connection is found between the point derivations and analytic structure [3]. In particular, for the disk algebra, there exist point derivations at a character if and only if the character corresponds to a point on the interior of the unit disk; see Section 9 of [11]. In the present paper, we find similar results when one looks at point derivations at characters.

On the other hand, when dealing with noncommutative algebras, it is clear that the characters are not enough to say much about the algebra. We have thus taken a look at point derivations at certain irreducible finite dimensional representations of directed graph operator algebras. Here, too, we find a sense of analytic structure, but we have not developed that theory to its full extent.

This work is developed as further strengthening of the connection between the disk algebra and directed graph operator algebras which was begun in [8] in the case of a directed cycle graph. There we developed the theory in direct analogy with the standard results for uniform algebras. Here, however, we take a different approach, since the finite dimensional representations are more diverse. More importantly, the general situation of $\mathcal{T}^+(Q)$ and $\mathcal{L}_Q$-valued derivations of $\mathcal{T}^+(Q)$ have not so far yielded to the approach of [8] where the finite dimensional representations are used to make statements about general derivations.

In the first part of the paper, we develop a description of a rich class of finite dimensional representations, generalizing a specific case from [5]. We then factor our representations through the algebras given by the directed cycle graphs. It is then a simple application of the results of [8] to describe the noncommutative point derivations into M_n for general directed graph algebras. This suggests a notion of noncommutative analyticity coming from the finite dimensional representations.

We take a similar approach to studying the point derivations at characters for a general directed graph operator algebra. Once again, we factor such a representation through the a well understood example, the noncommutative analytic Toeplitz algebras of [6]. Thus the discussion of point derivations and commutative analytic structure reduces to the discussion of point derivations on these well understood graph algebras.

We close the paper with a result concerning the range of a derivation $D : A_n \to A_n$. We show that such a derivation must have range contained in the commutator ideal of A_n. Of course, an inner derivation will satisfy this property. However, our result says nothing to suggest that every derivation on A_n is inner. In fact, we have little evidence at this point that such a result is even true.

2. Notation and background

We begin with a review of background material and we fix some notation. To a directed graph Q, there exists two non-self-adjoint operator algebras which we will study below. Both arise from the left regular representation of the graph acting on ℓ^2 of the finite path space. The first algebra, $\mathcal{T}^+(Q)$ will be the norm closure of this representation. The second $\mathcal{L}_Q$ will be the WOT closure of this representation.

For a directed graph Q, we denote the edge set of Q by $E(Q)$ and the vertex set of Q by $V(Q)$. To each edge there are maps $r: E(Q) \to V(Q)$ and $s: E(Q) \to V(Q)$ which give the range and source of an edge, respectively. We will write $\mathcal{C}_n$ for the cycle graph given by n distinct edges $\{e_i\}$ and n vertices $\{v_i\}$, with $s(e_i) = r(e_{i+1})$ for $1 \le i \le n-1$ and $s(e_n) = r(e_1)$. Recall that this algebra can be written as a matrix function algebra of the form

$$\begin{bmatrix} f_{1,1}(z^n) & zf_{1,2}(z^n) & z^2 f_{1,3}(z^n) & \cdots & z^{n-1} f_{1,n}(z^n) \\ z^{n-1} f_{2,1}(z^n) & f_{2,2}(z^n) & zf_{2,3}(z^n) & \cdots & z^{n-2} f_{2,n}(z^n) \\ z^{n-2} f_{3,1}(z^n) & z^{n-1} f_{3,2}(z^n) & f_{3,3}(z^n) & \cdots & z^{n-3} f_{3,n}(z^n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ zf_{n,1}(z^n) & z^2 f_{n,2}(z^n) & z^3 f_{n,3}(z^n) & \cdots & f_{n,n}(z^n) \end{bmatrix},$$

where $f_{i,j} \in A(\mathbb{D})$ for all $1 \le i, j \le n$.

We will denote by B_n the graph with 1 vertex and n edges. For shorthand, the algebras $\mathcal{T}^+(B_n)$ and $\mathcal{L}_{B_n}$ will be denoted by A_n and $\mathcal{L}_n$, respectively. In this case, we will assign an ordering to the edges and denote the isometry associated to the ith edge by L_i.

We now establish some standard definitions and notation. A path in Q will be a finite sequence $e_1 e_2 \cdots e_n$ with $e_i \in E(Q)$ and $r(e_i) = s(e_{i-1})$ for $2 \le i \le n$. Recall that a directed graph is *transitive* if for every pair of vertices, v and w there is a directed path beginning at v and ending at w. We say that a path w in a directed graph is *primitive* if $w \ne v^n$ for any paths v. We say that a path $w = e_1 e_2 \cdots e_n$ is a *cycle* if $r(e_1) = s(e_n)$.

We will write M_n to mean the $n \times n$ matrices with entries from $\mathbb{C}$. We denote by $e_{i,j}$ the elementary matrix in M_n with 1 in the i-j position and 0 everywhere else.

We now state a standard result concerning representations of graph operator algebras which will be useful in what follows. This result follows from work in [17] or [12] and is given explicitly in the case of countable directed graphs in Section 3 of [5].

Corollary 2.1. *Let Q be a directed graph and A an operator algebra. Let $\pi : Q \to A$ be a map with $\pi(v) = P_v$ a projection for all $v \in V(Q)$, $\pi(e) = L_e$ is a nonzero contraction for all edges $e \in E(Q)$, and:*

(1) *P_v is orthogonal to P_w for all vertices $v, w \in V(Q)$.*
(2) *$P_{r(e)} L_e P_{s(e)} = L_e$ for all edges $e \in E(Q)$.*
(3) *$[L_e]_{e \in E(Q)}$ is a row contraction in A.*

Then π extends to a completely contractive representation of $\mathcal{T}^+(Q)$.

3. Irreducible representations into M_n

Let Q be a directed graph and let $w = e_1 e_2 \cdots e_n$ be a finite path in Q. For $\lambda \in \overline{\mathbb{D}}$, $\mu \in \mathbb{T}$, $v \in V(Q)$, and $e \in E(Q)$ define

$$\pi_{w,\lambda,\mu}(P_v) = \sum_{s(e_j)=v} e_{j,j} \quad \text{and} \quad \pi_{w,\lambda,\mu}(L_e) = \sum_{e_j = e} \lambda e_{j-1,j},$$

where for the sake of the notation, we denote by $e_{0,1}$ the matrix $\mu e_{n,1}$. This map then extends to a representation $\pi_{w,\lambda,\mu} : \mathbb{F}^+(Q) \to M_n$.

Corollary 3.1. *Let w be a finite path in Q, $\lambda \in \overline{\mathbb{D}}$, and $\mu \in \mathbb{T}$, then the representation $\pi_{w,\lambda,\mu} : \mathbb{F}^+(Q) \to M_n$ extends to a completely contractive representation of $\mathcal{T}^+(Q)$ into M_n. Moreover, if w is a primitive cycle, and $\lambda \neq 0$, then the extension is onto.*

Proof. The extension to a completely contractive representation of the algebra $\mathcal{T}^+(Q)$ follows from Proposition 2.1. We discuss the other conclusion now. The details of the argument are in the proof of [5, Lemma 4.3]. We only sketch the proof here.

Let $w = e_1 e_2 \cdots e_n$ and notice that

$$\pi_{w,\lambda,\mu}(L_{e_j e_{j-1} e_{j-2} \cdots e_1 e_n e_{n-1} \cdots e_j}) = \lambda^{-k-1} \mu e_{j,j+1}.$$

Since $\lambda \neq 0$, letting j vary yields a generating set for M_n in the range of $\mathcal{T}^+(Q)$, and hence the extension is onto. $\square$

Remark 3.2. If $|\lambda| < 1$, then the map is w^*-continuous by Corollary 3.2 in [5] so that the representation extends to a w^*-continuous completely contractive representation of $\mathcal{L}_Q$.

Notice that in the special case of $\lambda = \frac{1}{2}$ this representation is the representation $\varphi_{w,\mu}$ given in Section 4 of [5], with respect to the usual orthonormal basis of $\mathbb{C}^n$ with reverse ordering. It follows by [5, Theorem 4.4] that in the case of transitive graphs, by letting w, φ, and μ vary, we get a family of irreducible representations which separate the points of $\mathcal{T}^+(Q)$. Here, however, the finite dimensional representations are richer and will allow for a more detailed discussion of the noncommutative point derivations.

DEFINITION 3.3. Let A, B, and C be operator algebras. Let $\pi : A \to B$ be a completely contractive representation of the operator algebra A. We say that π *factors through* C if there exist completely contractive representations, $\iota : A \to C$ and $\tilde{\pi} : C \to B$ such that $\pi(a) = \tilde{\pi} \circ \iota(a)$ for all $a \in A$.

We will show that the representations $\pi_{w,\lambda,\mu} : \mathcal{T}^+(Q) \to M_n$ factor through $\mathcal{T}^+(\mathcal{C}_n)$. We begin by constructing the map $\iota : \mathcal{T}^+(Q) \to \mathcal{T}^+(\mathcal{C}_n)$. For notation sake, with $1 \leq i < n$ let $Z_i \in \mathcal{T}^+(\mathcal{C}_n)$ be the matrix with z in the i-$(i+1)$ position and zeroes everywhere else. Denote by Z_n the matrix in $\mathcal{T}^+(\mathcal{C}_n)$ with a z in the n-1 position and zeroes everywhere else.

For a finite path w in Q given by $w = e_1 e_2 \cdots e_n$ define the representation $\iota_w : \mathbb{F}^+(Q) \to \mathcal{T}^+(\mathcal{C}_n)$ by first setting, for $v \in V(Q)$,

$$\iota_w(P_v) = \sum_{s(e_j)=v} e_{j,j} \in \mathcal{T}^+(\mathcal{C}_n).$$

Next, for $e \in E(Q)$, define

$$\iota_w(L_e) = \sum_{e_j = e} Z_j.$$

The map ι_w will then be the natural extension to $\mathbb{F}^+(Q)$. The next proposition follows immediately from Proposition 2.1.

COROLLARY 3.4. *Given a finite path w, the map ι_w extends to a completely contractive representation $\iota_w : \mathcal{T}^+(Q) \to \mathcal{T}^+(\mathcal{C}_n)$.*

REMARK 3.5. This map is also w^*-continuous and sends $\mathcal{L}_Q$ into $\mathcal{L}_{\mathcal{C}_n}$. This follows from Corollary 3.2 of [5], by noticing that the left regular representation of $\mathcal{L}_{\mathcal{C}_n}$ is pure.

An easy consequence of the definition is the following lemma.

LEMMA 3.6. *For a primitive cycle w, the map $\iota_w : \mathcal{T}^+(Q) \to \mathcal{T}^+(\mathcal{C}_n)$ is onto if and only if $s(e_i) \neq s(e_j)$ for all $i \neq j$.*

Proof. If ι_w is onto, then for each i there is $X_{i,i} \in \mathcal{T}^+(Q)$ with $\iota_w(X_{i,i}) = e_{i,i}$. But notice that by definition $\iota_w(X_{i,i}) = e_{i,i}$ if and only if $X_{i,i} = P_{v_i}$ where $v_i = s(e_i)$ and $v_i \neq v_j$ for all $i \neq j$.

Notice that if $s(e_i) \neq s(e_j)$ for all $i \neq j$ then in particular we know that $e_i \neq e_j$ for all $i \neq j$. It follows that $\iota_w(L_{e_i}) = Z_i$. Similarly, $\iota_w(P_{s(e_i)}) = e_{i+1,i+1}$, and hence the range of ι_w contains a generating set for $\mathcal{T}^+(\mathcal{C}_n)$. $\square$

Notice that even when ι_w is not onto the same argument as in the proof of Proposition 3.1 will tell us that if $A_0(z^n)$ is the nonunital subalgebra of $A(z^n)$

generated by z^n then

$$\begin{bmatrix} A_0(z^n) & zA_0(z^n) & z^2A_0(z^n) & \cdots & z^{n-1}A_0(z^n) \\ z^{n-1}A_0(z^n) & A_0(z^n) & zA_0(z^n) & \cdots & z^{n-2}A_0(z^n) \\ z^{n-2}A_0(z^n) & z^{n-1}A_0(z^n) & A_0(z^n) & \cdots & z^{n-3}A_0(z^n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ zA_0(z^n) & z^2A_0(z^n) & z^3A_0(z^n) & \cdots & A_0(z^n) \end{bmatrix} \subseteq \operatorname{ran}(\iota_w).$$

It follows that the complement of $\operatorname{ran}\iota_w$ is a finite dimensional subspace of $\mathcal{T}^+(\mathcal{C}_n)$, and hence is complemented. This will be important when we construct derivations.

Now, for $\lambda \in \overline{\mathbb{D}}$ and $\mu = e^{i\theta} \in \mathbb{T}$, we define a completely contractive representation $\pi_{\lambda,\mu} : \mathcal{T}^+(\mathcal{C}_n) \to M_n$. The map $\pi_{\lambda,\mu}$ will be chosen so that $\pi_{\lambda,\mu}(e_{i,i}) = e_{i,i}$, $\pi(Z_i) = \lambda e_{i,i+1}$ for $1 \le i < n$, and $\pi(Z_n) = \mu\lambda e_{n,1}$. We begin this by noticing that the inner automorphism $\pi_\mu : \mathcal{T}^+(\mathcal{C}_n) \to \mathcal{T}^+(\mathcal{C}_n)$ induced by the matrix

$$\begin{bmatrix} e^{i\frac{\theta}{n}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{-i\frac{\theta(n-2)}{n}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & e^{-i\frac{\theta(n-3)}{n}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-i\frac{\theta}{n}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

A technical calculation tells us that for all i, $\pi_\mu(e_{i,i}) = e_{i,i}$, $\pi_\mu(Z_i) = Z_i$ for $1 \le i < n$ and $\pi_\mu(Z_n) = \mu Z_n$. Following the automorphism by the completely contractive representation $\tau_\lambda : \mathcal{T}^+(\mathcal{C}_n) \to M_n$, which is given by evaluation at λ, yields a map $\pi_{\lambda,\mu}$ as described.

THEOREM 3.7. *For $\lambda \in \overline{\mathbb{D}}$, $\mu \in \mathbb{T}$ and w a finite path in Q, the representation $\pi_{w,\lambda,\mu} : \mathcal{T}^+(Q) \to M_n$ factors through $\mathcal{T}^+(\mathcal{C}_n)$ via the representation $\tau_\lambda \circ \pi_\mu \circ \iota_w$.*

Proof. Notice that ι_w is completely contractive by construction. Further, π_μ is completely contractive as it is given as an inner automorphism by an invertible of norm 1. For τ_λ, notice that the automorphism $a_\lambda : A(\mathbb{D}) \to A(\mathbb{D})$ given by $a_\lambda(f(z)) = f(\lambda)$ is contractive, and hence completely contractive. Thus, the matricial version of the automorphism $a_\lambda^{(n)} : M_n \otimes A(\mathbb{D}) \to M_n \otimes A(\mathbb{D})$ is completely contractive. Notice however that τ_λ is the restriction to $\mathcal{T}^+(\mathcal{C}_n)$ of the map $a_\lambda^{(n)}$, and hence τ_λ is completely contractive.

It follows that the composition, $\tau_\lambda \circ \pi_\mu \circ \iota_w$ is a completely contractive representation of $\mathcal{T}^+(Q)$ into M_n. Notice that $\tau_\lambda \circ \pi_\mu \circ \iota_w(L_e) = \pi_{w,\lambda,\mu}(L_e)$ for all $e \in E(Q)$ and $\tau_\lambda \circ \pi_\mu \circ \iota_w(P_v) = \pi_{w,\lambda,\mu}(P_v)$ for all $v \in V(Q)$. It follows that as $\mathcal{T}^+(Q)$ is generated by

$$\{P_v, L_e : v \in V(Q), e \in E(Q)\},$$

we know that $\tau_\lambda \circ \pi_\mu \circ \iota_w(a) = \pi_{w,\lambda,\mu}(a)$ for all $a \in \mathcal{T}^+(Q)$. □

REMARK 3.8. Notice that π_μ and ι_w are completely contractive and w^*-continuous. Thus, if $|\lambda| < 1$ we know that the w^*-continuous extension of $\pi_{w,\lambda,\mu}$ to all of $\mathcal{L}_Q$ factors through $\mathcal{L}_{\mathcal{C}_n}$ via w^*-continuous representations.

4. Noncommutative point derivations into M_n

Given a completely contractive representation $\pi : A \to B$ we say that a continuous linear map $D : A \to B$ is a *derivation at* π if $D(ab) = D(a)\pi(b) + \pi(a)D(b)$ for all $a, b \in A$. It was shown in [8, Proposition 1] that if $x \in B$ the linear map $\delta_X(a) = \pi(a)X - X\pi(a)$ for all $a \in A$ is a derivation at π. Any derivation at π of this form is said to be *inner at* π.

For w a finite primitive cycle, $\lambda \in \overline{\mathbb{D}}$, and $\mu \in \mathbb{T}$ we will be interested in continuous linear maps $D : \mathcal{T}^+(Q) \to M_n$ which are derivations at $\pi_{w,\lambda,\mu}$. We first find a method to recognize whether a derivation is inner.

LEMMA 4.1. *Let Q be a directed graph, w a primitive cycle in Q, $\lambda \in \overline{\mathbb{D}}$ with $\lambda \neq 0$, and $\mu \in \mathbb{T}$. For $D : \mathcal{T}^+(Q) \to M_n$, a continuous derivation at $\pi_{w,\lambda,\mu}$, D is inner if and only if $D(a) = 0$ for all $a \in \ker(\pi_{w,\lambda,\mu})$.*

Proof. If D is inner there exists, by definition, an $X \in M_n$ such that $D(a) = \pi_{w,\lambda,\mu}(a)X - X\pi_{w,\lambda,\mu}(a)$ for all $a \in \mathcal{T}^+(Q)$. Now if $a \in \ker \pi_{w,\lambda,\mu}$ then $D(a) = 0X - X0 = 0$, and hence $D|_{\ker(\pi_{w,\lambda,\mu})} = 0$.

Now assume that $D(a) = 0$ for all $a \in \ker(\pi_{w,\lambda,\mu})$. Define a map $\widehat{D} : M_n \to M_n$ by $\widehat{D}(x) = D(a)$ where $\pi_{w,\lambda,\mu}(a) = x$. Notice that if $\pi_{w,\lambda,\mu}(a) = \pi_{w,\lambda,\mu}(b) = x$ then $a - b \in \ker \pi_{w,\lambda,\mu}$, and hence $D(a-b) = 0$. Thus, $\widehat{D}$ is well defined. Since $\pi_{w,\lambda,\mu}$ is onto, for every $x, y \in M_n$, we have $a, b \in \mathcal{T}^+(Q)$ such that $\pi_{w,\lambda,\mu}(a) = x$ and $\pi_{w,\lambda,\mu}(b) = y$. Notice that $\widehat{D}(xy) = D(ab)$ by definition, but $D(ab) = D(a)\pi_{w,\lambda,\mu}(b) + \pi_{w,\lambda,\mu}(a)D(b) = \widehat{D}(x)y + x\widehat{D}(y)$, and hence $\widehat{D}$ is a derivation on M_n.

It is well known that every derivation on M_n is inner, and hence there is $X \in M_n$ with $\widehat{D}(y) = yX - Xy$ for all $x \in M_n$. By definition, for any a in $\mathcal{T}^+(Q)$ with $\pi_{w,\lambda,\mu}(a) = y$ we know that

$$\begin{aligned} D(a) &= \widehat{D}(y) \\ &= yX - Xy \\ &= \pi_{w,\lambda,\mu}(a)X - X\pi_{w,\lambda,\mu}(a), \end{aligned}$$

and hence D is inner. □

We now look at derivations in the special case where Q is the graph $\mathcal{C}_n$, as developed in [8].

COROLLARY 4.2. *Let w be a primitive cycle in $\mathcal{C}_n$ and $\lambda \in \overline{\mathbb{D}}, \mu \in \mathbb{T}$. Then every derivation of $\mathcal{T}^+(\mathcal{C}_n)$ at $\pi_{w,\lambda,\mu}$ is inner if and only if $|\lambda| = 1$.*

Proof. Notice that the only primitive cycles in $\mathcal{C}_n$ are given by

$$e_j e_{j+1} \cdots e_n e_1 e_2 \cdots e_{j-1},$$

where $1 \leq j \leq n$. In this case, notice that ι_w is a cyclic automorphism, in the sense of [1, Section 2]. It follows that $\pi_{w,\lambda,\mu}$ factors through $\mathcal{T}^+(\mathcal{C}_n)$ as a completely contractive automorphism followed by evaluation at λ. Let π_λ denote the representation of $\mathcal{T}^+(\mathcal{C}_n)$ given by evaluation at λ. It was shown in [8] that every continuous derivation of $\mathcal{T}^+(\mathcal{C}_n)$ at π_λ is inner if and only if $|\lambda| = 1$. The result now follows. □

Notice that there exist nonzero inner derivations for $\mathcal{T}^+(\mathcal{C}_n)$ at $\pi_{w,\lambda,\mu}$ for all λ and μ. Now, as we did in the case of representations, we will use the derivations of $\mathcal{T}^+(\mathcal{C}_n)$ to tell us about the derivations of $\mathcal{T}^+(Q)$ for an arbitrary graph.

DEFINITION 4.3. Let $\pi : A \to B$ be a completely contractive representation which factors through C, via $\tilde{\pi} \circ \iota$. We say that a continuous derivation, $D : A \to B$, at π *factors (continuously) through* C if there exists a (continuous) derivation, $\tilde{D} : C \to B$, at $\tilde{\pi}$ such that $D(a) = \tilde{D} \circ \iota(a)$ for all $a \in A$.

If w is a primitive cycle of length n in Q, $\lambda \in \overline{\mathbb{D}}$, and $\mu \in \mathbb{T}$, recall that the representation $\pi_{w,\lambda,\mu} : \mathcal{T}^+(Q) \to M_n$ factors through $\mathcal{T}^+(\mathcal{C}_n)$ via the map $\pi_{\lambda,\mu} \circ \iota_w$. Now, if $D : \mathcal{T}^+(\mathcal{C}_n) \to M_n$ is a continuous derivation at $\pi_{\lambda,\mu}$, then the map $D \circ \iota_w$ induces a derivation on $\mathcal{T}^+(Q)$. It is clear that the induced derivation will factor through $\mathcal{T}^+(\mathcal{C}_n)$. It follows that the derivations of $\mathcal{T}^+(Q)$ at $\pi_{w,\lambda,\mu}$ that factor through $\mathcal{T}^+(\mathcal{C}_n)$ are completely determined by the derivations of $\mathcal{T}^+(\mathcal{C}_n)$ at $\pi_{\lambda,\mu}$. Thus, the description of derivations that factor continuously through $\mathcal{T}^+(\mathcal{C}_n)$ can be easily understood from [8], where the continuous point derivations of $\mathcal{T}^+(\mathcal{C}_n)$ are studied. The following is just a restatement of the description of continuous derivations of $\mathcal{T}^+(\mathcal{C}_n)$ at the representation given by evaluation of $\mathcal{T}^+(\mathcal{C}_n)$ at λ.

COROLLARY 4.4. *Let w be a primitive cycle of length n in Q, $\lambda \in \overline{\mathbb{D}}$, and $\mu \in \mathbb{T}$. Assume that $D : \mathcal{T}^+(Q) \to M_n$ is a non-inner derivation at $\pi_{w,\lambda,\mu}$ which factors continuously through $\mathcal{T}^+(\mathcal{C}_n)$, then $|\lambda| < 1$.*

We now determine those derivations that do not factor through $\mathcal{T}^+(\mathcal{C}_n)$. We begin by describing a method to check whether a derivation factors through $\mathcal{T}^+(\mathcal{C}_n)$.

Corollary 4.5. *Let Q be a directed graph, w a primitive cycle of length n in Q, $\lambda \in \mathbb{D}$, and $\mu \in \mathbb{T}$. If $D : \mathcal{T}^+(Q) \to M_n$ is a continuous derivation at $\pi_{w,\lambda,\mu}$, then D factors through $\mathcal{T}^+(\mathcal{C}_n)$ if and only if $D|_{\ker \iota_w} \equiv 0$, where ι_w is the canonical map from $\mathcal{T}^+(Q)$ into $\mathcal{T}^+(\mathcal{C}_n)$.*

Proof. Certainly if $a \in \ker \iota_w$ and $D(a) \neq 0$ then D cannot factor through $\mathcal{T}^+(\mathcal{C}_n)$, else the induced derivation would send 0 to a nonzero element of M_n.

We next assume that $D|_{\ker \iota_w} = 0$. Since the range of ι_w is complemented as a Banach subspace of $\mathcal{T}^+(\mathcal{C}_n)$, every element $x \in \mathcal{T}^+(\mathcal{C}_n)$ can be written uniquely as $x_r + x_k$ where $x_r \in \operatorname{ran} \iota_w$ and x_k is in the orthogonal complement of $\operatorname{ran} \iota_w$. We need only define a continuous derivation on $\operatorname{ran} \iota_w$ and extend it continuously to $\mathcal{T}^+(\mathcal{C}_n)$ by sending the orthogonal complement of $\operatorname{ran} \iota_w$ to zero.

This follows by defining a map $\widehat{D} \colon \operatorname{ran}(\iota_w) \to M_n$ via the definition $\widehat{D}(\iota_w(x)) = D(x)$. We need only show that this map defines a derivation at $\tau_\lambda \circ \pi_\mu$. Notice that

$$\begin{aligned}
\widehat{D}(\iota_w(x)\iota_w(y)) &= \widehat{D}(\iota_w(xy)) \\
&= \pi_{w,\lambda,\mu}(x)D(y) + D(x)\pi_{w,\lambda,\mu}(y) \\
&= \tau_\lambda \circ \pi_\mu(\iota_w(x))\widehat{D}(\iota_w(y)) + \widehat{D}(\iota_w(x))\tau_\lambda \circ \pi_\mu(\iota_w(y)).
\end{aligned}$$

Thus, $\widehat{D}$ defines a derivation at $\tau_\lambda \circ \pi_\mu$, with $\widehat{D} \circ \iota_w = D$. □

Corollary 4.6. *For the representation $\pi_{w,\lambda,\mu} : \mathcal{T}^+(Q) \to M_n$ with w a primitive cycle in Q let D be a derivation at $\pi_{w,\lambda,\mu}$. If D is inner, then D factors continuously through $\mathcal{T}^+(\mathcal{C}_n)$, where n is the length of the primitive cycle w.*

Proof. Since D is inner we know that $D|_{\ker(\pi_{w,\lambda,\mu})} \equiv 0$. But notice that $\ker(\iota_w) \subseteq \ker(\pi_{w,\lambda,\mu})$ and the result follows. □

Thus, we do not get any new inner derivations on $\mathcal{T}^+(Q)$ besides those obtained via $\mathcal{T}^+(\mathcal{C}_n)$. We do, however, get noninner derivations that do not factor through $\mathcal{T}^+(\mathcal{C}_n)$. We describe some of these derivations now.

Theorem 4.7. *Let $w = e_1e_2\cdots e_n$ be a primitive cycle in Q, $\lambda \in \mathbb{D}$, and $\mu \in \mathbb{T}$. There exist derivations at $\pi_{w,\lambda,\mu} : \mathcal{T}^+(Q) \to M_n$ which do not factor through $\mathcal{T}^+(\mathcal{C}_n)$ if and only if either:*

(i) *there is an edge $e \neq e_i$ for all i such that there are j and k with $r(e) = s(e_j)$ and $s(e) = s(e_k)$ or*
(ii) *there is an edge e_i such that $r(e_i) = s(e_i)$.*

Proof. Notice that if v is a vertex with $v \neq s(e_j)$ for all j, then

$$\begin{aligned} D(P_v) &= D(P_v P_v) \\ &= \pi_{w,\lambda,\mu}(P_v)D(P_v) + D(P_v)\pi_{w,\lambda,\mu}(P_v) \\ &= 0D(P_v) + D(P_v)0 \\ &= 0. \end{aligned}$$

Now, if e is an edge with $r(e) \neq s(e_j)$ for all j, then

$$\begin{aligned} D(L_e) &= D\big(P_{r(e)} L_e\big) \\ &= \pi_{w,\lambda,\mu}\big(P_{r(e)}\big)D(L_e) + D\big(P_{r(e)}\big)\pi_{w,\lambda,\mu}(L_e) \\ &= 0D(L_e) + 0 \cdot 0 \\ &= 0. \end{aligned}$$

Similarly, if e is an edge with $s(e) \neq s(e_j)$ for all j, then $D(L_e) = 0$. Hence, if $D(L_e) \neq 0$, then $s(e) = s(e_j)$ for some j and $r(e) = s(e_k)$ for some k.

Thus, if there are no edges that satisfy case (i) or case (ii), then for any edge with $\iota_w(L_e) = 0$, we know that $D(L_e) = 0$. Similarly, if $\iota_w(P_v) = 0$, then $D(P_v) = 0$. Further, if X is in the ideal generated by such P_v and L_e, then $D(X) = 0$. But notice that the ideal generated by such P_v and L_e contains the ideal $\ker \iota_w$ and hence D factors through $\mathcal{T}^+(\mathcal{C}_n)$.

For the converse, we assume that $|\lambda| < 1$.

Let e be an edge with $e \neq e_i$ for all i such that $P_{r(e)}$ and $P_{s(e)}$ are not in $\ker \pi_{w,\lambda,\mu}$. Notice that the ideal generated by L_e is complemented as a Banach subspace of $\mathcal{L}_Q$, denote the ideal by $\langle L_e \rangle$ and the orthogonal complement by $\langle L_e \rangle^c$. We define a derivation on $\mathcal{L}_Q$ at $\pi_{w,\lambda,\mu}$ by first letting $D|_{\langle L_e \rangle^c}$ be constantly zero. We now define $D(L_e) = e_{j,i}$ where $s(e) = s(e_i)$ and $r(e) = s(e_j)$. We claim that this induces a continuous derivation on $\langle L_e \rangle$. The restriction of this derivation to A_Q will be a derivation at $\pi_{w,\lambda,\mu}$ which does not factor through $A_{\mathcal{C}_n}$. Notice first that $P_{r(e)} L_e P_{s(e)} = L_e$, and hence the derivation property implies that $D(L_e) = e_{j,j} D(L_e) e_{i,i}$ which is satisfied by $e_{j,i}$.

Next notice that every element $X \in \langle L_e \rangle$ can be written as $X = L_e \tilde{X}$ where $\tilde{X} \in \mathcal{L}_Q$; see [13]. As L_e is an isometry we also know that $\|X\| = \|\tilde{X}\|$. By the derivation property, we know that $D(X) = D(L_e)\pi_{w,\lambda,\mu}(\tilde{X})$. Now

$$\begin{aligned} \|D(X)\| &= \|D(L_e)\| \|\pi_{w,\lambda,\mu}(\tilde{X})\| \\ &\leq \|D(L_e)\| \|X\|, \end{aligned}$$

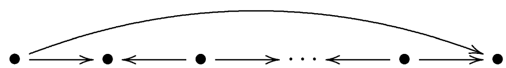
and hence D is continuous. Restricting the derivation to $\mathcal{T}^+(Q)$ gives a continuous noninner derivation at $\pi_{w,\lambda,\mu}$.

For the case in which $\pi_{w,\lambda,\mu}(L_e) \neq 0$, but $r(e) = s(e)$, we use the same proof as in the preceding case except here we let $D(L_e) = \sum_{r(e_i)=r(e)} e_{i,i}$ to construct the derivation. The same proof of continuity will work in this case as in the previous. Notice however that $D(L_e^{|\{i:e_i=e\}|+1}) \neq 0$ but $L_e^{|\{i:e_i=e\}|+1} \in \ker(\iota_w)$, and hence the derivation does not factor through $\mathcal{T}^+(\mathcal{C}_n)$. $\square$

Remark 4.8. In the previous proof, we needed to assume that $\lambda \in \mathbb{D}$ to use the special structure of ideals of $\mathcal{L}_Q$ to get continuity of the derivation. It is possible that the derivation will also be continuous at $\pi_{w,\lambda,\mu}$ with $|\lambda| = 1$, but we have not been able to construct a proof of continuity of the described derivation in this case.

It might be reasonable to expect a result along the lines of [8, Corollary 2]. However, it is not clear how one would piece together different copies of $\mathcal{T}^+(\mathcal{C}_n)$ arising from different primitive cycles to make a general statement about a transitive graph algebra. The previous proposition also suggests that dealing with loop edges will complicate the situation.

We close this section by noticing that this approach will provide little help in dealing with graphs which are not transitive. In fact, there are nontransitive graphs which have derivations that are not inner. As examples, notice that the algebras $\mathcal{A}_{2n}$ of [4], which have noninner derivations, can be viewed completely isometric isomorphically, as the graph algebra arising from nontransitive graphs with $2n$ vertices of the form



5. Representations and point derivations into $\mathbb{C}$

In this section, we will deal with point derivations at π where π is a multiplicative linear functional on $\mathcal{T}^+(Q)$. In this case, the only inner derivation at π is the zero derivation, since the range of a multiplicative linear functional is $\mathbb{C}$. We begin by looking at the multiplicative linear functionals of $\mathcal{T}^+(Q)$ for an arbitrary graph Q. These were described in [14] as an isomorphism invariant for the algebra $\mathcal{T}^+(Q)$. The next result is just a restatement of their description in a manner suitable for our analysis.

Corollary 5.1. *For a directed graph Q, let $\pi : \mathcal{T}^+(Q) \to \mathbb{C}$ be a multiplicative linear functional. Then there exists an n with $0 < n \leq \infty$ such that π factors through A_k, for all $k \geq n$.*

Proof. Notice that as π is a representation it will send projections to projections and hence $\pi(P_v) \in \{0,1\}$ for all $v \in V(Q)$. As the projections $\{P_v : v \in V(Q)\}$ are orthogonal, it follows that if there is a vertex v_0 with $\pi(P_{v_0}) = 1$ then $\pi(P_v) = 0$ for all $v \in V(Q) \setminus \{v_0\}$. If, however, $\pi(P_v) = 0$ for all $v \in V(Q)$, then $\pi(L_e) = \pi(L_e P_{s(e)}) = 0$ for all edges e, and hence π is identically 0 and not a multiplicative linear functional.

Now fix v_0 the unique vertex with $\pi(P_{v_0})=1$. Notice that if e is an edge with either $r(e)$ or $s(e)$ not equal to v_0 then $\pi(L_e)=\pi(P_{r(e)}L_eP_{s(e)})=0$. It follows that $\pi(L_e)\neq 0$ only if $r(e)=s(e)=v_0$. Let n be the number of edges with $r(e)=s(e)=v_0$ and put an label these edges $e_1,e_2,\dots,e_n$. Now, for $k\geq n$, let $f_1,f_2,\dots,f_k$ be the edges in B_k and define a representation $\iota:\mathcal{T}^+(Q)\to A_k$ by $\iota(P_v)=1$ if and only if $v=v_0$, $\iota(L_{e_i})=f_i$ for all i and $\iota(L_e)=0$ if $e\neq e_i$.

It is easy to see that this map extends to a completely contractive representation $\iota:\mathcal{T}^+(Q)\to A_k$. Now look at the multiplicative linear functional, call it $\tilde{\pi}:A_k\to\mathbb{C}$, which satisfies $\tilde{\pi}(f_i)=\pi(e_i)$. This multiplicative linear functional completes the result. □

Next, we notice that the same is true of a derivation at π, where π is a multiplicative linear functional.

Corollary 5.2. *For Q a countable directed graph, let π be a multiplicative linear functional for $\mathcal{T}^+(Q)$, and assume that $D:\mathcal{T}^+(Q)\to\mathbb{C}$ is a continuous derivation at π. Then there exists an $0<n\leq\infty$ such that D factors continuously through A_n.*

Proof. As before, there exists a vertex v_0 with $\pi(P_{v_0})=1$ such that $\{e:r(e)=s(e)=v_0\}=\{e_1,e_2,\dots,e_n\}$, recall that $\pi(P_v)=0$ for any $v\neq v_0$. Now let $\iota:\mathcal{T}^+(Q)\to A_n$ be as in the preceding proposition. Notice that $D(P_v)=D(P_vP_v)=\pi(P_v)D(P_v)+D(P_v)\pi(P_v)=0$ for all vertices with $v\neq v_0$. Similarly, if e is an edge with either $r(e)\neq v_0$ or $s(e)\neq 0$, then $D(L_e)=D(P_{r(e)}L_e)=D(L_eP_{s(e)})$ one of which must be zero since L_e and one of $P_{r(e)}$ or $P_{s(e)}$ must be in $\ker\pi$.

Now notice that $D(P_{v_0})=D(P_{v_0}P_{v_0})=D(P_{v_0})\pi(P_{v_0})+\pi(P_{v_0})D(P_{v_0})$ and since $\mathbb{C}$ is a field it follows that $D(P_{v_0})=0$. Define a map $\tilde{D}:A_n\to\mathbb{C}$ by first assigning $\tilde{D}(1)=0$ and $\tilde{D}(L_{f_i})=D(L_{e_i})$ and extending using linearity and the definition of a derivation at $\tilde{\pi}$ (i.e., $\tilde{D}(xy)=\tilde{D}(x)\tilde{\pi}(y)+\tilde{\pi}(x)\tilde{D}(y)$). It is easy to see that $\tilde{D}\circ\iota(a)=D(a)$ for all $a\in\mathcal{T}^+(Q)$. We need only see that $\tilde{D}$ is a continuous linear functional. This, however, is not difficult as $\|\iota(a)\|=\|a\|$ for all a in the subalgebra of $\mathcal{T}^+(Q)$ generated by P_{v_0} and L_{e_i}. Thus, for any $b\in A_n$, there is $\widehat{b}\in\mathcal{T}^+(Q)$ with $\iota(\widehat{b})=b$. By construction $\|\tilde{D}(b)\|=\|D(\widehat{b})\|\leq\|D\|\|\widehat{b}\|=\|D\|\|b\|$, and hence $\tilde{D}$ is bounded. □

It follows that we need only understand the multiplicative linear functionals and derivations for the algebras A_n. In what follows, we will use notation as if n is finite since the infinite case will follow in a manner similar to the finite case.

Recall from [6, Theorem 3.3] or [18] that a multiplicative linear functional $\pi:A_n\to\mathbb{C}$ is uniquely given by evaluation at a point $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\overline{\mathbb{B}_n}$

where $\pi(L_{e_i}) = \lambda_i$ for all $1 \le i \le n$. Here, by $\overline{\mathbb{B}_n}$, we mean the unit ball in $\mathbb{C}_n$ with the usual norm. We will denote the multiplicative linear functional arising from evaluation at λ by π_λ.

A further consequence of [6, Theorem 3.3] is that the Gelfand map, $\widehat{a}(\lambda) = \pi_\lambda(a)$, induces a homomorphism from A_n into $A(\mathbb{B}_n)$, the analytic functions on $\mathbb{B}_n$ with continuous extensions to $\overline{\mathbb{B}_n}$.

Lastly, notice following the arguments of [6, Proposition 2.4] that the commutator ideal of A_n, denoted $\mathfrak{C}_n$, is equal to $\bigcap_{\lambda \in \overline{\mathbb{B}_n}} \pi_\lambda$. Putting these facts together, we see that $A_{n,\mathfrak{C}} := A_n/\mathfrak{C}_n$ is a semisimple subalgebra of $A(\mathbb{B}_n)$. This will allow us to use well-known results about point derivations of uniform algebras to discuss the point derivations of A_n. We summarize the preceding discussion in the following proposition.

Corollary 5.3. *Let $\pi_\lambda : A_n \to \mathbb{C}$ be a multiplicative linear functional, then π_λ factors through $A_{n,\mathfrak{C}}$.*

A similar result holds for derivations.

Corollary 5.4. *Let $\pi_\lambda : A_n \to \mathbb{C}$ be a multiplicative linear functional and D be a derivation at π_λ. Then D factors through $A_{n,\mathfrak{C}}$.*

Proof. Notice that

$$\begin{aligned} D(ab - ba) &= D(ab) - D(ba) \\ &= \big(D(a)\pi(b) + \pi(a)D(b)\big) - \big(D(b)\pi(a) + \pi(b)D(a)\big) \\ &= 0 \end{aligned}$$

as $\mathbb{C}$ is commutative. It follows that $D|_{\mathfrak{C}} \equiv 0$. Further define $\tilde{D} : A_{n,\mathfrak{C}} \to \mathbb{C}$ by $\tilde{D}(a) = D(\widehat{a})$ where $\widehat{a} = a + k$ for some $k \in \mathfrak{C}$. Notice that $\tilde{D}$ is well defined since $D(k) = 0$ for all $k \in \mathfrak{C}$.

Now, there exist $k, k_1, k_2 \in \mathfrak{C}$ with

$$\begin{aligned} \tilde{D}(ab) &= D(\widehat{ab}) \\ &= D(ab + k) \\ &= D(ab) \\ &= D(a)\pi_\lambda(b) + \pi_\lambda(a)D(b) \\ &= D(a + k_1)\pi_\lambda(b + k_2) + \pi_\lambda(a + k_1)D(b + k_2) \\ &= \tilde{D}(a)\tilde{\pi_\lambda} \circ \iota(b + k_2) + \tilde{\pi_\lambda} \circ \iota(a + k_1)\tilde{D}(b) \\ &= \tilde{D}(a)\tilde{\pi_\lambda}(b) + \tilde{\pi_\lambda}(a)\tilde{D}(b). \end{aligned}$$

Hence, $\tilde{D}$ is a derivation at π_λ satisfying the appropriate property for factoring through A_n. □

It is not clear that the factorization must be continuous, we will see later that it is. On the other hand, we will not need continuity of the induced derivation for what follows. We now describe the point derivations of $A_{n,\mathfrak{C}}$ by viewing it as a subalgebra of $A(\mathbb{B}_n)$. As a corollary, we will pull back the derivations and describe when nontrivial derivations can occur for A_n at a representation π_λ.

LEMMA 5.5. *Let $\lambda \in \mathbb{B}_n$, then $\pi_\lambda : A_n \to \mathbb{C}$ factors through $A_{m,\mathfrak{C}}$ where m is the number of λ_i with $\lambda_i \neq 0$.*

Proof. Let $\{e_i : 1 \leq i \leq n\}$ be the edges in B_n and assume without loss of generality that $\pi_\lambda(e_i) = 0$ if and only if $m+1 \leq i \leq n$. Now let $\Omega : A_n \to A_m$ be the completely contractive representation that sends, for $1 \leq i \leq m$, L_{e_i} to L_{f_i} where $\{f_i\}$ is the set of edges in B_m, and sends $L(e_i)$ to zero when $m+1 \leq i \leq n$. Then notice that π_λ will factor through A_m via $\pi_{(\lambda_1,\lambda_2,\ldots,\lambda_m)} \circ \Omega$. But now $\pi_{(\lambda_1,\lambda_2,\ldots,\lambda_m)} : A_m \to \mathbb{C}$ factors through $A_{m,\mathfrak{C}}$. Putting the appropriate maps together, we get that $\pi_\lambda : A_n \to \mathbb{C}$ factors through $A_{m,\mathfrak{C}}$. □

Notice that if A is a commutative operator algebra with identity, and $\pi : A \to \mathbb{C}$ is a completely contractive representation, then $\ker \pi$ is complemented as a Banach subspace of A. Denote by $\overline{(\ker \pi)^2}$ the norm closure of the ideal generated by $\{fg : f, g \in \ker \pi\}$. If $D : A \to \mathbb{C}$ is a linear functional such that $D(1) = 0$ and $D|_{\overline{(\ker \pi)^2}} = 0$ then we claim that D is a derivation at π. To see this, let $f, g \in A$ and notice that $(f - \pi(f) \cdot 1)(g - \pi(g) \cdot 1) \in (\ker \pi)^2$. Hence, $D((f - \pi(f) \cdot 1)(g - \pi(g) \cdot 1)) = 0$. Multiplying out and using linearity of D, we get that $D(fg) = D(f)\pi(g) + \pi(f)D(g)$, and hence D is a derivation at π. This argument, which appears in [3], will be used in the proof of the next result.

COROLLARY 5.6. *If $\pi_\lambda(L_i) = 0$ for some i, then there is a unique continuous derivation induced by sending $D(L_i)$ to 1, and $D(L_k) = 0$ for all $k \neq i$.*

Proof. Since the range of π_λ is finite dimensional, we know that $\ker(\pi_\lambda)$ is complemented as a Banach subspace of A_n. Notice further that $L_i \in \ker(\pi_\lambda)$ and yet $L_i \notin (\ker(\pi_\lambda))^2$, and hence the map which sends L_i to 1 and all other L_j to zero extends by Hahn–Banach, to a continuous derivation on A_n. □

We say that a derivation at π_λ of this form is the canonical derivation at L_i, denoted D_i. Notice that the canonical derivation does not factor through $A_{m,\mathfrak{C}}$, where m is the number of nonzero λ_i.

COROLLARY 5.7. *Let $D : A_n \to \mathbb{C}$ be a continuous derivation at π_λ, and let m be the number of λ_i such that $\lambda_i \neq 0$, then $D = D_1 + D_2$ where D_1 factors through $A_{m,\mathfrak{C}}$ and D_2 is a linear combination of canonical derivations at L_j where $\lambda_j = 0$.*

Proof. For each λ_i with $\lambda_i = 0$, let $\omega_i = D(L_i)$. Then notice that

$$D_1 = D - \sum_{\lambda_i = 0} \omega_i \cdot D_i$$

is a derivation on A_n such that D_1 factors through $A_{m,\mathfrak{C}}$ and the result follows. □

THEOREM 5.8. *Let $D : A_n \to \mathbb{C}$ be a nontrivial point derivation at λ which factors through $A_{m,\mathfrak{C}}$ where m is the number of λ_i with $\lambda_i \neq 0$. Then $|\lambda| < 1$. Further the nontrivial point derivation factors continuously through $A_{m,\mathfrak{C}}$.*

Proof. We will assume without loss of generality that $m = n$.

Let $f(z_1, z_2, \ldots, z_n)$ denote an arbitrary element of $A_{n,\mathfrak{C}}$ and assume that $|\lambda| = 1$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. By assumption $\lambda_1 \neq 0$. We will see that this implies that any point derivation at λ, call it D, sends $z_1 - \lambda_1$ to zero. It will then follow by linearity of D that $D(z_1) = 0$. A similar argument will then prove that any point derivation at λ is the zero derivation. Notice that the subalgebra $A := \{f \in A_{n,\mathfrak{C}} : f(z_1, z_2, \ldots, z_n) = f(z_1, \lambda_2, \ldots, \lambda_n)\}$ is a function algebra in the variable z_1. Further, $z_i - \mu \in A$, and hence A separates the points of $\{z_1 : \|(z_1, \lambda_2, \lambda_3, \ldots, \lambda_n)\| \leq 1\}$.

Now define $\pi : A \to \mathbb{C}$ by $\pi(f) = f(\lambda_1, \lambda_2, \ldots, \lambda_n)$. This is a multiplicative linear functional on A, and by [3] there does not exist a nonzero point derivation at π if there exists a function $g \in A$ with $\|g(z, \lambda_2, \lambda_3, \ldots, \lambda_n)\| < \|g(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)\|$ for all $z \neq \lambda_1$. Notice that the function $g(z) = \frac{z+\lambda_1}{2\lambda_1}$ satisfies this property, and hence any point derivation at π is the zero derivation. Now, since any point derivation on $A_{m,\mathfrak{C}}$ at λ will induce a point derivation on A at π_λ, the point derivation must send L_1 to zero.

To see that a nontrivial point derivation factors continuously through $A_{n,\mathfrak{C}}$ we need only see that any nonzero point derivation of $A_{n,\mathfrak{C}}$ at λ is unique. Notice that since $|\lambda| < 1$ we know that the representation at λ extends to a wk^* continuous representation of $\mathcal{L}_n$. Now notice, from [6, Theorem 2.10 and Theorem 1.3], that the ideal $\ker(\pi_\lambda) \subset \mathcal{L}_n$ is equal to the algebraic ideal generated by n elements, $\{X_1, X_2, \ldots, X_n\}$. In particular, every element $a \in A_n$ can be written uniquely as $a_0 + Y_1 X_1 Z_1 + Y_2 X_2 Z_2 + \cdots + Y_n X_n Z_n + a_1$ where $a_0 \notin \ker(\pi_\lambda)$, $Y_i, X_i \notin \ker \pi_\lambda$ for all i, and $a_1 \in \ker(\pi_\lambda)^2$. Now applying the quotient map q, we get a decomposition of every element of $f \in \mathcal{L}_{n,\mathfrak{C}}$ as

$$f = f(\lambda) + \sum_{i=1}^{n} q(X_i) g_i(z) + q(a_1),$$

where $g_i(z) \notin \ker(\pi_\lambda)$ and $q(a_1) \in (\ker(\pi_\lambda))^2$.

Notice that any derivation at π_λ will send $q(a_1)$ to zero. Now

$$\begin{aligned} D(f) &= D(f(\lambda)) + \sum_{i=1}^n D(q(X_i)g_i(z)) + D(q(a_1)) \\ &= 0 + \sum_{i=1}^n \big(D(q(X_i))g_i(\lambda) + \pi_\lambda(X_i)D(g_i(z))\big) + 0 \\ &= \sum_{i=1}^n D(q(X_i))g_i(\lambda). \end{aligned}$$

Hence, every derivation when restricted to $A_{n,\mathfrak{C}}$ is a linear combination of scalar multiples of the continuous derivation that sends X_i to 1 and every other X_j to zero. ☐

COROLLARY 5.9. *Let $D\colon A_n \to \mathbb{C}$ be a nontrivial continuous point derivation at π_λ then, either $\lambda_i = 0$ for some i, or $|\lambda| < 1$.*

Also notice that any derivation that factors through $A_{m,\mathfrak{C}}$ is unique and the above results describe all point derivations of A_n, and hence of $\mathcal{T}^+(Q)$ where Q is a directed graph. This extends the results of Popescu [18], where a description of the point derivations was given for the representation sending L_i to zero for all i.

We close with a result concerning A_n-valued derivations of A_n. This is a simple application of an idea in [2, Theorem 16, page 92].

COROLLARY 5.10. *Let $D\colon A_n \to A_n$ be a continuous derivation, then $D(A_n) \subseteq \mathfrak{C}$.*

Proof. For all $z \in \mathbb{C}$, we know that e^{zD} is a continuous automorphism of A_n. Thus, for lambda in $\overline{\mathbb{B}_n}$, the mapping $\pi_\lambda \circ (e^{zD})$ is a multiplicative linear functional on A_n, and hence $|\pi_\lambda(e^{zD})(a)| \le \|a\|$ for all $a \in A_n$. Now, for a in A_n, the mapping $z \mapsto \pi_\lambda(e^{zD})(a)$ is a bounded entire function, and hence is constant. But examining the power series of this function tells us that the coefficient of z is $\pi_\lambda(Da)$ which must be zero. As λ was arbitrary, the result follows. ☐

The upshot of this result is that if D is a derivation on A_n then the induced automorphism is quasi-inner in the sense of [6] (i.e., the automorphism is trivial modulo the commutator ideal). Of course, it is not the case that every quasi-inner automorphism arises via a derivation.

REMARK 5.11. The above proof can be extended to arbitrary graphs using a characterization of the commutator ideal [13, Corollary 5.5] and noting that this ideal is the intersection of the kernels of all multiplicative linear functionals as is the case for A_n.

It is of course left open whether one can describe the continuous $\mathcal{T}^+(Q)$-valued derivations of $\mathcal{T}^+(Q)$ for an arbitrary transitive graph Q, as was done in [8] for the graph $\mathcal{C}_n$. We have also not made an attempt to discuss the higher point cohomology for the graph algebras as an analogue of the results in Section 9 of [11].

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