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AREA OF IDEAL TRIANGLES AND GROMOV HYPERBOLICITY IN HILBERT GEOMETRY

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ABSTRACT. We show that a Hilbert geometry is hyperbolic in the sense of Gromov if and only if there is an upper bound on the area of ideal triangles.

Introduction and statements

The aim of this paper is to show, in the context of Hilbert geometry, the equivalence between the existence of an upper bound on the area of ideal triangles and the Gromov-hyperbolicity.

Let us recall that a Hilbert geometry $(\mathcal{C}, d_{\mathcal{C}})$ is a nonempty bounded open convex set $\mathcal{C}$ on $\mathbb{R}^n$ (that we shall call *convex domain*) with the Hilbert distance $d_{\mathcal{C}}$ defined as follows: for any distinct points p and q in $\mathcal{C}$, the line passing through p and q meets the boundary $\partial\mathcal{C}$ of $\mathcal{C}$ at two points a and b, such that one walking on the line goes consecutively by a, p, q, b (Figure 1). Then we define

$$d_{\mathcal{C}}(p,q) = \frac{1}{2}\ln[a,p,q,b],$$

where $[a,p,q,b]$ is the cross-product of (a,p,q,b), i.e.,

$$[a,p,q,b] = \frac{\|q-a\|}{\|p-a\|} \times \frac{\|p-b\|}{\|q-b\|} > 1,$$

with $\|\cdot\|$ the canonical Euclidean norm in $\mathbb{R}^n$.

Note that the invariance of the cross-ratio by a projective map implies the invariance of $d_{\mathcal{C}}$ by such a map.

These geometries are naturally endowed with a C^0 Finsler metric $F_{\mathcal{C}}$ as follows: if $p \in \mathcal{C}$ and $v \in T_p\mathcal{C} = \mathbb{R}^n$ with $v \neq 0$, the straight line passing by p and directed by v meets $\partial\mathcal{C}$ at two points $p_{\mathcal{C}}^{+}$ and $p_{\mathcal{C}}^{-}$ (see Figure 2); we then

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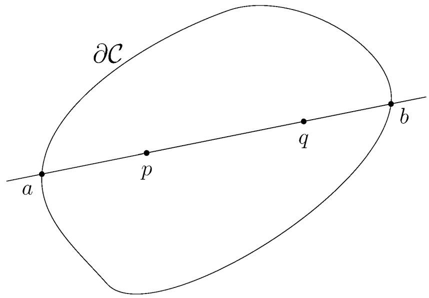

FIGURE 1. The Hilbert distance.

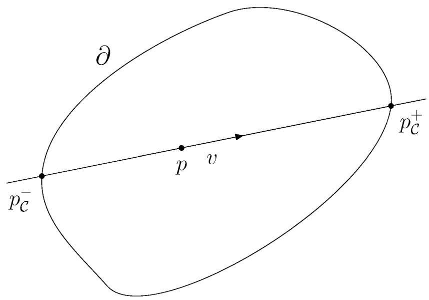

FIGURE 2. The Finsler structure.

define

$$F_{\mathcal{C}}(p,v) = \frac{1}{2}\|v\|\left(\frac{1}{\|p-p_{\mathcal{C}}^{-}\|} + \frac{1}{\|p-p_{\mathcal{C}}^{+}\|}\right) \quad \text{and} \quad F_{\mathcal{C}}(p,0)=0.$$

The Hilbert distance $d_{\mathcal{C}}$ is the length distance associated to $F_{\mathcal{C}}$.

Thanks to that Finsler metric, we can built a Borel measure $\mu_{\mathcal{C}}$ on $\mathcal{C}$ (which is actually the Hausdorff measure of the metric space $(\mathcal{C}, d_{\mathcal{C}})$; see [BBI01], Example 5.5.13) as follows.

To any $p \in \mathcal{C}$, let $B_{\mathcal{C}}(p) = \{v \in \mathbb{R}^n | F_{\mathcal{C}}(p,v) < 1\}$ be the open unit ball in $T_p\mathcal{C} = \mathbb{R}^n$ of the norm $F_{\mathcal{C}}(p,\cdot)$ and ω_n the Euclidean volume of the open unit ball of the standard Euclidean space $\mathbb{R}^n$. Consider the (density) function $h_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbb{R}$ given by $h_{\mathcal{C}}(p) = \omega_n / \operatorname{Vol}(B_{\mathcal{C}}(p))$, where Vol is the canonical Lebesgue measure of $\mathbb{R}^n$. We define $\mu_{\mathcal{C}}$, which we shall call the *Hilbert Measure* on $\mathcal{C}$, by

$$\mu_{\mathcal{C}}(A) = \int_A h_{\mathcal{C}}(p)\,\mathrm{d}\operatorname{Vol}(p)$$

for any Borel set A of $\mathcal{C}$.

A fundamental result of Y. Benoist [Ben03] gives an extrinsic characterization of Gromov-hyperbolic Hilbert geometries, that is sufficient and necessary conditions on the boundary $\partial\mathcal{C}$ of a convex domain $\mathcal{C}$ to insure that the associate Hilbert geometry $(\mathcal{C}, d_{\mathcal{C}})$ is Gromov-hyperbolic.

The goal of this paper is to give an intrinsic condition equivalent to the Gromov-hyperbolicity in terms of the area of the ideal triangles of $(\mathcal{C}, d_{\mathcal{C}})$.

We define an ideal triangle $T \subset \mathcal{C}$ as the affine convex hull of three points a, b, c of $\partial\mathcal{C}$ not on a line, and such that $T \cap \partial\mathcal{C} = a \cup b \cup c$. (Note that the affine convex hull coincide with the geodesic convex hull when the space is uniquely geodesic, which is the case of Gromov-hyperbolic Hilbert geometry.) The area of a triangle T (ideal or not) of $(\mathcal{C}, d_{\mathcal{C}})$, denoted by $\mathrm{Area}_{\mathcal{C}}(T)$, is its area for the Hilbert measure of $(\mathcal{C} \cap P, d_{\mathcal{C}\cap P})$, where P is the unique plane in $\mathbb{R}^n$ containing the triangle (in dimension 2, as $\mathcal{C} \cap P = \mathcal{C}$, we will also denote it by $\mu_{\mathcal{C}}(T)$).

In this paper, we prove the following theorem.

THEOREM 1. *A Hilbert Geometry $(\mathcal{C}, d_{\mathcal{C}})$ is hyperbolic in the sense of Gromov if and only if there is a bound on the area of ideal triangles. Precisely,*

(1) *for any $M > 0$ there exists $\delta(M) > 0$ such that if the area of any ideal triangle $T \subset \mathcal{C}$ is bounded above by M, then $(\mathcal{C}, d_{\mathcal{C}})$ is $\delta(M)$-hyperbolic;*
(2) *for any $\delta > 0$, there exists $M(\delta) > 0$ such that if the Hilbert geometry $(\mathcal{C}, d_{\mathcal{C}})$ is δ-hyperbolic then the area of any ideal triangle $T \subset \mathcal{C}$ is bounded above by $M(\delta)$.*

To show that the bound on the area of ideal triangles implies the δ-hyperbolicity is quite straightforward and its proof is in the first part of the paper (Theorem 2). The converse is much more delicate: we show it on the second part of the paper (Theorem 7). The main ingredient of the proof is a cocompacity lemma (Theorem 8, whose idea goes back in some sense to Benzecri [Ben60]) and the results of Benoist's paper [Ben03]. To make the proof readable, we allowed some technical lemmas in the Appendix at the end of the paper, in particular Lemma 21 deduced from [Ben03], which implies an α-Hölder regularity of the boundary of a convex domain whose Hilbert geometry is δ-hyperbolic, with α depending only on δ, and Lemma 18, where we show that the α-Hölder regularity implies the finiteness of the area of ideal triangles.

Note that the results of this Appendix are used many times in the proof of Theorem 7.

In the sequel, we will switch between affine geometry (where our results are stated) and projective geometry (where Benoist's results are stated). We will use the following two classical facts (see [Sam88], Section 1.3, pages 8–11)

(1) Any affine space can be embedded into a projective space (by "adding an hyperplane at infinity"). Furthermore any one-to-one affine map extends to a homography keeping the "hyperplane at infinity" globally invariant.

(2) The complement of a projective hyperplane in a projective space is an affine space. Furthermore, all homographies keeping this hyperplane globally invariant are naturally identified with an affine map on the complement.

1. Bounded area implies δ-hyperbolicity

In this part, we prove the following theorem.

THEOREM 2. *Let $M > 0$. There exists $\delta = \delta(M) > 0$ with the following property: Let $(\mathcal{C}, d_{\mathcal{C}})$ be a convex domain with its induced Hilbert distance. If any ideal triangle in $(\mathcal{C}, d_{\mathcal{C}})$ has its area less than M, then $(\mathcal{C}, d_{\mathcal{C}})$ is δ-hyperbolic.*

This theorem is a straightforward consequence of the following proposition.

PROPOSITION 3. *There exist a constant $C > 0$ with the following property: for any δ, if $(\mathcal{C}, d_{\mathcal{C}})$ is not δ-hyperbolic, then there exists an ideal triangle $T \subset \mathcal{C}$, whose area satisfies $\mu_{\mathcal{C}}(T) \geq C \cdot \delta$.*

Indeed, if the assumption of Theorem 2 is satisfied, then $\mathcal{C}$ has to be δ-hyperbolic for any $\delta > M/C$, otherwise we would get a contradiction with the Proposition 3.

Now let us prove Proposition 3. We already know that if $\partial\mathcal{C}$ is not strictly convex, then there is an ideal triangle of arbitrarily large area ([CVV04], Corollaire 6.1, page 210). Hence, we can assume that $\partial\mathcal{C}$ is strictly convex, which implies that all the geodesics of $(\mathcal{C}, d_{\mathcal{C}})$ are straight segments (see [dlH93], Proposition 2, page 99).

Each triangle $T \subset \mathcal{C}$ determines a plane section of $\mathcal{C}$, and is contained in an ideal triangle of this plane section. So, it suffices to exhibit a triangle (not necessarily ideal) such that $\mu_{\mathcal{C}}(T) \geq C \cdot \delta$.

This is done thanks to the two following lemmas.

LEMMA 4. *If $(\mathcal{C}, d_{\mathcal{C}})$ is not δ-hyperbolic, there is a plane P and a triangle T in $P \cap \mathcal{C}$ such that a point in the triangle is at a distance greater than $\delta/4$ from its sides.*

Proof. If $(\mathcal{C}, d_{\mathcal{C}})$ is not δ-hyperbolic, there exists a triangle $T \in \mathcal{C}$ of vertices a, b, c, a point $p \in \partial T$, say between a and b, such that the distance from p to the two opposite sides of ∂T is greater than δ. The end of the proof takes place in the plane determined by the triangle T.

Let $R = \delta/2$. Consider a circle S of center p and radius R. Let $p_1, p_2 = S \cap \partial T$. We have $d_{\mathcal{C}}(p_1, p_2) = 2R$.

If $q \in S$, then $d_{\mathcal{C}}(p_1, q) + d(q, p_2) \geq 2R$ by the triangle inequality. By continuity, we can choose $q \in S \cap T$, with $d_{\mathcal{C}}(q, p_1) \geq R$; $d_{\mathcal{C}}(q, p_2) \geq R$. From this fact, and by the classical triangular inequality, we deduce $d_{\mathcal{C}}(q, \partial T) \geq R/2$:

to see it, let p_3 be the middle of the segment pp_1. We have $d_{\mathcal{C}}(p,p_3) = d_{\mathcal{C}}(p_3,p_1) = R/2$.

- If $q' \in pp_3$, $d_{\mathcal{C}}(q,q') \geq d_{\mathcal{C}}(p,q) - d_{\mathcal{C}}(p,p_3) \geq R/2$.
- If $q' \in p_3p_1$, then $d_{\mathcal{C}}(q,q') \geq d_{\mathcal{C}}(q,p_1) - d_{\mathcal{C}}(q'q_1) \geq R/2$ and this show also that if $d_{\mathcal{C}}(q',p_1) \leq R/2$ then $d_{\mathcal{C}}(q,q') \geq R/2$.
- If q' is such that $d_{\mathcal{C}}(q',p) \geq 3R/2$, then $d_{\mathcal{C}}(q,q') \geq R/2$.

This allows to conclude for the half-line issue from p through p_1 and we can do the same for the other half line. $\square$

LEMMA 5. *There exists a constant C_n such that any ball of radius $R > 2$ in any Hilbert geometry of dimension n has a volume greater or equal to $C_n \cdot R$.*

Proof. Let B a ball centered at q of radius R. Consider a geodesic segment starting at q: it has length R and lies inside B. We can cover it by $N =$ integer part of R, pairwise disjoint balls of radius 1 contained in B, with $N \to \infty$ with δ. But we know (Theorem 12, [CV]) that the volume of a radius 1 ball is uniformly bounded below for all the Hilbert geometries by a constant $c(n)$. Hence, the volume of the ball of radius $R \geq 2$ is greater than $(R-1) \cdot c(n) \geq R \cdot c(n)/2$. $\square$

Hence, if $(\mathcal{C}, d_{\mathcal{C}})$ is not δ hyperbolic thanks to Lemma 4, we would find a triangle T containing a two-dimensional ball of radius $\delta/4$, hence its area would be greater than $\delta/4 \cdot C_2$ thanks to Lemma 5, which ends the proof of Proposition 3.

A consequence of Theorem 2, already proved with different approaches by Karlsson and Noskov [KN02], Benoist [Ben03], and Colbois and Verovic [CV04], is the following corollary.

COROLLARY 6. *If the boundary of $\mathcal{C}$ is C^2 with strictly positive curvature, then $(\mathcal{C}, d_{\mathcal{C}})$ is Gromov hyperbolic.*

This is a consequence of Theorem 4 in [CVV04] which says that if the boundary is C^2 with strictly positive curvature, then the assumptions of Theorem 2 are satisfied.

2. From δ-hyperbolicity to bounded area

The aim of this section is to prove the following

THEOREM 7. *Let $\delta > 0$. Then there exists $V = V(\delta) > 0$ with the following property: Let $\mathcal{C}$ be a convex domain such that $(\mathcal{C}, d_{\mathcal{C}})$ is δ-hyperbolic. Then for any ideal triangle T of $\mathcal{C}$, we have* $\mathrm{Area}_{\mathcal{C}}(T) \leq V$.

Though the ideas to prove Theorem 7 are quite simple, the proof itself is somewhat technical. The bound on the area of ideal triangle depends only on the δ of the Gromov hyperbolicity. Therefore, it suffices to prove Theorem 7 in the two-dimensional case. Thus, from this point on, everything will be done in the two-dimensional case.

2.1. Cocompactness of triangle-pointed convex. Let us begin with some notations.

Let $G_n := \mathrm{PGL}(\mathbb{R}^n)$, $\mathbb{P}^n := \mathbb{P}(\mathbb{R}^{n+1})$ the projective space of $\mathbb{R}^{n+1}$. A properly open convex subset Ω of $\mathbb{P}^n$ is an open convex set such that there is a projective hyperplane who does not meet its closure. Denote by X_n the set of properly open convex sets. Let X_n^δ be the set of δ-hyperbolic properly open convex sets in $\mathbb{P}^n$

In X_n, we will consider the topology induced by the Hausdorff distance between sets, denoted by d.

We will say that a convex domain $\mathcal{C}$ is *triangle-pointed* if one fixes an ideal triangle in $\mathcal{C}$. Let

$$T_2^\delta = \{(\omega, x, y, z) \in X_2^\delta \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \mid \\ x, y, z \in \partial\omega, x \neq y, y \neq z, z \neq x\}$$

be the set of triangle-pointed convex sets $\mathcal{C}$ with $\mathcal{C} \in X_2^\delta$.

One of the main steps of our proof will rely upon the following cocompactness result.

THEOREM 8. *G_2 acts cocompactly on T_2^δ, i.e., for any sequence $(\omega_n, \Delta_n)_{n\in\mathbb{N}}$ in T_2^δ, there is a sequence $(g_n)_{n\in\mathbb{N}}$ in G_2 and a sub-sequence of $(g_n\omega_n, g_n\Delta_n)_{n\in\mathbb{N}}$ that converges to $(\omega, \Delta) \in T_2^\delta$.*

Actually, Theorem 8 is a corollary of the following more precise statement.

PROPOSITION 9. *Let $(\omega_n, T_n)_{n\in\mathbb{N}}$ be a sequence in T_2^δ, then*

(1) *There is a sequence $(g_n)_{n\in\mathbb{N}}$ in G_2 and a number $0 < e \leq 1/2$ such that $g_nT_n = \Delta \subset \mathbb{R}^2$ the triangle whose coordinates are the points $(1,0)$, $(0,1)$, $(1,1)$, and $g_n\omega_n \subset \mathbb{R}^+ \times \mathbb{R}^+$ is tangent at $(1,0)$ to the x-axe, at $(0,1)$ to the y-axe and at $(1,1)$ to the line passing through the points $(1/\alpha_n, 0)$ and $(0, 1/(1-\alpha_n))$ for some $0 < e \leq \alpha_n \leq 1/2$;*
(2) *From the previous sequence we can extract a subsequence converging to some $(\omega, \Delta) \in T_2^\delta$.*

Proof. Step 1: A first transformation

According to [CVV04] (see the proof of Théorème 3, page 215 and Lemma 9, page 216), for each $n \in \mathbb{N}$, there is a number $\alpha_n \in (0, 1/2]$ and an affine transformation A_n of $\mathbb{R}^2$ such that:

(1) The bounded open convex domain $\Omega_n := A_n(\omega_n)$ is contained in the triangle $\mathcal{T} \subset \mathbb{R}^2$ whose vertices are the points $(0,0)$, $(1,0)$ and $(0,1)$.
(2) The points $(\alpha_n, 0)$, $(0, 1-\alpha_n)$ and $(\alpha_n, 1-\alpha_n)$ are in $\partial\Omega_n$ and the ideal triangle Δ_n they define in (Ω_n, d_{Ω_n}) is equal to $A_n(T_n)$.
(3) The x-axis, the y-axis, and the line passing through $(1,0)$ and $(0,1)$ are tangent to $\partial\Omega_n$ at the points $(\alpha_n, 0)$, $(0, 1-\alpha_n)$ and $(\alpha_n, 1-\alpha_n)$, respectively (see Figure 3).

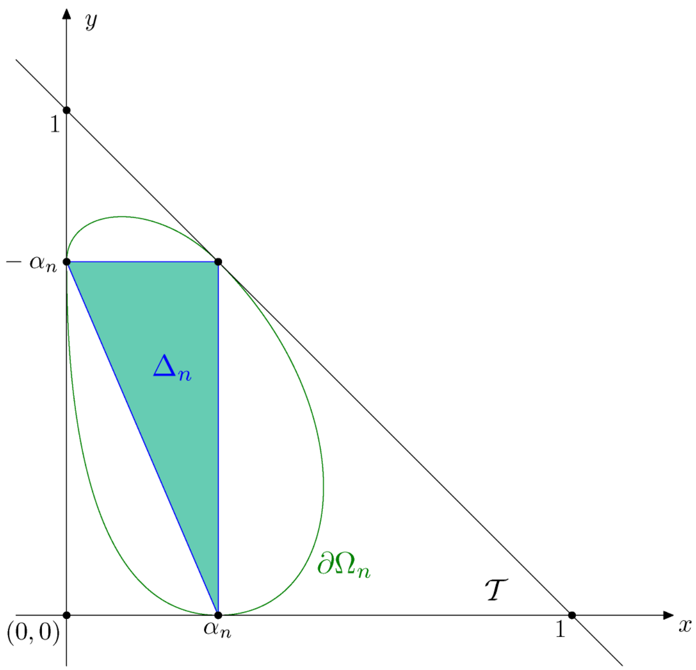

FIGURE 3. The Ω_n are convex sets included in a fixed triangle.

Remark that we may have to take out different projective lines to see the proper convex sets ω_n as convex sets in an affine space. But up to some homography, we can suppose that we always took the same. The geometries involved will not be changed.

Step 2: Proof of the first part of (1)

In this part, we show the first part of point (1). The second point of (1), that is to see that the set of $\{\alpha_n\}$ is uniformly bounded below by $e>0$, will be done at the step 4.

For each $n\in\mathbb{N}$, if we consider the linear transformation L_n of $\mathbb{R}^2$ defined by

$$L_n(1,0)=(1/\alpha_n,0) \quad \text{and} \quad L_n(0,1)=\big(0,1/(1-\alpha_n)\big),$$

we have:

(1) The bounded open convex domain $\mathcal{C}_n := L_n(\Omega_n)$ is contained in the triangle $\mathcal{T}_n\subset\mathbb{R}^2$ whose vertices are the points $(0,0)$, $(1/\alpha_n,0)$ and $(0,1/(1-\alpha_n))$.

(2) The points $(1,0)$, $(0,1)$, and $(1,1)$ are in $\partial\mathcal{C}_n$ and the ideal triangle Δ they define in $(\mathcal{C}_n,d_{\mathcal{C}_n})$ is equal to $L_n(\Delta_n)$.

(3) The x-axis, the y-axis, and the line passing through $(1/\alpha_n, 0)$ and $(0, 1/(1-\alpha_n))$ are tangent to $\partial\mathcal{C}_n$ at the points $(1,0)$, $(0,1)$ and $(1,1)$, respectively (see Figure 4).

For each $n \in \mathbb{N}$, the affine transformation $L_n \circ A_n$ of $\mathbb{R}^2$ induces an isometry from (ω_n, d_{ω_n}) onto the metric space $(\mathcal{C}_n, d_{\mathcal{C}_n})$. Hence, we have that $(\mathcal{C}_n, d_{\mathcal{C}_n})$ is δ-hyperbolic for all $n \in \mathbb{N}$.

Step 3: Convergence of a subsequence

All the convex domains $\mathcal{C}_n \subset \mathbb{R}^2$ contain the fixed triangle Δ and are by construction contained in the convex subset $\mathcal{B} = \{(x,y) \in \mathbb{R}^2 : x \geq 0;\ 0 \leq y \leq 2\}$. The convex $\mathcal{B}$ correspond to a properly convex set of the projective plane because it does not contain the line $\{x + y = -1\}$.

From Lemma 2.2, page 189 in [Ben03], the set of all the bounded open convex domains in the projective plane $\mathbb{P}^2$ contained in $\mathcal{B}$ and containing the image of Δ is compact for the Hausdorff distance d. Thus, there exist a proper convex domain Ω in $\mathbb{P}^2$ such that $\Omega \subset \mathcal{B}$ and a subsequence of $(\mathcal{C}_n)_{n\in\mathbb{N}}$, still denoted by $(\mathcal{C}_n)_{n\in\mathbb{N}}$, such that $d(\mathcal{C}_n, \Omega) \to 0$ as $n \to +\infty$.

Point (a) of Proposition 2.10, page 12, in Benoist [Ben03], then implies that Ω is δ-hyperbolic and strictly convex.

Note that since the points $(1,0)$, $(0,1)$, and $(1,1)$ are in $\partial\mathcal{C}_n$ for all $n \in \mathbb{N}$, they also are in $\partial\Omega$.

Step 4: The bound on the α_n

By contradiction: Suppose $\inf\{\alpha_n : n \in \mathbb{N}\} = 0$.

By considering a subsequence, we can assume that

$$\lim_{n\to+\infty} \alpha_n = 0.$$

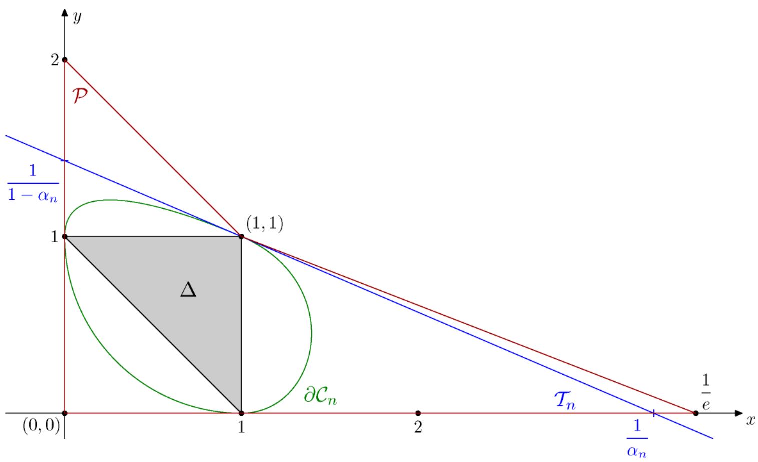

FIGURE 4. The $\mathcal{C}_n$ are convex sets with a fixed ideal triangle.

Then we have that for any $\mathcal{C}_n$, a part of its boundary is in the triangle $(0,1)$, $(1,1)$, and $(0,1/(1-\alpha_n))$. When $n \to +\infty$, the last point converges toward $(0,1)$, i.e., the triangle collapses on the segment defined by $(0,1)$ and $(1,1)$. Hence, this segment is on $\partial\Omega$, which contradicts the strict convexity of step 3.

This implies that there exists a constant $e > 0$ such that $\alpha_n \in [e, 1/2]$ for all $n \in \mathbb{N}$, and that Ω is bounded in $\mathbb{R}^2$. $\square$

PROPOSITION 10. *Let $\mathcal{C}$ be a bounded open convex domain in $\mathbb{R}^2$ such that $\partial\mathcal{C}$ is α-Hölder for some $\alpha > 1$. Then for any ideal triangle T in $(\mathcal{C}, d_{\mathcal{C}})$, $\mu_{\mathcal{C}}(T)$ is finite.*

Proof. Let T be an ideal triangle in $(\mathcal{C}, d_{\mathcal{C}})$ whose boundary $\partial\mathcal{C}$ is of regularity α-Hölder for some $\alpha > 1$. Let a, b, and c be the vertices of T. Let D_a, D_b, and D_c be the tangent at a, b, and c respectively to $\partial\mathcal{C}$. For any two points p, q in the plane, let D_{pq} be the straight line passing by p and q. Let us focus on the vertex a, and choose a system of coordinates in $\mathbb{R}^2$ such that the x-axes is the straight line D_a and the convex $\mathcal{C}$ lies in $\mathbb{R} \times [0, +\infty)$.

Then Lemma 16 implies that for ρ small enough, there is a function $f : [-\rho, \rho] \to \mathbb{R}$ and a real number $h > 0$ such that $\partial\mathcal{C} \cap ([-\rho, \rho] \times [0, h])$ is the graph of f. Now choose $a' \in D_{ab}$ and $a'' \in D_{ac}$ such that $D_{a'a''}$ is parallel to D_{bc} and $[a', a''] \subset [-\rho, \rho] \times [0, h]$. Lemma 18 implies that the area of the triangle $T_a = aa'a''$ is finite.

In the same way, we built two other triangles $bb'b''$, $cc'c''$ which are of finite area. Now the hexagon $\mathcal{H} = (a'a''b'b''c''c')$ is a compact set in $(\mathcal{C}, d_{\mathcal{C}})$, hence of finite area.

Thus, the ideal triangle T which is the union of the hexagon $\mathcal{H}$ and the triangles T_a, T_b, and T_c is of finite area. $\square$

From Benoist work, mainly Corollaire 1.5(a), page 184 in [Ben03], we know that if $(\mathcal{C}, d_{\mathcal{C}})$ is Gromov-hyperbolic, then there is some $\alpha \in]1, 2]$ such that $\partial\mathcal{C}$ is C^{α}. Hence, the corollary follows.

COROLLARY 11. *Let $\mathcal{C}$ be a bounded open convex domain in $\mathbb{R}^2$ such that $(\mathcal{C}, d_{\mathcal{C}})$ is Gromov-hyperbolic. Then for any ideal triangle T in $(\mathcal{C}, d_{\mathcal{C}})$, we have that $\mu_{\mathcal{C}}(T)$ is finite.*

2.2. Proof of Theorem 7. The proof is done by contradiction.

Assume that we can find a sequence $(\omega_n, T_n) \in T_2^{\delta}$ such that

$$\sup\{\mu_{\omega_n}(T_n) : n \in \mathbb{N}\} = +\infty$$

and prove this is not possible.

The main idea is to use the fact that G_2 acts cocompactly by isometries on the triangle-pointed convex, to transform a converging subsequence of $(\omega_n, T_n) \in T_2^{\delta}$ into a sequence of convex sets $(\mathcal{C}_n, \Delta) \in T_2^{\delta}$ evolving around a fixed ideal triangle Δ.

Then in a perfect world we would be able to find a convex set $\mathcal{C}_{\text{perfect}}$ containing Δ as an ideal triangle with finite area and included in all $\mathcal{C}_n$, and then we would get a contradiction.

Things are not that easy, but almost. Actually, we will cut Δ into 4 pieces, and for each of these pieces, we will show that there is a convex set for which it is of finite volume and included in $\mathcal{C}_n$ for all $n \in \mathbb{N}$.

Before going deeper into the proof, let us first make an overview of the different steps.

Step 1: We transform the problem to obtain a converging sequence $(\mathcal{C}_n, \Delta) \in T_2^\delta$ to (Ω, Δ), where Δ is a fixed ideal triangle, and $\mathcal{C}_n$ are convex sets tangent to two fixed lines at two of the vertices of Δ.

Step 2: In this step, we built a small convex set $\mathcal{G}_1 \subset \mathcal{C}_n$ around the vertex $(1,0)$ of Δ, which is tangent to the x-axe at $(1,0)$ and such that a sufficiently small section T_1 of Δ containing the vertex $(1,0)$ is of finite volume V_1 in $\mathcal{G}_1$.

Step 3: Reasoning as in the previous step we built a small convex set $\mathcal{G}_2 \subset \mathcal{C}_n$ around the vertex $(0,1)$ of Δ, which is tangent to the y-axe at $(0,1)$ and such that a sufficiently small section T_2 of Δ containing the vertex $(0,1)$ is of finite volume V_2 in $\mathcal{G}_2$.

Step 4: We built a small triangle A which is a section of Δ admitting the vertex $(1,1)$ as one of its vertices and whose volume is bounded by a finite number V_3 for any $\mathcal{C}_n$.

Step 5: We built a convex set $\mathcal{U}$ and a compact set S such that

(a) for all n, $\mathcal{U} \subset \mathcal{C}_n$;
(b) $\mu_{\mathcal{U}}(S) = V_4$ is finite; and
(c) $S \cup A \cup T_1 \cup T_2 = \Delta$.

We then conclude that for all n

$$\begin{aligned} \mu_{\mathcal{C}_n}(\Delta) &\le \mu_{\mathcal{C}_n}(T_1) + \mu_{\mathcal{C}_n}(T_2) + \mu_{\mathcal{C}_n}(A) + \mu_{\mathcal{C}_n}(S) \\ &\le \underbrace{\mu_{\mathcal{G}_1}(T_1)}_{\le V_1 \text{ by step 2}} + \underbrace{\mu_{\mathcal{G}_2}(T_2)}_{\le V_2 \text{ by step 3}} + \underbrace{\mu_{\mathcal{C}_n}(A)}_{\le V_3 \text{ by step 4}} + \underbrace{\mu_{\mathcal{U}}(S)}_{\le V_4 \text{ by step 5}} \\ &\le V_1 + V_2 + V_3 + V_4 < +\infty \end{aligned} \tag{1}$$

which is absurd.

Step 1: Extraction of the subsequence.

Following the proof of Proposition 9, and keeping its notations, we can find a sequence (g_n) in G_2 such that $g_n T_n = \Delta$ and $g_n \omega_n = \mathcal{C}_n$, which therefore satisfies

$$\mu_{\mathcal{C}_n}(\Delta) = \mu_{\omega_n}(T_n). \tag{2}$$

This implies that

$$\sup\{\mu_{\mathcal{C}_n}(\Delta) : n \in \mathbb{N}\} = +\infty. \tag{3}$$

Furthermore, always by Proposition 9, we have $(\mathcal{C}_n, \Delta)_{n\in\mathbb{N}}$ which converges toward some Ω. Recall that (for all $n \in \mathbb{N}$) $\mathcal{C}_n$ and Ω are tangent to the x-axe at $(1,0)$, to the y-axe at $(0,1)$, and at $(1,1)$ to some line.

Step 2: We will need the following theorem.

THEOREM 12. *Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of convex sets in $\mathbb{R}^2$ whose Hilbert geometry is δ-hyperbolic, for some fixed δ, and a straight line L. Assume that*

- *The sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ converges to some open convex set $\mathcal{D}$;*
- *There is some $p \in L$ such that for all n, $\mathcal{D}_n$ lies in the same half plane determined by L, and is tangent at p to L;*

then taking as origin the point p, as x-axe the line L, and as y-axe an orthogonal line to L,

(1) *There is a number $3a = \rho > 0$ such that for all $n \in \mathbb{N}$, there is a convex function $f_n : [-3a, 3a] \to \mathbb{R}$ and numbers $b_n > 0$ and $s_n \in \mathbb{R}$ such that*

$$\partial\mathcal{D}_n \cap \{(x,y) \in \mathbb{R}^2 : x \in [-3a, 3a] \text{ and } y < s_n x + b_n\} = \operatorname{Graph} f_n. \tag{4}$$

(2) *There is some $\mu > 0$ and $\alpha > 0$ (which will be made explicit in the proof) such that*

$$f_n(x) \leq \mu|x|^\alpha \quad \text{for all } x \in [-a, a].$$

(3) *Let*

$$m(f_n) = \min\{f_n(-a), f_n(a)\}$$

then we have $u_0 := \inf\{m(f_n) : n \in \mathbb{N}\} > 0$.

We first show how to use Theorem 12 to achieve the second step of the proof of Theorem 7.

We just have to exhibit the part T_1 of the triangle whose area will be bounded from above, independently of the δ-hyperbolic convex set $\mathcal{C}_n$ we consider.

Let

$$\mathcal{D} := \Omega + (-1, 0) \tag{5}$$

(translate of $\mathcal{C}$ by the vector $(-1,0)$) and

$$\mathcal{D}_n := \mathcal{C}_n + (-1, 0) \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

Note that since (Ω, d_Ω) is Gromov-hyperbolic, the same is true for $(\mathcal{D}, d_\mathcal{D})$.

We thus apply Theorem 12 to this sequence $\mathcal{D}_n$ in order to use Lemma 18 to see that a fixed triangle T_1 has finite area.

To do this, let us consider $a_0 := (\alpha\mu)^{\frac{1}{1-\alpha}} > 0$. The tangent line to $\{(x, \mu|x|^\alpha) : x \in \mathbb{R}\}$ at the point $(-a_0, \mu a_0^\alpha)$ is then parallel to the line $\{(x,y) \in \mathbb{R}^2 : y = -x\}$.

Define

$$u_1 := \min\{u_0, \mu a_0^\alpha\} > 0$$

and pick any $u \in (0, u_1/3]$. Applying the linear transformation of $\mathbb{R}^2$ given by

$$(x,y) \mapsto \big(-x(3u/\mu)^{-1/\alpha}, y/3u\big),$$

we are in the situation of Lemma 18 with

$$\lambda = (3u/\mu)^{1/\alpha}/3u \geq 1,$$

from which we can deduce with $\tau := 2/3 \in (0,1)$ that the triangle

$$\{(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } -x < y < 2u\}$$

is included in the bounded open convex domain

$$\{(x,y) \in \mathbb{R}^2 : \mu|x|^\alpha < 3u \text{ and } \mu|x|^\alpha < y < 3u\}$$

and has a finite Hilbert area.

So, if we consider the triangle

$$T_1(u) := \{(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } -x < y < 2u\} + (1,0)$$

and the bounded open convex domain

$$\mathcal{G}_1(u) := \{(x,y) \in \mathbb{R}^2 : \mu|x|^\alpha < 3u \text{ and } \mu|x|^\alpha < y < 3u\} + (1,0),$$

we have $T_1(u) \subset \mathcal{G}_1(u)$ and $V_1 := \mu_{\mathcal{G}_1(u)}(T_1(u))$ is finite.

In addition, since $3u < u_0$, we get that for all $n \in \mathbb{N}$, $\mathcal{G}_1(u)$ is contained in the convex set $\mathcal{C}_n$, and thus $\mu_{\mathcal{C}_n}(T_1(u)) \leq V_1$ by Proposition 14 of the Appendix.

Proof of Theorem 12. Let us postpone the proof of claim (1), and prove the other two claims.

Claim (2). First note that we have $f_n \geq 0$ and $f_n(0) = 0$ since $(0,0) \in \partial\mathcal{D}_n$. In addition, as $(\mathcal{D}_n, d_{\mathcal{D}_n})$ is δ-hyperbolic, Lemma 6.2, page 216, and Proposition 6.6, page 219, in Benoist [Ben03] implies there is a number $H = H(\delta) \geq 1$, independent of n such that f_n is H-quasi-symmetrically convex on the compact set

$$[-2a, 2a] \subset (-3a, 3a).$$

Therefore, by Lemma 21, we have for $H_2 = (4H(H+1))^{\frac{1+a}{a}}$,

$$f_n(x) \leq 160(H_2+1)M(f_n)|x|^\alpha \quad \text{for all } x \in [-a,a], \tag{7}$$

where

$$\alpha = 1 + \log_2(1 + H_2^{-1}) > 1 \tag{8}$$

and

$$M(f_n) = \max\{f_n(-a), f_n(a)\}. \tag{9}$$

We next claim that the sequence $(M(f_n))_{n \in \mathbb{N}}$ is bounded from above.

Indeed, suppose that $\sup\{M(f_n) : n \in \mathbb{N}\} = +\infty$. If

$$\pi_2 : \mathbb{R}^2 \to \mathbb{R}$$

denotes the projection onto the second factor, since $\mathcal{D}$ is bounded and included in $\mathbb{R}\times[0,+\infty)$, there is a number $R>0$ such that

$$\pi_2(\mathcal{D})\subset[0,R]$$

(this latter point is a consequence of the fact that $\mathcal{D}_n\subset\mathbb{R}\times[0,+\infty)$ for all $n\in\mathbb{N}$). Then using $d_H(\pi_2(\overline{\mathcal{D}}_n),\pi_2(\overline{\mathcal{D}}))\to 0$ as $n\to+\infty$, there is an integer $n_1\in\mathbb{N}$ such that $\pi_2(\overline{\mathcal{D}}_n)\subset[0,3R]$ for all $n\geq n_1$.

Now there exists $n\geq n_1$ such that $M(f_n)\geq 4R$, that is,

$$\pi_2(-a,f_n(-a))=f_n(-a)\geq 4R \quad\text{or}\quad \pi_2(a,f_n(a))=f_n(a)\geq 4R.$$

As the points $(-a,f_n(-a))$ and $(a,f_n(a))$ are both in $\overline{\mathcal{D}}_n$, we get that $\pi_2(\overline{\mathcal{D}}_n)\cap[4R,+\infty)\neq\emptyset$, which is not possible.

Hence, there is a constant $M>0$ such that for all $n\in\mathbb{N}$, we have $M(f_n)\leq M$, and thus

$$f_n(x)\leq\mu|x|^\alpha \quad\text{for all } x\in[-a,a],$$

where $\mu:=160(H+1)M>0$. Which proves our second claim.

Claim (3). Now, for any $n\in\mathbb{N}$, recall that

$$m(f_n)=\min\{f_n(-a),f_n(a)\}. \tag{10}$$

We have to show that

$$\inf\{f_n(-a):n\in\mathbb{N}\}\quad\text{and}\quad\inf\{f_n(a):n\in\mathbb{N}\} \tag{11}$$

are both positive numbers. So, assume that one of them, for example, the second one is equal to zero.

Therefore, there would exist a subsequence of $((a,f_n(a)))_{n\in\mathbb{N}}$ that converges to $(a,0)$, and hence $(a,0)\in\overline{\mathcal{D}}$, since

$$d((a,f_n(a)),\mathcal{D})\to 0\quad\text{as } n\to+\infty.$$

As $(0,0)\in\overline{\mathcal{D}}$, the whole line segment $\{0\}\times[0,a]$ would then be included in the convex set $\overline{\mathcal{D}}$. But $\mathcal{D}$ is included in $\mathbb{R}\times[0,+\infty)$ whose boundary contains $\{0\}\times[0,a]$.

This would imply that $\{0\}\times[0,a]\subset\partial\mathcal{D}$, and thus $\mathcal{D}$ would not be strictly convex, contradicting the Gromov hyperbolicity of $(\mathcal{D},d_{\mathcal{D}})$ by Socié–Méthou [SM00].

Claim (1). Now back to the first claim. It suffices to use the following lemma.

LEMMA 13. *There is a number $\rho>0$ such that for all $n\in\mathbb{N}$, we have*

$$\mathcal{D}_n\cap\big((-\infty,-\rho)\times\mathbb{R}\big)\neq\emptyset\quad\textit{and}\quad\mathcal{D}_n\cap\big((\rho,+\infty)\times\mathbb{R}\big)\neq\emptyset.$$

Thus, given any $n\in\mathbb{N}$, as $\mathcal{D}_n\subset\mathbb{R}\times[0,+\infty)$, it suffices to apply Lemma 16 with $a:=\rho/3$ and $\mathcal{S}=\mathcal{D}_n$ in order to get numbers $b_n>0$ and $s_n\in\mathbb{R}$ and a convex function $f_n:[-3a,3a]\to\mathbb{R}$ such that

$$\partial\mathcal{D}_n\cap\{(x,y)\in\mathbb{R}^2: x\in[-3a,3a]\text{ and } y<s_nx+b_n\}=\operatorname{Graph}f_n. \tag{12}$$

□

Proof of Lemma 13. From the Gromov hyperbolicity of $(\mathcal{D}, d_{\mathcal{D}})$, we get that the boundary $\partial\mathcal{D}$ is a 1-dimensional submanifold of $\mathbb{R}^2$ of class C^1 by Karlson–Noskov [KN02].

As $(0,0) \in \partial\mathcal{D}$, Lemma 15 then implies that $\mathcal{D}$ neither lies in $(0,+\infty) \times (0,+\infty)$, nor in $(-\infty,0) \times (0,+\infty)$.

Hence, denoting by $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first factor, $\pi_1(\mathcal{D})$ is an open set in $\mathbb{R}$ that contains 0, and thus there exists a number $r > 0$ such that

$$[-2r, 2r] \subset \pi_1(\mathcal{D}).$$

Since π_1 is continuous and

$$d_H(\mathcal{D}_n, \mathcal{D}) \to 0 \quad \text{as } n \to +\infty, \tag{13}$$

we get that

$$d_H(\pi_1(\mathcal{D}_n), \pi_1(\mathcal{D})) \to 0 \quad \text{as } n \to +\infty, \tag{14}$$

which implies there is an integer $n_0 \in \mathbb{N}$ such that for all $n > n_0$, one has

$$\pi_1(\mathcal{D}_n) \cap (-\infty, -r) \neq \emptyset \quad \text{and} \quad \pi_1(\mathcal{D}_n) \cap (r, +\infty) \neq \emptyset. \tag{15}$$

Finally, given any $n \in \{0, \dots, n_0\}$, there exists $r_n > 0$ such that $[-2r_n, 2r_n] \subset \pi_1(\mathcal{D}_n)$ by applying to $\mathcal{D}_n$ the same argument as the one used for $\mathcal{D}$ above.

But this implies that

$$\pi_1(\mathcal{D}_n) \cap (-\infty, -r_n) \neq \emptyset \quad \text{and} \quad \pi_1(\mathcal{D}_n) \cap (r_n, +\infty) \neq \emptyset. \tag{16}$$

Then choosing $\rho = \min\{r, r_0, \dots, r_{n_0}\} > 0$, Lemma 13 is proved. □

Step 3: Using the translation by the vector $(0,-1)$ and reasoning as in Step 1 with x and y exchanged, we get numbers $\beta > 1$, $\nu > 0$, $b_0 > 0$ and $0 < v_1 \leq \nu b_0^\beta$ such that the following holds:

(1) The tangent line to $\{(\nu|y|^\beta, y) : y \in \mathbb{R}\}$ at the point $(\nu b_0^\beta, b_0)$ is parallel to the line $\{(x,y) \in \mathbb{R}^2 : y = -x\}$.

(2) For each $v \in (0, v_1/3]$, the triangle

$$T_2(v) := \{(x,y) \in \mathbb{R}^2 : y < 0 \text{ and } -y < x < 2v\} + (0,1)$$

and the bounded open convex domain

$$\mathcal{G}_2(v) := \{(x,y) \in \mathbb{R}^2 : \nu|y|^\beta < 3v \text{ and } \nu|y|^\alpha < x < 3v\} + (0,1),$$

satisfy

$$T_2(v) \subset \mathcal{G}_2(v) \subset \mathcal{C}_n$$

for all $n \in \mathbb{N}$ and

$$V_2 := \mu_{\mathcal{G}_2(v)}(T_2(v))$$

is finite.

Therefore, we deduce that $\mu_{\mathcal{C}_n}(T_2(v)) \le V_2$ for all $n \in \mathbb{N}$.

Step 4: The geometric idea is similar to the two precedent steps. The only difficulty is that the tangent line to $\mathcal{C}_n$ is not always the same, and we have to be sure to make a uniform choice.

For each $n \in \mathbb{N}$, consider the affine transformation Φ_n of $\mathbb{R}^2$ defined by

$$\Phi_n(1,1) = (0,0), \qquad \Phi_n(\alpha_n - 1, \alpha_n) = (1,0),$$
$$\Phi_n(-\alpha_n, \alpha_n - 1) = (0,1).$$

As

$$\Phi_n\big((1,1) + \mathbb{R}_+(-1,0)\big) = \mathbb{R}_+(1-\alpha_n, \alpha_n)$$

and

$$\Phi_n\big((1,1) + \mathbb{R}_+(0,-1)\big) = \mathbb{R}_+(-\alpha_n, 1-\alpha_n),$$

since $\alpha_n \in [e, 1/2]$ (see Proposition 9), we also have

$$\Phi_n(\Delta) \subset \{(x,y) \in \mathbb{R}^2 : y \ge e : |x|/(1-e)\}. \tag{17}$$

Then applying Theorem 12 to $\Phi_n(\mathcal{C}_n) \subset \mathbb{R} \times [0, +\infty)$, we get numbers $c > 0$, $\gamma > 1$ and $\kappa > 0$ and a convex function $g_n : [-c, c] \to \mathbb{R}$ such that

$$g_n(x) \le \kappa |x|^\gamma \quad \text{for all } x \in [-c, c].$$

Next, as in claim (12) of Theorem 12 in Step 2, there exists a constant $w_0 > 0$ such that for all $n \in \mathbb{N}$, we have both

$$g_n(-c) \ge w_0 \quad \text{and} \quad g_n(c) \ge w_0.$$

Let

$$c_0 := \big(e/\big(\kappa(1-e)\big)\big)^{1/(\gamma-1)} > 0. \tag{18}$$

The point $(c_0, \kappa c_0^\gamma)$ is then the intersection point between the curve

$$\{(x, \kappa|x|^\gamma) : x \in \mathbb{R}\}$$

and the half line

$$\{(x,y) \in \mathbb{R}^2 : y = ex/(1-e), x \ge 0\}.$$

Define

$$w_1 := \min\{w_0, \kappa c_0^\gamma\} > 0 \tag{19}$$

and pick any $w \in (0, w_1/4]$.

Applying the linear transformation of $\mathbb{R}^2$ given by

$$(x,y) \mapsto \big(-x(4w/\kappa)^{-1/\gamma}, y/4w\big),$$

we are in the situation of Lemma 18 with

$$\lambda = \frac{e}{1-e}(4w/\kappa)^{1/\gamma}/4w.$$

If $\lambda \geq 1$, it is an immediate application of Lemma 18. If $\lambda < 1$, we have to do a new linear transformation given by

$$(x, y) \mapsto (\alpha x, \alpha^\gamma y)$$

with $\alpha = \lambda^{\frac{1}{1-\gamma}}$, which allow to be in situation of Lemma 18 with $\lambda = 1$.

From this, we can deduce with $\tau := 3/4 \in (0,1)$ that the triangle

$$\{(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } -x < y < 3w\}$$

is included in the bounded open convex domain

$$\{(x,y) \in \mathbb{R}^2 : \kappa|x|^\gamma < 4w \text{ and } \kappa|x|^\gamma < y < 4w\}$$

and has a finite Hilbert area, we denote by V_3.

So, for every $n \in \mathbb{N}$, if we consider the triangle

$$A_n(w) := \Phi_n^{-1}\big(\{(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } -x < y < 3w\}\big)$$

and the bounded open convex domain

$$\mathcal{G}_n(w) := \Phi_n^{-1}\big(\{(x,y) \in \mathbb{R}^2 : \mu|x|^\alpha < 4w \text{ and } \mu|x|^\alpha < y < 4w\}\big),$$

we have

$$A_n(w) \subset \mathcal{G}_n(w) \quad \text{and} \quad \mu_{\mathcal{G}_n(w)}(A_n(w)) = V_3.$$

In addition, since $4w < w_0$, we get that for all $n \in \mathbb{N}$, $\mathcal{G}_n(w)$ is contained in the convex set $\mathcal{C}_n$, and thus $\mu_{\mathcal{C}_n}(A_n(w)) \leq V_3$ by Proposition 14.

Now, fix $n \in \mathbb{N}$.

The edge of the triangle

$$\{(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } -x < y < 3w\}$$

that does not contain $(0,0)$ lies in the line $\ell := (0, 3w) + \mathbb{R}(1,0)$. Hence, the edge of the triangle $A_n(w)$ that does not contain $(1,1)$ lies in the line

$$\begin{aligned} \ell_n &:= \Phi_n^{-1}(\ell) = \Phi_n^{-1}(0, 3w) + \mathbb{R}(\Phi_n)^{-1}(1,0) \\ &= \big(1 - 3\alpha_n w, 1 + 3(\alpha_n - 1)w\big) + \mathbb{R}(\alpha_n - 1, \alpha_n). \end{aligned} \tag{20}$$

The x-coordinate x_n of the intersection point between ℓ_n and the line

$$\{(x,y) \in \mathbb{R}^2 : y = 1\}$$

is then equal to

$$x_n = 1 - 3\alpha_n w + s(\alpha_n - 1)$$

with $s = 3(1 - \alpha_n)w/\alpha_n$. From $\alpha_n \in [e, 1/2]$, we get that $s > 0$, and thus

$$x_n < 1 - 3\alpha_n w < 1 - 3ew. \tag{21}$$

On the other hand, the y-coordinate y_n of the intersection point between ℓ_n and the line

$$\{(x,y) \in \mathbb{R}^2 : x = 1\}$$

is equal to

$$y_n = 1 + 3(\alpha_n - 1)w + t\alpha_n$$

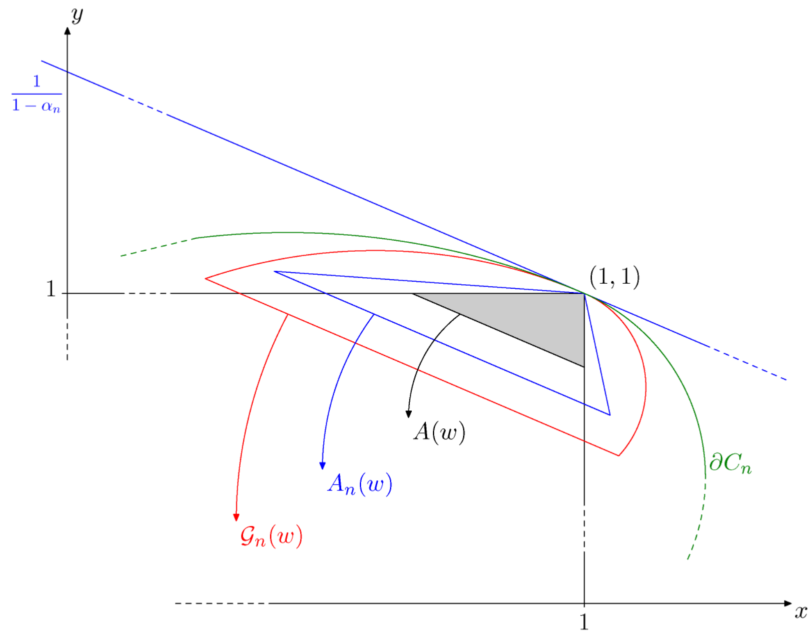

FIGURE 5. The triangle $A(w)$.

with $t = 3\alpha_n w/(\alpha_n - 1)$. As $\alpha_n \in [e, 1/2]$, we have $t < 0$, and thus

$$y_n < 1 + 3(\alpha_n - 1)w < 1 - w < 1 - 3ew. \tag{22}$$

Using equation (17), this proves that the fixed triangle $A(w)$ whose vertices are the points $(1,1)$, $(1-3ew,1)$, and $(1,1-3ew)$ is included in the triangle $A_n(w)$ (see Figure 5).

Conclusion: By Proposition 14, $\mu_{\mathcal{C}_n}(A(w)) \leq V_3$ for all $n \in \mathbb{N}$.

*Step 5:*Let us introduce the points

$$p_3 := (1, 1-ew), \qquad p_4 := (1-ew, 1),$$

by step 4, p_3 and p_4 are in $A(w)$ (see Figure 6).

Now consider $\mathcal{U}$ the convex hull of $\mathcal{G}_1(u) \cup \mathcal{G}_2(v) \cup \{p_3, p_4\}$, recall that for all $n \in \mathbb{N}$ we have

$$\begin{aligned} \mathcal{G}_1(u) \subset \mathcal{C}_n, &\quad \text{by step 2;} \\ \mathcal{G}_2(v) \subset \mathcal{C}_n, &\quad \text{by step 3;} \\ p_3, p_4 \in A(w) \subset \mathcal{C}_n, &\quad \text{by step 4,} \end{aligned}$$

hence, the fixed bounded open convex domain $\mathcal{U}$ is included in the convex set $\mathcal{C}_n$, for all $n \in \mathbb{N}$.

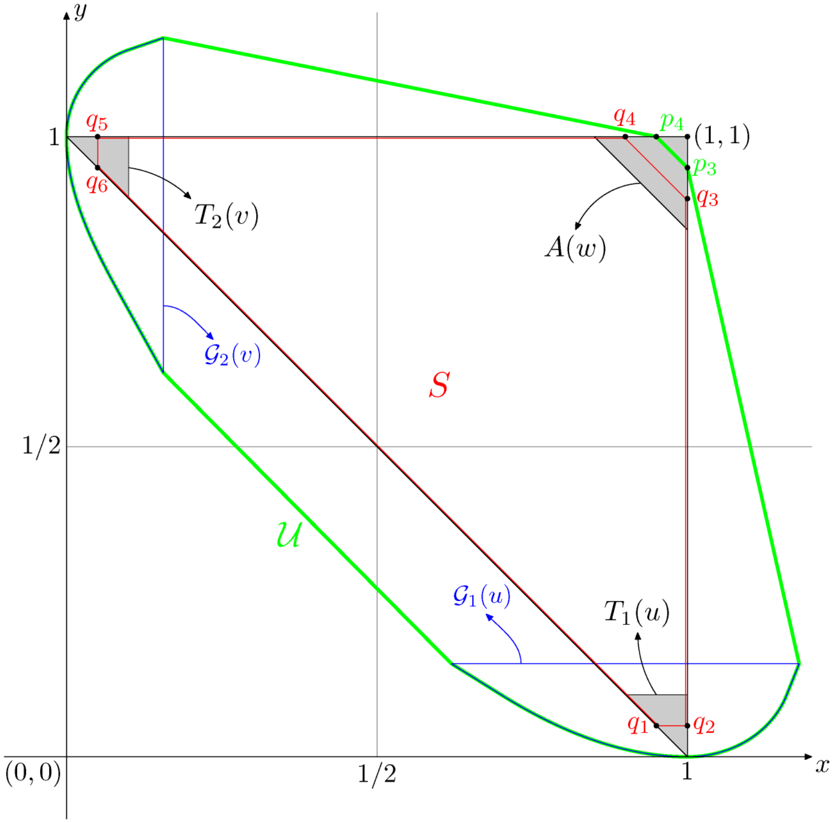

FIGURE 6. The convex set $\mathcal{U}$ and the compact set S.

Now, if S is the closed convex hull in $\mathbb{R}^2$ of the points (see Figure 6)

$$\begin{aligned} &q_1 := (1-u, u), && q_2 := (1, u), && q_3 := (1, 1-2ew), \\ &q_4 := (1-2ew, 1), && q_5 := (v, 1), && q_6 := (v, 1-v), \end{aligned}$$

we have $S \in \mathcal{U}$, and thus $V_4 := \mu_{\mathcal{U}}(S)$ is finite, since S is compact.

This gives that $S \subset \mathcal{C}_n$ with $\mu_{\mathcal{C}_n}(S) \leq V_4$ for all $n \in \mathbb{N}$.

Finally, as $\Delta \subset T_1(u) \cup T_2(v) \cup A(w) \cup S$, we get

$$\begin{aligned} \mu_{\mathcal{C}_n}(\Delta) &\leq \mu_{\mathcal{C}_n}(T_1(u)) + \mu_{\mathcal{C}_n}(T_2(v)) \\ &\quad + \mu_{\mathcal{C}_n}(A(w)) + \mu_{\mathcal{C}_n}(S) \\ &\leq V_1 + V_2 + V_3 + V_4 =: V < +\infty. \end{aligned} \tag{23}$$

But this is in contradiction with the assumption

$$\sup\{\mu_{\mathcal{C}_n}(\Delta) : n \in \mathbb{N}\} = +\infty,$$

hence, Theorem 7 is proved.

Appendix. Technical lemmas

We recall, without proof, Proposition 5 in [CVV04], page 208.

PROPOSITION 14. *Let $(\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{B}, d_{\mathcal{B}})$ be two Hilbert's geometries such that $\mathcal{A} \subset \mathcal{B} \subset \mathbb{R}^n$. Then*

(1) *The Finsler metrics $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ satisfy $F_{\mathcal{B}}(p,v) \leq F_{\mathcal{A}}(p,v)$ for all $p \in \mathcal{A}$ and all none null $v \in \mathbb{R}^n$ with equality, if and only if $p^-_{\mathcal{A}} = p^-_{\mathcal{B}}$ and $p^+_{\mathcal{A}} = p^+_{\mathcal{B}}$.*
(2) *If $p,q \in \mathcal{A}$, we have $d_{\mathcal{B}}(p,q) \leq d_{\mathcal{A}}(p,q)$.*
(3) *For all $p \in \mathcal{A}$, we have $\mu(B_{\mathcal{A}}(p)) \leq \mu(B_{\mathcal{B}}(p))$ with equality, if, and only if $\mathcal{A} = \mathcal{B}$.*
(4) *For any Borel set A in $\mathcal{A}$, we have $\mu_{\mathcal{B}}(A) \leq \mu_{\mathcal{A}}(A)$ with equality, if and only if $\mathcal{A} = \mathcal{B}$.*

LEMMA 15. *Fix $s \in \mathbb{R}$ and consider the half-closed cone $C = \{(x,y) \in \mathbb{R}^2 : x \geq 0$ and $y \leq sx\}$ in $\mathbb{R}^2$. Then we have:*

(1) *For any $\varepsilon > 0$ and any parameterized curve $\sigma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ that is differentiable at $t = 0$, if $\sigma(0) = (0,0)$ and $\sigma((-\varepsilon,\varepsilon)) \subset C$, then $\sigma'(0) = (0,0)$.*
(2) *For any 1-dimensional topological submanifold Γ of $\mathbb{R}^2$, if $(0,0) \in \Gamma$ and $\Gamma \subset C$, then Γ is not a differentiable submanifold of $\mathbb{R}^2$ at $(0,0)$.*

Proof. Point 1: Let $\sigma(t) = (x(t), y(t))$ for all $t \in (-\varepsilon,\varepsilon)$. As $x(t) \geq 0 = x(0)$ for all $t \in (-\varepsilon,\varepsilon)$, the function $x : (-\varepsilon,\varepsilon) \to \mathbb{R}$ has a local minimum at $t = 0$, and thus $x'(0) = 0$.

On the other hand, for all $t \in (-\varepsilon,\varepsilon)$, we have $y(t) \leq sx(t)$, or equivalently $y(t) - sx(t) \leq 0 = y(0) - sx(0)$. This shows that the function $y - sx : (-\varepsilon,\varepsilon) \to \mathbb{R}$ has a local minimum at $t = 0$, and thus $(y - sx)'(0) = 0$. But $x'(0) = 0$, and hence $y'(0) = 0$, which proves the first point of the lemma.

Point 2: Assume that Γ is a 1-dimensional differentiable submanifold of $\mathbb{R}^2$ at $(0,0)$.

Then we can find open sets U and V in $\mathbb{R}^2$ that contain $(0,0)$ together with a diffeomorphism $\Phi : U \to V$ satisfying $\Phi(U \cap \Gamma) = V \cap (\mathbb{R} \times \{0\})$ and $\Phi(0,0) = (0,0)$.

Let $\varepsilon > 0$ such that $(-\varepsilon,\varepsilon) \times \{0\} \subset V \cap (\mathbb{R} \times \{0\})$, and consider the parameterized curve $\sigma : (-\varepsilon,\varepsilon) \to \mathbb{R}^2$ defined by $\sigma(t) = \Phi^{-1}(t,0)$.

As σ is differentiable at $t = 0$ and satisfies $\sigma((-\varepsilon,\varepsilon)) \subset U \cap \Gamma \subset \Gamma \subset C$ and $\sigma(0) = (0,0)$, we get from Point (1) above that $\sigma'(0) = (0,0)$, which implies that $(\Phi \circ \sigma)'(0) = (0,0)$ by the chain rule.

But a direct calculation gives $(\Phi \circ \sigma)(t) = (t,0)$ for all $t \in (-\varepsilon,\varepsilon)$, and hence $(\Phi \circ \sigma)'(0) = (1,0) \neq (0,0)$.

So Γ cannot be a 1-dimensional differentiable submanifold of $\mathbb{R}^2$ at $(0,0)$, proving the second point of the lemma. ∎

LEMMA 16. *Let* $\rho > 0$ *and* $\mathcal{S}$ *be an open convex domain in* $\mathbb{R}^2$ *that lies in* $\mathbb{R} \times (0, +\infty)$.

If $\mathcal{S} \cap ((-\infty, -\rho) \times \mathbb{R}) \neq \emptyset$ *and* $\mathcal{S} \cap ((\rho, +\infty) \times \mathbb{R}) \neq \emptyset$, *then there exist* $s \in \mathbb{R}$, $b > 0$ *and a function* $f : [-\rho, \rho] \to \mathbb{R}$ *such that*

$$\partial\mathcal{S} \cap \{(x, y) \in \mathbb{R}^2 : x \in [-\rho, \rho] \text{ and } y < sx + b\} = \operatorname{Graph} f.$$

Proof. Pick

$$p_0 = (x_0, y_0) \in \mathcal{S} \cap \big((-\infty, -\rho) \times \mathbb{R}\big)$$

and

$$p_1 = (x_1, y_1) \in \mathcal{S} \cap \big((\rho, +\infty) \times \mathbb{R}\big).$$

The closed line segment L with vertices p_0 and p_1 then lies in the convex set $\mathcal{S}$. Denoting by $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first factor, we thus get $[-\rho, \rho] \subset [x_0, x_1] = \pi_1(L) \subset \pi_1(\mathcal{S})$. Since $\mathcal{S} \subset \mathbb{R} \times (0, +\infty)$, this allows us to consider the function $f : [-\rho, \rho] \to \mathbb{R}$ defined by

$$f(x) = \inf\{y \geq 0 : (x, y) \in \overline{\mathcal{S}}\}.$$

Fix $x \in [-\rho, \rho]$.

Given any $z \geq 0$ such that $(x, z) \in \overline{\mathcal{S}}$, we have by compactness

$$f(x) = \inf\{y \in [0, z] : (x, y) \in \overline{\mathcal{S}}\} = \min\{y \in [0, z] : (x, y) \in \overline{\mathcal{S}}\},$$

and thus $(x, f(x)) \in \overline{\mathcal{S}}$.

If $(x, f(x))$ were in $\mathcal{S}$, there would exist $\varepsilon > 0$ such that

$$[x - \varepsilon, x + \varepsilon] \times [f(x) - \varepsilon, f(x) + \varepsilon] \subset \mathcal{S} \subset \mathbb{R} \times [0, +\infty),$$

and thus we would get $f(x) - \varepsilon \in \{y \geq 0 : (x, y) \in \overline{\mathcal{S}}\}$. But this contradicts the very definition of $f(x)$. Therefore, we have $(x, f(x)) \in \partial\mathcal{S}$.

Now let $s = (y_1 - y_0)/(x_1 - x_0)$ and $b = (x_1 y_0 - x_0 y_1)/(x_1 - x_0) > 0$. The equation of the straight line containing L is then $y = sx + b$.

Since for all $x \in [-\rho, \rho]$, the point $(x, sx + b) \in L \subset \overline{\mathcal{S}}$, we get $f(x) \leq sx + b$ from the definition of f. As $(x, f(x)) \in \partial\mathcal{S}$ and $L \cap \partial\mathcal{S} = \emptyset$, we also have $f(x) \neq sx + b$. Hence,

$$\operatorname{Graph} f \subset \partial\mathcal{S} \cap \{(x, y) \in \mathbb{R}^2 : x \in [-\rho, \rho] \text{ and } y < sx + b\}.$$

On the other hand, for any given $(x, z) \in \partial\mathcal{S} \cap \{(x, y) \in \mathbb{R}^2 : x \in [-\rho, \rho]$ and $y < sx + b\}$, assume there is $y \geq 0$ with $(x, y) \in \overline{\mathcal{S}}$ satisfying $y < z$. Then (x, z) is in the triangle whose vertices are p_0, p_1 and (x, y), which is not possible since this triangle lies in $\mathcal{S}$ (the interior of the closure of a convex set in $\mathbb{R}^n$ is equal to the interior of that convex set in $\mathbb{R}^n$) and $(x, z) \in \partial\mathcal{S}$. Therefore, $z \leq y$, which shows that $z = f(x)$ by the definition of f. This proves that

$$\partial\mathcal{S} \cap \{(x, y) \in \mathbb{R}^2 : x \in [-\rho, \rho] \text{ and } y < sx + b\} \subset \operatorname{Graph} f. \qquad \square$$

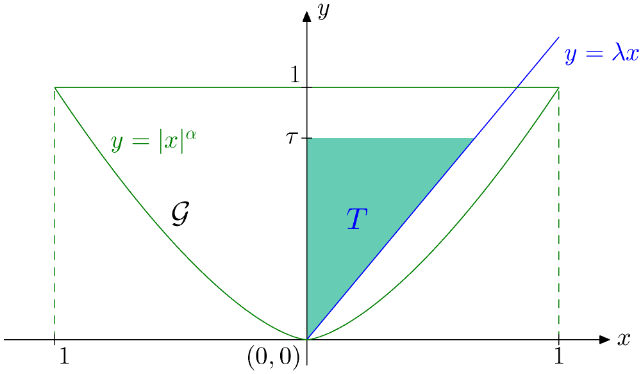

FIGURE 7. Lemma 18 illustrated.

REMARK 17. The function f obtained in Lemma 16 satisfies $f \geq 0$ and is automatically convex since its epigraph is equal to the convex set in $\mathbb{R}^2$ equal to the union of the convex set

$$\overline{\mathcal{S}} \cap \{(x,y) \in \mathbb{R}^2 : x \in [-\rho, \rho] \text{ and } y < sx + b\} \subset \mathbb{R}^2$$

(intersection of two convex sets in $\mathbb{R}^2$) and the convex set

$$\{(x,y) \in \mathbb{R}^2 : x \in [-\rho, \rho] \text{ and } y \geq sx + b\} \subset \mathbb{R}^2.$$

LEMMA 18. *Let $\alpha > 1$, $\lambda \geq 1$ and $\tau \in (0,1)$. Consider the bounded open convex domain*

$$\mathcal{G} = \{(x,y) \in \mathbb{R}^2 : \ -1 < x < 1 \ \text{and} \ |x|^{\alpha} < y < 1\} \tag{24}$$

and the triangle

$$T = \{(x,y) \in \mathbb{R}^2 : x > 0 \ \text{and} \ \lambda x < y < \tau\}. \tag{25}$$

Then we have $T \subset \mathcal{G}$ and the area $\mu_{\mathcal{G}}(T)$ is finite (see Figure 7).

Proof. Step 1: For each $p = (x,y) \in T$, let $B_{\mathcal{G}}(p) = \{v \in \mathbb{R}^2 : F_{\mathcal{G}}(p,v) < 1\}$ be the open unit ball in $T_p\mathcal{G} = \mathbb{R}^2$ of the norm $F_{\mathcal{G}}(p,\cdot)$.

An easy computation shows that the vectors

$$v_1 = \big((y^{2/\alpha} - x^2)/y^{1/\alpha}, 0\big) \quad \text{and} \quad v_2 = \big(0, 2(1-y)(y-x^{\alpha})/(1-x^{\alpha})\big)$$

are in the boundary $\partial B_{\mathcal{G}}(p)$ of $B_{\mathcal{G}}(p)$.

As $B_{\mathcal{G}}(p)$ is convex and symmetric about $(0,0)$ in $T_p\mathcal{G} = \mathbb{R}^2$, we get that the rhombus defined as the convex hull of v_1, v_2, $-v_1$ and $-v_2$ is included in the closure of $B_{\mathcal{G}}(p)$ in $T_p\mathcal{G} = \mathbb{R}^2$.

Therefore, the Euclidian volume of this rhombus is less than or equal to that of $B_{\mathcal{G}}(p)$, which writes

$$\operatorname{Vol}(B_{\mathcal{G}}(p)) \geq 4\frac{(1-y)(y-x^\alpha)(y^{2/\alpha}-x^2)}{y^{1/\alpha}(1-x^\alpha)}.$$

Since $1-x^\alpha \leq 1$ and $1-y \geq 1-\tau$, we then deduce

$$\operatorname{Vol}(B_{\mathcal{G}}(p)) \geq 4(1-\tau)\frac{(y-x^\alpha)(y^{2/\alpha}-x^2)}{y^{1/\alpha}}.$$

Step 2: From the inequality obtained in step 1, we have

$$\mu_{\mathcal{G}}(T) = \pi\int\int_T \frac{\mathrm{d}x\,\mathrm{d}y}{\operatorname{Vol} B_{\mathcal{G}}(p)} \leq \frac{\pi}{4(1-\tau)} I, \tag{26}$$

where

$$I := \int\int_T \frac{y^{1/\alpha}\mathrm{d}x\,\mathrm{d}y}{(y-x^\alpha)(y^{2/\alpha}-x^2)}.$$

Now, using the change of variables $\Phi : (0,+\infty)^2 \to (0,+\infty)^2$ defined by

$$(s,t) = \Phi(x,y) := (x/y^{1/\alpha}, x),$$

whose Jacobian at any $(x,y) \in (0,+\infty)^2$ is equal to $x/(\alpha y^{1+1/\alpha})$, we get

$$I = \alpha\int\int_{\Phi(T)} \frac{\mathrm{d}s\,\mathrm{d}t}{t(1-s^\alpha)(1-s^2)}$$

with

$$\begin{aligned}\Phi(T) = \{&(s,t) \in \mathbb{R}^2 : 0 < t < \tau/\lambda \\ &\text{and } t\cdot\tau^{-1/\alpha} < s < \lambda^{-1/\alpha}\cdot t^{1-1/\alpha}\}.\end{aligned}$$

So,

$$\begin{aligned} I &= \alpha\int_0^{\frac{\tau}{\lambda}} \frac{1}{t}\left(\int_{\tau^{-1/\alpha}t}^{\lambda^{-1/\alpha}t^{1-1/\alpha}} \frac{1}{(1-s^\alpha)(1-s^2)}\,\mathrm{d}s\right)\mathrm{d}t \\ &\leq \alpha\int_0^{\frac{\tau}{\lambda}} \frac{1}{t}\left(\int_{\tau^{-1/\alpha}t}^{\lambda^{-1/\alpha}t^{1-1/\alpha}} \frac{1}{(1-\tau^{\alpha-1}\lambda^{-\alpha})(1-\tau^{2-2/\alpha}\lambda^{-2})}\,\mathrm{d}s\right)\mathrm{d}t, \end{aligned}$$

since $(s,t) \in \Phi(T)$ implies

$$\begin{aligned} &1-s^\alpha \geq 1-t^{\alpha-1}/\lambda \geq \tau^{\alpha-1}\lambda^{-\alpha} \quad \text{and} \\ &1-s^2 \geq 1-\lambda^{-2/\alpha}t^{2-2/\alpha} \geq \tau^{2-2/\alpha}\lambda^{-2}. \end{aligned}$$

Therefore, one has

$$\begin{aligned} I &\leq \alpha\int_0^{\tau/\lambda} \frac{\lambda^{-1/\alpha}t^{-1/\alpha}-\tau^{-1/\alpha}}{(1-\tau^{\alpha-1}\lambda^{-\alpha})(1-\tau^{2-2/\alpha}\lambda^{-2})}\,\mathrm{d}t \\ &\leq \Lambda\int_0^{\tau/\lambda} \frac{1}{t^{1/\alpha}}\,\mathrm{d}t, \end{aligned}$$

where $\Lambda := \frac{\alpha\lambda^{-1/\alpha}}{(1-\tau^{\alpha-1}\lambda^{-\alpha})(1-\tau^{2-2/\alpha}\lambda^{-2})}$.

Since $1/\alpha < 1$, this shows that $I < +\infty$, and equation (26) proves the lemma. $\square$

DEFINITION 19. Given a number $K \geq 1$ and an interval $I \subset \mathbb{R}$, a function $f : I \to \mathbb{R}$ is said to be K-quasi-symmetric if and only if one has:

$$\begin{aligned} &\forall x \in I,\ \forall h \in \mathbb{R}\ (x+h \in I \text{ and } x-h \in I) \\ &\qquad \Longrightarrow \quad |f(x+h) - f(x)| \leq K|f(x) - f(x-h)|. \end{aligned} \tag{27}$$

DEFINITION 20. Given a number $H \geq 1$ and an interval $I \subset \mathbb{R}$, a function $f : I \to \mathbb{R}$ is said to be H-quasi-symmetrically convex if and only if it is convex, differentiable and has the following property:

$$\begin{aligned} &\forall x \in I,\ \forall h \in \mathbb{R}\ (x+h \in I \text{ and } x-h \in I) \\ &\qquad \Longrightarrow \quad D_x(h) \leq H D_x(-h), \end{aligned} \tag{28}$$

where

$$D_x(h) := f(x+h) - f(x) - f'(x)h.$$

LEMMA 21. *Let $a > 0$, $H \geq 1$ and $f : [-2a, 2a] \to \mathbb{R}$ a H-quasi-symmetrically convex function that satisfies $f \geq 0$ and $f(0) = 0$. Define*

$$H_2 = \big(4H(H+1)\big)^{\frac{1+a}{a}} > 1 \tag{29}$$

and

$$\alpha = 1 + \log_2(1 + H_2^{-1}) > 1 \tag{30}$$

and $M(f) = \max\{f(-a), f(a)\}$. Then we have

$$f(x) \leq 160(H_2 + 1)M(f)|x|^\alpha \quad \textit{for all } x \in [-a, a]. \tag{31}$$

Before proving this lemma, recall the two following results due to Benoist [Ben03].

LEMMA 22 ([Ben03], Lemma 5.3(b), page 204). *Let $a > 0$, $H \geq 1$ and $f : [-2a, 2a] \to \mathbb{R}$ a H-quasi-symmetrically convex function. Then the restriction of the derivative f' to $[-a, a]$ is K-quasi-symmetric, where $K = (4H(H+1))^{\frac{1+a}{a}} \geq 1$.*

LEMMA 23 ([Ben03], Lemma 4.9(a), page 203). *Let $a > 0$, $K \geq 1$ and $f : [-a, a] \to \mathbb{R}$ a differentiable convex function.*

If the derivative f' is K-quasi-symmetric, then for all $x, y \in [-a, a]$, we have

$$|f'(x) - f'(y)| \leq 160(1 + K)\|f\|_\infty |x - y|^{\alpha - 1},$$

where $\alpha = 1 + \log_2\big(1 + 1/K\big) > 1$.

Proof of Lemma 21. Using Lemma 22, we have that the derivative f' of f is K-quasi-symmetric when restricted to $[-a,a]$. But then according to Lemma 23 with $y=0$ and the fact that $f'(0)=0$, since 0 is the minimum of f, we get that

$$|f'(x)| \leq 160(1+K) \max_{t\in[-a,a]} |f(t)||x|^{\alpha-1} \quad \text{for all } x \in [-a,a].$$

Now, the convexity of f implies that f' is a nondecreasing function. As $f'(0)=0$, we have that $f'(x)\leq 0$ for all $x\in[-a,0]$ and $f'(x)\geq 0$ for all $x\in[0,a]$. Hence, f is a function that is nonincreasing on $[-a,0]$ and nondecreasing on $[0,a]$, which yields to $\max_{t\in[-a,a]}|f(t)| = M(f)$ and

$$|f'(x)| \leq 160(1+K)M(f)|x|^{\alpha-1} \quad \text{for all } x \in [-a,a]. \tag{32}$$

Choosing an arbitrary $u\in[-a,a]$ and applying Taylor's theorem to f between 0 and u, we get the existence of $\vartheta\in(0,1)$ such that

$$f'(\vartheta u)u = f(u) - f(0) = f(u).$$

Therefore, plugging $x=\vartheta u\in[-a,a]$ in equation (32) and multiplying by $|u|$, one has

$$\begin{aligned} |f(u)| = |f'(\vartheta u)||u| &\leq 160(1+K)M(f)|\vartheta u|^{\alpha-1}|u| \\ &\leq 160(1+K)M(f)|u|^{\alpha} \end{aligned} \tag{33}$$

since $|\vartheta u|\leq|u|$. This proves Lemma 21. □

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