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PROPERTY (P) AND STEIN NEIGHBORHOOD BASES ON C^1 DOMAINS

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ABSTRACT. Let Ω be a bounded domain in $\mathbb{C}^n$ satisfying Catlin's Property (P). Sibony has shown that $\overline{\Omega}$ possesses a Stein neighborhood basis when the boundary is of class C^3. In this paper, we use an alternative characterization of such domains to show that Sibony's result holds when the boundary is of class C^1.

1. Introduction

Let Ω be a pseudo-convex domain in $\mathbb{C}^n$, $n \geq 2$. We say that $\overline{\Omega}$ has a Stein neighborhood basis if for every open set U containing $\overline{\Omega}$ there exists a pseudo-convex domain Ω_U such that $\overline{\Omega} \subset \Omega_U \subset U$ (see [11] for the necessary background on pseudo-convex domains and Stein manifolds). Diederich and Fornaess have shown in [6] that there exist smooth bounded pseudo-convex domains with no Stein neighborhood basis, even though all pseudo-convex domains can be exhausted from within by strictly pseudo-convex domains.

Catlin's Property (P) was introduced in [2] as a sufficient condition for compactness of the $\overline{\partial}$-Neumann operator (an overview of the $\overline{\partial}$-Neumann problem can be found in [1] with more detailed accounts in [8] or [3]; see [9] for details on compactness). A pseudo-convex domain Ω satisfies Property (P) if for every $M > 0$ there exists a smooth plurisubharmonic function λ on $\overline{\Omega}$ such that $0 \leq \lambda \leq 1$ on $\overline{\Omega}$ and $i\partial\overline{\partial}\lambda \geq iM\partial\overline{\partial}|z|^2$ on $\partial\Omega$. As demonstrated in [15], Property (P) implies compactness even if λ is not smooth.

In [14], Sibony shows that Property (P) also implies the existence of a Stein neighborhood basis on all C^3 pseudo-convex domains. In fact, Sibony demonstrates that Property (P) implies the stronger condition of uniform H-convexity (as defined by Cirka in [4]). Let $\delta(z)$ denote the geodesic distance from z to $\partial\Omega$. We say that $\partial\Omega$ is uniformly H-convex if there exist constants

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$b > a > 0$ such that for every $\varepsilon > 0$ there is a pseudo-convex domain Ω_ε containing $\partial\Omega$ such that $b\varepsilon > \delta(z) > a\varepsilon$ for all $z \in \partial\Omega_\varepsilon$. Sibony's argument also works for McNeal's more general Property $(\tilde{P})$ [12] (see [13] for a more general sufficient condition).

Our goal is to demonstrate that Property (P) implies uniform H-convexity even when the boundary is only of class C^1. To that end, we will use the equivalent formulation of Property (P) proved in [10]. Let Ω be a bounded Lipschitz domain in $\mathbb{C}^n$ satisfying Property (P). Then there exists a function $\rho \in C(\overline{\Omega})$ satisfying $\frac{1}{c}\delta(z) < -\rho(z) < c\delta(z)$ in $\overline{\Omega}$ for some $c > 1$, and $i\partial\overline{\partial}(-\log(-\rho)) \geq i\phi(-\rho)\partial\overline{\partial}|z|^2$ for some positive function $\phi \in C(0,\infty)$ satisfying $\lim_{x\to 0^+}\phi(x) = \infty$. In effect, ρ provides an exhaustion of Ω by strictly pseudo-convex domains with Levi forms that decay at a singular rate. In this paper, we will show that such an exhaustion can be "pushed out" of Ω to provide a Stein neighborhood basis. Our main theorem is as follows.

THEOREM 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a pseudo-convex domain with C^1 boundary such that for every $p \in \partial\Omega$ there exists an open neighborhood $p \in U$ and a function $\rho \in C(U \cap \Omega)$ satisfying:*

$$\frac{1}{c}\delta(z) < -\rho(z) < c\delta(z)$$

for some $c > 1$ and:

$$i\partial\overline{\partial}(-\log(-\rho)) \geq i\phi(-\rho)\partial\overline{\partial}|z|^2$$

in $U \cap \Omega$ for some positive function $\phi \in C(0,\infty)$ such that $\lim_{x\to 0^+}\phi(x) = \infty$. Then $\partial\Omega$ is uniformly H-convex, and hence $\overline{\Omega}$ has a Stein neighborhood basis.

The existence of Stein neighborhood bases is particularly relevant on domains with low boundary regularity. For example, uniform H-convexity can be used to solve the inhomogeneous Cauchy–Riemann equation $\overline{\partial}u = f$ in $C^\infty_{(p,q)}(\overline{\Omega})$ when $\overline{\partial}f = 0$ (as shown in Dufresnoy [7]). On smooth domains this result can be obtained from Kohn's method using the weighted $\overline{\partial}$-Neumann operator [8], but this approach will not work on C^1 domains. Hence, the above theorem implies that Property (P) suffices for solvability of the $\overline{\partial}$ operator in $C^\infty_{(p,q)}(\overline{\Omega})$ when $\partial\Omega$ is only C^1. See [16] for applications of stronger conditions on Stein neighborhood bases to the $\overline{\partial}$-Neumann problem.

2. Special defining functions

In the proof of the main theorem, we will need to work with three different defining functions: a local defining function $\rho(z)$ satisfying a strong form of Oka's lemma, the geodesic distance function $\delta(z)$ for the boundary, and a local defining function derived by representing the boundary as a graph. When comparing these three functions, the constants of comparison will need to be small. On C^1 domains, this is always possible. When working in special coordinate charts $\{z^1, \dots, z^n\}$, we will use z' to denote $\{z^1, \dots, z^{n-1}\}$.

LEMMA 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain with C^1 boundary, and let $p \in \partial\Omega$ be a point with a neighborhood U and a local defining function ρ on $\Omega \cap U$ satisfying:*

$$i\partial\overline{\partial}(-\log(-\rho)) \geq i\phi(-\rho)\partial\overline{\partial}|z|^2,$$

in the sense of currents on $\Omega \cap U$ where ϕ is a positive continuous function on $(0,\infty)$ satisfying $\lim_{x\to 0^+} \phi(x) = \infty$. Then for any constant $\tilde{c} > 1$, there exists an open neighborhood $p \in \tilde{U} \subset U$ with local orthonormal coordinates centered at p and a local defining function $\tilde{\rho}$ on $\Omega \cap \tilde{U}$ satisfying the following:

(1) *There is a C^1 function φ on $\mathbb{C}^{n-1} \times \mathbb{R}$ such that*

$$\Omega \cap \tilde{U} = \{z \in \tilde{U} : \operatorname{Im} z^n < \varphi(z', \operatorname{Re} z^n)\}.$$

(2) *On $\Omega \cap \tilde{U}$, $-\frac{1}{\tilde{c}}\tilde{\rho}(z) < |\operatorname{Im} z^n - \varphi(z', \operatorname{Re} z^n)| < -\tilde{c}\tilde{\rho}(z)$ and*

$$i\partial\overline{\partial}(-\log(-\tilde{\rho})) \geq i\tilde{\phi}(-\tilde{\rho})\partial\overline{\partial}|z|^2,$$

in the sense of currents where $\tilde{\phi}$ is a positive continuous function on $(0,\infty)$ satisfying $\lim_{x\to 0^+} \tilde{\phi}(x) = \infty$.

(3) *On $\tilde{U}$, $\frac{1}{\tilde{c}}\delta(z) < |\operatorname{Im} z^n - \varphi(z', \operatorname{Re} z^n)| < \tilde{c}\delta(z)$.*

Proof. Since $\partial\Omega$ is C^1, we can choose orthonormal coordinates $\{z^1, \dots, z^n\}$ such that $p = 0$ and the unit outward normal at p is $\frac{\partial}{\partial y^n}$, where $z^n = x^n + iy^n$. Let $\tilde{\delta}(z)$ denote the signed distance function for $\partial\Omega$, i.e., $\tilde{\delta}(z) = -\delta(z)$ inside Ω and $\tilde{\delta}(z) = \delta(z)$ outside Ω. If we represent $\partial\Omega$ locally as the graph of a function φ, then $\operatorname{Im} z^n - \varphi(z', \operatorname{Re} z^n)$ and $\tilde{\delta}(z)$ are both C^1 defining functions for Ω near 0, so $h(z) = \frac{\operatorname{Im} z^n - \varphi(z', \operatorname{Re} z^n)}{\tilde{\delta}(z)}$ is a continuous positive function near 0 and $hd\tilde{\delta} = d(\operatorname{Im} z^n - \varphi(z', \operatorname{Re} z^n))$ on $\partial\Omega$. At 0, we have $hd\tilde{\delta} = dy^n$, and since $|d\tilde{\delta}| \equiv 1$, we can conclude $h(0) = 1$. We can now choose a neighborhood $\tilde{U}$ of 0 where $\frac{1}{\sqrt{\tilde{c}}} < h(z) < \sqrt{\tilde{c}}$. Since $\sqrt{\tilde{c}} < \tilde{c}$, statement (3) follows immediately.

By Oka's lemma, we can shrink $\tilde{U}$ as necessary so that $i\partial\overline{\partial}(-\log\delta) \geq 0$ on $\Omega \cap \tilde{U}$ in the sense of currents. Since ρ is a local defining function, there is a constant $c > 1$ such that $\frac{1}{c}\delta(z) < -\rho(z) < c\delta(z)$ on $\Omega \cap \tilde{U}$. Setting $\rho_t = -(-\rho)^t\delta^{1-t}$ for any $0 < t \leq 1$, we have $\frac{1}{c^t}\delta(z) < -\rho_t(z) < c^t\delta(z)$ and $i\partial\overline{\partial}(-\log(-\rho_t)) \geq it\phi(-\rho)\partial\overline{\partial}|z|^2$. If we choose t small enough so that $c^t \leq \sqrt{\tilde{c}}$ and set $\tilde{\rho} = \rho_t$, statement (2) will follow. □

REMARK 2.2. We will see that Lemma 2.1 is only needed for a single value of $\tilde{c}$ satisfying $1 < \tilde{c} < \sqrt{\frac{4}{3}}$. Hence, Theorem 1.1 will hold even for Lipschitz domains if Lemma 2.1 is satisfied for such a $\tilde{c}$. For example, if Ω is a piecewise C^1 domain such that all interior and exterior angles are greater than $\frac{2\pi}{3}$, then Theorem 1.1 still follows.

3. Proof of the main theorem

We will begin by constructing plurisubharmonic functions near each boundary point satisfying certain estimates. In order to patch these functions together we will adapt an idea from [5]. Fix some constant $1<c<\sqrt{\frac{4}{3}}$. For each $p\in\partial\Omega$, let U_p be a neighborhood where the conclusions of Lemma 2.1 hold for $\tilde{c}=c$. Let $B(p,r_p)\subset U_p$ denote a ball of radius r_p centered at p. Choose some finite subcollection of $\{B(p,\frac{r_p}{3})\}$ covering $\partial\Omega$ indexed by $P\subset\partial\Omega$ and for each $p\in P$ let $\chi_p\in C_0^\infty(B(p,\frac{r_p}{2}))$ be a function satisfying $0\leq\chi_p\leq 1$ and $\chi_p\equiv 1$ on $\overline{B(p,\frac{r_p}{3})}$.

Fix some $p\in P$ and let $\{z^1,\dots,z^n\}$, $\tilde{\rho}$, and φ be as in Lemma 2.1. Henceforth, we will suppress the subscript p and take $\rho=\tilde{\rho}$ and $\phi=\tilde{\phi}$. Since $1<c^2<\frac{4}{3}$, we can choose a satisfying $\frac{4-3c^2}{c}>a>0$ and b satisfying $\frac{2-c^2}{c}>b>a+\frac{2c^2-2}{c}$. For $\varepsilon>0$ sufficiently small, let $K_\varepsilon=\{z\in\overline{B(0,\frac{r}{2})}\setminus\Omega: a\varepsilon\leq\delta(z)\leq b\varepsilon\}$ and define $\tilde{\rho}_\varepsilon(z)=\rho(z',z^n-i\varepsilon)$ on K_ε. Note that on K_ε:

$$\operatorname{Im}(z^n-i\varepsilon)-\varphi(z',\operatorname{Re}z^n)<c\delta(z)-\varepsilon\leq(bc-1)\varepsilon<(1-c^2)\varepsilon<0,$$

so $(z',z^n-i\varepsilon)\in\Omega$ and $\tilde{\rho}_\varepsilon$ is well defined on K_ε. In fact, on K_ε we have the estimates:

$$-\tilde{\rho}_\varepsilon(z)<c\big(\varphi(z',\operatorname{Re}z^n)-\operatorname{Im}(z^n-i\varepsilon)\big)<c\left(\varepsilon-\frac{1}{c}\delta(z)\right)=c\varepsilon-\delta(z),\tag{3.1}$$

and

$$-\tilde{\rho}_\varepsilon(z)>\frac{1}{c}\big(\varphi(z',\operatorname{Re}z^n)-\operatorname{Im}(z^n-i\varepsilon)\big)>\frac{1}{c}\big(\varepsilon-c\delta(z)\big)=\frac{1}{c}\varepsilon-\delta(z).\tag{3.2}$$

Since $\tilde{\rho}_\varepsilon$ is only a translation of ρ, we have

$$i\partial\overline{\partial}(-\log(-\tilde{\rho}_\varepsilon))\geq i\phi(-\tilde{\rho}_\varepsilon)\partial\overline{\partial}|z|^2$$

on K_ε. We may assume ϕ is decreasing (if not, we may replace ϕ with the function whose graph is the convex hull of the graph of the original ϕ), so

$$i\partial\overline{\partial}(-\log(-\tilde{\rho}_\varepsilon))>i\phi(c\varepsilon-\delta)\partial\overline{\partial}|z|^2\geq i\phi\big((c-a)\varepsilon\big)\partial\overline{\partial}|z|^2.\tag{3.3}$$

Since the relevant estimates [(3.1), (3.2), and (3.3)] are strict inequalities, they will apply to $\tilde{\rho}_\varepsilon$ in a small neighborhood of K_ε. Using the standard construction to regularize $-\log(-\tilde{\rho}_\varepsilon)$ by convolution, we can obtain a smooth function ρ_ε on K_ε satisfying

$$\frac{1}{c}\varepsilon-\delta(z)<-\rho_\varepsilon(z)<c\varepsilon-\delta(z),\tag{3.4}$$

$$i\partial\overline{\partial}(-\log(-\rho_\varepsilon))>i\phi\big((c-a)\varepsilon\big)\partial\overline{\partial}|z|^2.\tag{3.5}$$

Next, we let $f(x)=-\frac{1}{x}$ and $g(x)=f(x-(\frac{c^2-1}{c})\varepsilon)-f(x)$. Note that $f(x)$ and $f''(x)$ are increasing while $f'(x)$ is decreasing, and hence $g(x)$ and $g''(x)$ are negative while $g'(x)$ is positive. Set

$$\lambda_\varepsilon=-f(-\rho_\varepsilon)-g(-\rho_\varepsilon)\chi.$$

On $\partial B(0, \frac{r}{2})$, when $\chi \equiv 0$, we estimate

$$\lambda_\varepsilon = -f(-\rho_\varepsilon) < -f\left(\frac{1}{c}\varepsilon - \delta\right),$$

while on $\overline{B(0, \frac{r}{3})}$, where $\chi \equiv 1$, we have

$$\begin{aligned}\lambda_\varepsilon = -f(-\rho_\varepsilon) - g(-\rho_\varepsilon) &= -f\left(-\rho_\varepsilon - \left(\frac{c^2-1}{c}\right)\varepsilon\right) \\ > -f\left(c\varepsilon - \delta - \left(\frac{c^2-1}{c}\right)\varepsilon\right) &= -f\left(\frac{1}{c}\varepsilon - \delta\right).\end{aligned}$$

Note that both bounds on λ_ε are independent of p. This will be crucial when patching the local functions together to obtain a global function.

Since g is negative, we can estimate λ_ε when $\delta(z) = a\varepsilon$ by

$$\begin{aligned}\lambda_\varepsilon \le -f(-\rho_\varepsilon) - g(-\rho_\varepsilon) &= -f\left(-\rho_\varepsilon - \left(\frac{c^2-1}{c}\right)\varepsilon\right) \\ < -f\left(\frac{1}{c}\varepsilon - a\varepsilon - \left(\frac{c^2-1}{c}\right)\varepsilon\right) &= -f\left(\left(\frac{2-c^2}{c} - a\right)\varepsilon\right).\end{aligned}$$

Similarly, when $\delta(z) = b\varepsilon$, we have

$$\begin{aligned}&\lambda_\varepsilon \ge -f(-\rho_\varepsilon) > -f(c\varepsilon - b\varepsilon) \\ &\quad > -f\left(\left(c - a - \frac{2c^2-2}{c}\right)\varepsilon\right) = -f\left(\left(\frac{2-c^2}{c} - a\right)\varepsilon\right).\end{aligned}$$

Now we can set $k_\varepsilon = -f((\frac{2-c^2}{c} - a)\varepsilon)$ and have $\lambda_\varepsilon > k_\varepsilon$ when $\delta(z) = b\varepsilon$ and $\lambda_\varepsilon < k_\varepsilon$ when $\delta(z) = a\varepsilon$.

It remains to see that λ_ε is strictly plurisubharmonic on K_ε when ε is sufficiently small. Note that

$$i\partial\overline{\partial}(-\log(-\rho_\varepsilon)) = i\partial((-\rho_\varepsilon)^{-1}\overline{\partial}\rho_\varepsilon) = i(-\rho_\varepsilon)^{-1}\partial\overline{\partial}\rho_\varepsilon + i(-\rho_\varepsilon)^{-2}\partial\rho_\varepsilon \wedge \overline{\partial}\rho_\varepsilon,$$

so:

$$i\partial\overline{\partial}\rho_\varepsilon = i(-\rho_\varepsilon)\partial\overline{\partial}(-\log(-\rho_\varepsilon)) - i(-\rho_\varepsilon)^{-1}\partial\rho_\varepsilon \wedge \overline{\partial}\rho_\varepsilon.$$

We can now compute

$$\begin{aligned}i\partial\overline{\partial}\lambda_\varepsilon &= i\partial\big(\big(f'(-\rho_\varepsilon) + g'(-\rho_\varepsilon)\chi\big)\overline{\partial}\rho_\varepsilon - g(-\rho_\varepsilon)\overline{\partial}\chi\big) \\ &= i\big(f'(-\rho_\varepsilon) + g'(-\rho_\varepsilon)\chi\big)\partial\overline{\partial}\rho_\varepsilon - i\big(f''(-\rho_\varepsilon) + g''(-\rho_\varepsilon)\chi\big)\partial\rho_\varepsilon \wedge \overline{\partial}\rho_\varepsilon \\ &\quad - ig(-\rho_\varepsilon)\partial\overline{\partial}\chi + ig'(-\rho_\varepsilon)(\partial\rho_\varepsilon \wedge \overline{\partial}\chi + \partial\chi \wedge \overline{\partial}\rho_\varepsilon) \\ &= i\big(f'(-\rho_\varepsilon) + g'(-\rho_\varepsilon)\chi\big)(-\rho_\varepsilon)\partial\overline{\partial}(-\log(-\rho_\varepsilon)) \\ &\quad - i\big(f''(-\rho_\varepsilon) + (-\rho_\varepsilon)^{-1}f'(-\rho_\varepsilon)\big)\partial\rho_\varepsilon \wedge \overline{\partial}\rho_\varepsilon \\ &\quad - i\big(\big(g''(-\rho_\varepsilon) + (-\rho_\varepsilon)^{-1}g'(-\rho_\varepsilon)\big)\chi\big)\partial\rho_\varepsilon \wedge \overline{\partial}\rho_\varepsilon \\ &\quad - ig(-\rho_\varepsilon)\partial\overline{\partial}\chi + ig'(-\rho_\varepsilon)(\partial\rho_\varepsilon \wedge \overline{\partial}\chi + \partial\chi \wedge \overline{\partial}\rho_\varepsilon).\end{aligned}$$

Since

$$-\left(f''(x)+\frac{1}{x}f'(x)\right)=\frac{2}{x^3}-\frac{1}{x^3}=\frac{1}{x^3}$$

is a decreasing function, we can conclude that $-(g''(x)+\frac{1}{x}g'(x))$ is a positive function, so we have

$$\begin{aligned}i\partial\overline{\partial}\lambda_\varepsilon > i(-\rho_\varepsilon)^{-1}\phi\big((c-a)\varepsilon\big)\partial\overline{\partial}|z|^2 + i(-\rho_\varepsilon)^{-3}\partial\rho_\varepsilon\wedge\overline{\partial}\rho_\varepsilon \\ - ig(-\rho_\varepsilon)\partial\overline{\partial}\chi + ig'(-\rho_\varepsilon)(\partial\rho_\varepsilon\wedge\overline{\partial}\chi+\partial\chi\wedge\overline{\partial}\rho_\varepsilon).\end{aligned}$$

Clearly,

$$i\big((-\rho_\varepsilon)^{-\frac{3}{2}}\partial\rho_\varepsilon+(-\rho_\varepsilon)^{\frac{3}{2}}g'(-\rho_\varepsilon)\partial\chi\big)\wedge\big((-\rho_\varepsilon)^{-\frac{3}{2}}\overline{\partial}\rho_\varepsilon+(-\rho_\varepsilon)^{\frac{3}{2}}g'(-\rho_\varepsilon)\overline{\partial}\chi\big)\geq 0,$$

so

$$\begin{aligned}&i(-\rho_\varepsilon)^{-3}\partial\rho_\varepsilon\wedge\overline{\partial}\rho_\varepsilon + ig'(-\rho_\varepsilon)(\partial\rho_\varepsilon\wedge\overline{\partial}\chi+\partial\chi\wedge\overline{\partial}\rho_\varepsilon)\\ &\quad\geq -i(-\rho_\varepsilon)^3(g'(-\rho_\varepsilon))^2\partial\chi\wedge\overline{\partial}\chi,\end{aligned}$$

and

$$i\partial\overline{\partial}\lambda_\varepsilon > i(-\rho_\varepsilon)^{-1}\phi\big((c-a)\varepsilon\big)\partial\overline{\partial}|z|^2 - ig(-\rho_\varepsilon)\partial\overline{\partial}\chi - i(-\rho_\varepsilon)^3(g'(-\rho_\varepsilon))^2\partial\chi\wedge\overline{\partial}\chi.$$

From (3.4), we know that

$$\left(\frac{1}{c}-b\right)\varepsilon < -\rho_\varepsilon < (c-a)\varepsilon,$$

so since $g(-\rho_\varepsilon)=O(\varepsilon^{-1})$ and $g'(-\rho_\varepsilon)=O(\varepsilon^{-2})$, there exist positive constants A and B independent of ε such that

$$-ig(-\rho_\varepsilon)\partial\overline{\partial}\chi > -iA\varepsilon^{-1}\partial\overline{\partial}|z|^2, \tag{3.6}$$

$$-i(-\rho_\varepsilon)^3(g'(-\rho_\varepsilon))^2\partial\chi\wedge\overline{\partial}\chi > -iB\varepsilon^{-1}\partial\overline{\partial}|z|^2. \tag{3.7}$$

Hence, we conclude

$$i\partial\overline{\partial}\lambda_\varepsilon > i\Big(\big((c-a)\varepsilon\big)^{-1}\phi\big((c-a)\varepsilon\big) - A\varepsilon^{-1} - B\varepsilon^{-1}\Big)\partial\overline{\partial}|z|^2.$$

By assumption, $\phi((c-a)\varepsilon) > (c-a)(A+B)$ for all sufficiently small $\varepsilon>0$, so λ_ε is strictly plurisubharmonic on K_ε for all such $\varepsilon>0$.

Let $\tilde{K}_\varepsilon=\{z\in\mathbb{C}^n\setminus\Omega : a\varepsilon\leq\delta(z)\leq b\varepsilon\}$ and define

$$\tilde{\lambda}_\varepsilon(z)=\sup_{\{p\in P: z\in K_{\varepsilon,p}\}}\lambda_{\varepsilon,p}(z).$$

If we choose $\varepsilon>0$ sufficiently small so that $\{B(p,\frac{r_p}{3})\}$ cover $\tilde{K}_\varepsilon$ and each $\lambda_{\varepsilon,p}$ is strictly plurisubharmonic then $\tilde{\lambda}_\varepsilon$ is strictly plurisubharmonic since it is locally the supremum over a finite collection of strictly plurisubharmonic functions (this is guaranteed since each $\lambda_{\varepsilon,p} > -f(\frac{1}{c}\varepsilon-\delta)$ on $\overline{B(p,\frac{r_p}{3})}$, but $\lambda_{\varepsilon,p} < -f(\frac{1}{c}\varepsilon-\delta)$ on $\partial B(p,\frac{r_p}{2})$).

Since $\tilde{\lambda}_\varepsilon > k_\varepsilon$ when $\delta(z)=b\varepsilon$ and $\tilde{\lambda}_\varepsilon < k_\varepsilon$ when $\delta(z)=a\varepsilon$, we can conclude that the level curve $\{z\in\tilde{K}_\varepsilon : \tilde{\lambda}_\varepsilon(z)=k_\varepsilon\}$ is contained in the interior of $\tilde{K}_\varepsilon$, and hence defines a piecewise smooth strictly pseudoconvex neighborhood of Ω. Since this construction holds for all sufficiently small $\varepsilon > 0$, we have a Stein neighborhood basis for Ω.

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