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TAUT REPRESENTATIONS OF COMPACT SIMPLE LIE GROUPS

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Abstract. The concept of taut submanifold of Euclidean space is due to Carter and West, and can be traced back to the work of Chern and Lashof on immersions with minimal total absolute curvature and the subsequent reformulation of that work by Kuiper in terms of critical point theory. In this paper, we classify the reducible representations of compact simple Lie groups, all of whose orbits are tautly embedded in Euclidean space, with respect to $\mathbf{Z}_2$-coefficients.

1. Introduction

The concept of taut submanifold of Euclidean space is due to Carter and West [CW72], and can be traced back to the work of Chern and Lashof [CL57] on immersions with minimal total absolute curvature and the subsequent reformulation of that work by Kuiper [Kui58] in terms of critical point theory. Although there is an extensive literature on taut submanifolds of Euclidean space (see, e.g., [CR85, Cec97]), the general classification problem is very difficult and remains open [TT97, Problem 3.12]. In view of previous results, this paper can be seen as one step more toward the classification of taut homogeneous submanifolds of Euclidean space. In fact, a representation of a compact Lie group is called *taut* if all of its orbits are taut submanifolds of the representation space. The main result of this paper is the following classification theorem, which extends the classification in the irreducible case completed in [GT03].

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THEOREM 1. *A taut reducible representation of a compact simple Lie group is one of the following representations:*

1	$\mathbf{SU}(n), n \geq 3$	$\mathbf{C}^n \oplus \cdots \oplus \mathbf{C}^n$	*k copies, where $1 < k < n$*
2	$\mathbf{SO}(n), n \geq 3, n \neq 4$	$\mathbf{R}^n \oplus \cdots \oplus \mathbf{R}^n$	*k copies, where $1 < k$*
3	$\mathbf{Sp}(n), n \geq 1$	$\mathbf{C}^{2n} \oplus \cdots \oplus \mathbf{C}^{2n}$	*k copies, where $1 < k$*
4	$\mathbf{G}_2$	$\mathbf{R}^7 \oplus \mathbf{R}^7$	—
5	$\mathbf{Spin}(6)$	$\mathbf{R}^6 \oplus \mathbf{C}^4$	$\mathbf{R}^6 = (\text{vector}), \mathbf{C}^4 = (\text{spin})$
6		$\mathbf{R}^7 \oplus \mathbf{R}^8$	
7	$\mathbf{Spin}(7)$	$\mathbf{R}^8 \oplus \mathbf{R}^8$	$\mathbf{R}^7 = (\text{vector}), \mathbf{R}^8 = (\text{spin})$
8		$\mathbf{R}^8 \oplus \mathbf{R}^8 \oplus \mathbf{R}^8$	
9		$\mathbf{R}^7 \oplus \mathbf{R}^7 \oplus \mathbf{R}^8$	
10		$\mathbf{R}^8_0 \oplus \mathbf{R}^8_+$	
11	$\mathbf{Spin}(8)$	$\mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_+$	$\mathbf{R}^8_0 = (\text{vector}), \mathbf{R}^8_+ = (\text{halfspin})$
12		$\mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_+$	
13	$\mathbf{Spin}(9)$	$\mathbf{R}^{16} \oplus \mathbf{R}^{16}$	$\mathbf{R}^{16} = (\text{spin})$

We make some remarks regarding the representations appearing in this table. In case 1, any number of summands $\mathbf{C}^n$ in the sum can be replaced by the same number of summands of the dual representation $\mathbf{C}^{n*}$, and the resulting representation remains taut; and similarly, $\mathbf{C}^4$ in case 5 can be replaced by $\mathbf{C}^{4*}$. Another thing is that the representations of $\mathbf{Spin}(8)$ are listed up to composition with an outer automorphism of the Lie group, so the pair $(\mathbf{R}^8_0, \mathbf{R}^8_+)$ appearing in the list can be replaced by any pair of inequivalent 8-dimensional representations of $\mathbf{Spin}(8)$, and the resulting representations for $\mathbf{Spin}(8)$ will still be taut. Finally, we point out that cases 1 through 8 and case 10 were or could have been known previously to be taut, whereas cases 9, 11, 12, and 13 definitely give new examples of taut representations and taut submanifolds.

In the following, we present the relevant definitions and a historical perspective of the subject. Carter and West introduced in [CW72] the concept of tautness for submanifolds (see also the monograph [CR85]). Fix a field of coefficients F (herein assumed to be $\mathbf{Z}_2$). Let M be a properly embedded submanifold of an Euclidean space $\mathbf{R}^m$. For each $p \in \mathbf{R}^m$, consider the squared distance function $L_p : M \to \mathbf{R}$ given by $L_p(x) = \|x - p\|^2$. It is a consequence of the Morse index theorem that the critical points of L_p are nondegenerate, i.e., L_p is a Morse function, if and only if p is not a focal point of M. Now M is called F-*taut*, or simply *taut*, if L_p is a perfect Morse function for every p in $\mathbf{R}^m$ that is not a focal point of M. We recall that a Morse function is said to be *perfect* if the Morse inequalities are equalities for the function restricted to any sublevel set. As a consequence of the proof of the Morse inequalities, one sees that an equivalent definition of F-tautness for a submanifold $M \subset \mathbf{R}^m$ is that the induced homomorphism

$$H_*(M \cap B; F) \to H_*(M; F)$$

in singular homology is injective for almost every closed ball B in $\mathbf{R}^m$. It is then clear that tautness is conformally invariant.

A compact surface in $\mathbf{R}^3$ which is taut is either a round sphere or a cyclide of Dupin [Ban70]; the latter can all be constructed as the image of a torus of revolution under a Möbius transformation. Pinkall and Thorbergsson found in [PT89] the diffeomorphism classes of the compact 3-dimensional manifolds that admit taut embeddings, and their list consists of seven manifolds. The first three are $S^1 \times S^2$ and its quotients $S^1 \times \mathbf{R}P^2$ and $S^1 \times_{\mathbf{Z}_2} S^2$. The next three are S^3 and its quotients $\mathbf{R}P^3$ and $S^3/\{\pm 1, \pm i, \pm j, \pm k\}$ (the so-called quaternion space). The last example is the torus T^3. It follows from the Chern–Lashof theorem [CL57] that a taut substantial (namely, nor contained in an affine hyperplane) embedding of a sphere must be spherical and of codimension one. If M is an n-dimensional taut hypersurface in $\mathbf{R}^{n+1}$ which has the same integral homology as $S^k \times S^{n-k}$, then Cecil and Ryan proved in [CR78] that M has precisely two principal curvatures at each point and that the principal curvatures are constant along the corresponding curvature distributions. Bott and Samelson proved in [BS58] that the orbits of the isotropy representations of the symmetric spaces, sometimes called generalized flag manifolds, are tautly embedded submanifolds, although they did not use this terminology. The generalized flag manifolds are homogeneous examples of submanifolds which belong to another very important, more general class of submanifolds called isoparametric submanifolds. Hsiang, Palais, and Terng studied in [HPT88] the topology of isoparametric submanifolds and proved, among other things, that they and their focal submanifolds are taut.

The class of taut submanifolds of Euclidean space is also closely related to the class of Dupin hypersurfaces. Pinkall introduced this class in [Pin81] (see also [Pin85]) as a simultaneous generalization of the classical cyclides of Dupin and of isoparametric hypersurfaces. Thorbergsson showed in [Tho83] that a complete Dupin hypersurface embedded in $\mathbf{R}^n$ with constant number of distinct principal curvatures is taut. Pinkall [Pin86] and Miyaoka [Miy84] then independently showed that a taut hypersurface is Dupin (not necessarily with a constant number of distinct principal curvatures). More generally, a tube around a taut submanifold is Dupin.

Most of the examples of taut embeddings known are homogeneous spaces. In [Tho88], Thorbergsson posed some questions regarding the problem of which homogeneous spaces admit taut embeddings and derived some necessary topological conditions for the existence of a taut embedding which allowed him to conclude that certain homogeneous spaces cannot be tautly embedded, among others, the lens spaces distinct from the real projective space (see also [Heb88]). Many proofs have been given of the tautness of special cases of generalized flag manifolds where the arguments are easier. No new examples of taut embeddings of homogeneous spaces besides the generalized flag manifolds were known until Gorodski and Thorbergsson classified in [GT03] (see also [GT]) the taut irreducible representations of compact Lie groups. It turns out that the classification includes three new representations which are not isotropy representations of symmetric spaces, thereby supplying many new examples of tautly embedded homogeneous spaces. In [GT02], Gorodski and Thorbergsson provided another proof of the tautness of those representations by adapting the proof of Bott and Samelson to that case. It is interesting to remark that those three representations precisely coincide with the representations of cohomogeneity three of the compact Lie groups which are not orbit equivalent to the isotropy representation of a symmetric space. (Recall that two representations are said to be *orbit equivalent* if there is an isometry between the representation spaces mapping the orbits of the first representation onto the orbits of the second one.) As mentioned above, in this paper, we extend the classification in [GT03] to the case in which the representation is reducible and the group is simple.

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2. Preliminary material

In this section, we collect results that will be used later to prove that certain representations are or are not taut. We start with a following simple remark, namely, *every summand of a taut reducible representation is taut.* Indeed, this is because an orbit of a summand is also an orbit of the sum, and it implies that taut reducible representations are sums of taut irreducible ones. So, in order to classify taut reducible representations, we need just to decide which of those sums are allowed. We shall do that for simple groups.

We begin by recalling the main result of [GT03].

THEOREM 2 ([GT03]). *A taut irreducible representation of a compact connected Lie group is either orbit equivalent to the isotropy representation of a symmetric space or it is one of the following orthogonal representations ($n \geq 2$):*

$\mathbf{SO}(2) \times \mathbf{Spin}(9)$	(vector) $\otimes_{\mathbf{R}}$ (spin)
$\mathbf{U}(2) \times \mathbf{Sp}(n)$	(vector) $\otimes_{\mathbf{C}}$ (vector)
$\mathbf{SU}(2) \times \mathbf{Sp}(n)$	$(\text{vector})^3 \otimes_{\mathbf{H}}$ (vector)

Since the groups appearing in the table of Theorem 2 are nonsimple, now we can refine the remark above and state that *every summand of a taut reducible representation is orbit equivalent to the isotropy representation of a symmetric space.* Throughout the paper, we shall make use of the tables of isotropy representations of a symmetric spaces given in [Wol84]. The irreducible representations orbit equivalent to the isotropy representation of a symmetric space are also classified [EH99]. Lists with some of the principal isotropy subgroups of these representations can be found in [HPT88, Str96].

The fundamental result about taut sums of representations is contained in the following proposition.

Proposition 1 ([GT03]). *Let ρ_1 and ρ_2 be representations of a compact connected Lie group G on V_1 and V_2, respectively. Assume that $\rho_1 \oplus \rho_2$ is F-taut. Then the restriction of ρ_2 to the isotropy group G_{v_1} is taut for every $v_1 \in V_1$. Furthermore, we have that $p(G(v_1,v_2);F) = p(Gv_1;F)p(G_{v_1}v_2;F)$, where $p(M;F)$ denotes the Poincaré polynomial of M with respect to the field F. In particular, $G_{v_1}v_2$ is connected and $b_1(G(v_1,v_2);F) = b_1(Gv_1;F) + b_1(G_{v_1}v_2;F)$, where $b_1(M;F)$ denotes the first Betti number of M with respect to F.*

We give examples of how Proposition 1 can be used. These are taken from [GT].

Examples 1. (i) Let $G = \mathbf{SO}(n)$ and let ρ_1 be the $\mathbf{SO}(n)$-conjugation on the space V_1 of real traceless symmetric $n \times n$ matrices. Then ρ_1 is taut since it is the isotropy representation of the symmetric space $\mathbf{SL}(n,\mathbf{R})/\mathbf{SO}(n)$. Let ρ_2 be any other nontrivial representation of $\mathbf{SO}(n)$ with representation space V_2. Then $\rho_1 \oplus \rho_2$ cannot be taut if $n \geq 3$. To see this, let $v_1 \in V_1$ be a regular point. Then G_{v_1} is the discrete group consisting of all diagonal matrices with determinant one and entries ± 1 on the diagonal. The kernel of ρ_2 is contained in the center of $\mathbf{SO}(n)$. Since $n \geq 3$, we see that G_{v_1} cannot be contained in the kernel of ρ_2. Hence, there is an element $v_2 \in V_2$ that is not fixed by G_{v_1}. It follows that $G_{v_1}v_2$ is disconnected. Now Proposition 1 implies that $\rho_1 \oplus \rho_2$ is not taut. The same argument applies more generally whenever ρ_1 is a taut representation of a compact connected Lie group G such that its principal isotropy subgroup is discrete and not central.

(ii) Now let G be a compact connected simple Lie group of rank at least two and let ρ_1 denote the adjoint representation of G. We assume that G is simply connected. Let ρ_2 be any other nontrivial representation of G. Then $\rho_1 \oplus \rho_2$ is not taut. To see this, let T be a maximal torus in G. We denote the representation spaces of ρ_1 and ρ_2 by V_1 and V_2, respectively. There is a regular element $v_1 \in V_1$ with $G_{v_1} = T$. The restriction of ρ_2 to T has a discrete kernel that is contained in the center of G. If $v_2 \in V_2$ is a T-regular point then the isotropy subgroup T_{v_2} coincides with the kernel of $\rho_2|T$. Hence, $G_{v_1}v_2$ is diffeomorphic to T and it follows that $b_1(G_{v_1}v_2;F)$ is equal to the rank of G. In particular, $b_1(G_{v_1}v_2;F) \geq 2$. Now notice that the isotropy group of (v_1,v_2) is also T_{v_2}. Hence, $\pi_1(G(v_1,v_2)) = T_{v_2}$ which implies $H_1(G(v_1,v_2);\mathbf{Z}) = T_{v_2}$ since T_{v_2} is Abelian. If $G \neq \mathbf{Spin}(4k)$ then the center of G is a cyclic group and it follows that $b_1(G(v_1,v_2);F) \leq 1$. If $G = \mathbf{Spin}(4k)$, then $k \geq 2$ and we get $b_1(G_{v_1}v_2;F) = 2k \geq 4$; since the center of $\mathbf{Spin}(4k)$ is $\mathbf{Z}_2 \times \mathbf{Z}_2$, we have $b_1(G(v_1,v_2);F) \leq 2$. In either case, $b_1(G_{v_1}v_2;F) > b_1(G(v_1,v_2);F)$ which implies by Proposition 1 that $\rho_1 \oplus \rho_2$ is not taut.

Recall that the *slice representation* of a representation $\rho : G \to \mathbf{O}(V)$ at a point $p \in V$ is the representation induced by the isotropy G_p on the normal space to the orbit Gp at p. The following result often works as a kind of induction.

PROPOSITION 2 ([GT03]). *Let $\rho: G \to \mathbf{O}(V)$ be a taut representation of a compact connected Lie group G. Then the slice representation of ρ at any $p \in V$ is taut.*

We now discuss a reduction principle which in many cases considerably simplifies the problem of deciding whether a representation is taut or not. Let $\rho: G \to \mathbf{O}(V)$ be a representation of a compact Lie group G which is not assumed to be connected. Denote by H a fixed principal isotropy subgroup of the G-action on V and let V^H be the subspace of V that is left pointwise fixed by the action of H. Let N be the normalizer of H in G. Then the group $\bar{N} = N/H$ acts on V^H with trivial principal isotropy subgroup. Moreover, the following result is known [GS00], [Lun75], [LR79], [Sch80], [SS95], [Str94].

THEOREM 3 (Luna–Richardson). *The inclusion $V^H \to V$ induces a stratification preserving homeomorphism between orbit spaces*

$$V^H/\bar{N} \to V/G.$$

The relation to tautness is expressed by the following result.

PROPOSITION 3 ([GT03]). *Suppose there is a subgroup $L \subset H$ which is a finitely iterated $\mathbf{Z}_2$-extension of the identity and such that the fixed point sets $V^L = V^H$. Suppose also that the reduced representation $\bar{\rho}: \bar{N}^0 \to \mathbf{O}(V^H)$ is $\mathbf{Z}_2$-taut, where $\bar{N}^0$ denotes the connected component of the identity of $\bar{N}$. Then $\rho: G \to \mathbf{O}(V)$ is $\mathbf{Z}_2$-taut.*

We close this section with some very useful remarks.

REMARK 1. (a) It follows from the discussion of Kuiper in [Kui61] that *if M is a taut substantial submanifold of an Euclidean space, then there exists $p \in M$ such that the image of the second fundamental form of M at p spans the normal space of M at p.* As a corollary, the codimension of M is at most $n(n+1)/2$, where $n = \dim M$.

(b) One defines a submanifold of an Euclidean space to be *F-tight*, or simply *tight*, similarly as was done for tautness, except that one replaces distance functions by height functions $h_\xi(x) = \langle x, \xi \rangle$, ξ a nonzero vector. It turns out that tightness is invariant under linear transformations, and a taut sumanifold of an Euclidean space is tight. Moreover, a tight submanifold of an Euclidean space which is contained in a round sphere is taut, and in this situation the set of critical points of a distance function will also occur as the set of critical points of a height function [CR85, PT88].

(c) Ozawa proved in [Oza86] that the set of critical points of a distance function of a taut submanifold decomposes into critical submanifolds which are nondegenerate in the sense of Bott; it follows that the so-called Morse–Bott inequalities are equalities for the function restricted to any sublevel set; namely, the number of critical points of the function is equal to the sum of the Betti numbers of the critical submanifolds; see [Bot54].

3. The classification

We first prove two lemmas for later use.

LEMMA 1. *The following representations are not taut:*

(a) $S^1 \times S^1 \to \mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1)$ *given by* $(e^{i\alpha}, e^{i\beta}) \mapsto (e^{i\alpha}, e^{i\beta}, e^{i(\alpha+\beta)})$.

(b) $\mathbf{Sp}(1) \times \mathbf{Sp}(1) \to \mathbf{SO}(4) \times \mathbf{SO}(4) \times \mathbf{SO}(4)$ *given by* $(p,q) \mapsto (l_p, r_{\bar{q}}, l_p r_{\bar{q}})$, *where* l_x *(resp.* r_x*) denotes left (resp. right) translation by the unit quaternion* x.

Proof. We will prove (a); assertion (b) is similar. Let M denote the orbit through $p = (1,1,1) \in \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C}$. We will show that M is not taut by exhibiting a height function which is not perfect; see Remark 1(b). The normal space $\nu_p M$ is easily seen to be spanned over $\mathbf{R}$ by $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(i,i,-i)$. Let $h : M \to \mathbf{R}$ be the height function defined by p. Note that gp, $g \in S^1 \times S^1$, is a critical point of h if and only if $p \in \nu_{gp} M$, or equivalently, $g^{-1}p \in \nu_p M$. One immediately computes that $g = (\pm 1, \pm 1)$ or $(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2})$, so there are 6 critical points. Since M is a 2-torus, h is not perfect. $\square$

Since a representation of real or quaternionic type is necessarily self-dual, we may assume that the representation ρ_2 in the next lemma is of complex type.

LEMMA 2. *Let* $\rho_i : G \to \mathbf{O}(V_i)$, *where* $i = 1, 2$, *be taut representations of a compact Lie group* G *such that* $\rho = \rho_1 \oplus \rho_2 : G \to \mathbf{O}(V_1 \oplus V_2)$ *is a taut representation. Assume that* ρ_2 *is a representation of complex type. Then* $\tau = \rho_1 \oplus \rho_2^*$ *is also a taut representation.*

Proof. Fix a point $p = (p_1, p_2) \in V_1 \oplus V_2$. We will show that the orbit $\tau(G)p$ is taut. Denote by $\theta : V_2 \to V_2$ the conjugation with respect to the invariant complex structure on V_2. Then due to the fact that ρ_2 viewed as a complex representation is unitary, we have that

$$\theta \circ \rho_2^*(g) = \rho_2(g) \circ \theta$$

for all $g \in G$. Define $F : V_1 \oplus V_2 \to V_1 \oplus V_2$ by setting $F = \mathrm{id}_{\mathrm{V}_1} \oplus \theta$. Then F is an isometry of $V_1 \oplus V_2$. Moreover,

$$F(\tau(g)p) = (\rho_1(g)p_1, \theta\rho_2^*(g)p_2) = (\rho_1(g)p_1, \rho_2(g)\theta p_2) = \rho(g)(p_1, \theta p_2)$$

for all $g \in G$. It follows that

$$\tau(G)p = F^{-1}(\rho(G)(F(p))).$$

Since $\rho(G)(F(p))$ is taut, this shows that $\tau(G)p$ is also taut. □

Now we start the proof of Theorem 1. Throughout the rest of this paper, we let $\rho: G \to \mathbf{O}(V)$ be a taut reducible representation where G is a compact connected simple Lie group. Of course, we may assume that ρ does not contain trivial summands. Write $\rho = \rho_1 \oplus \rho_2$, where ρ_1 is irreducible. Then ρ_1 is orbit equivalent to the isotropy representation of an irreducible symmetric space. We shall run through all the possibilities for G and ρ_1, where we find it convenient to consider separately the cases $G = \mathbf{Spin}(n)$ and $G = \mathbf{SO}(n)$.

3.1. The case $G = \mathbf{SO}(n)$, $n = 3$ or $n \geq 5$. Here, ρ_1 is one of the following:

(a) the vector representation on $\mathbf{R}^n$;
(b) the adjoint representation on $\Lambda^2\mathbf{R}^n$, where $n \geq 5$;
(c) the representation on the space of traceless symmetric matrices $S_0^2\mathbf{R}^n$.

The possibilities (b) and (c) are ruled out by Examples 1. **Spin**(8) Example 1(i) Now possibility (a) is taken care of by the following proposition (compare [TT97], Examples 3.14).

PROPOSITION 4. *Assume that $n \geq 3$ and ρ is the sum of $k > 1$ copies of the vector representation. Then ρ is taut.*

Proof. Let $V = \mathbf{R}^n \oplus \cdots \oplus \mathbf{R}^n$, k copies. Suppose first that $k \leq n$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbf{R}^n$, and let $p = (e_1, \ldots, e_k) \in V$. View V as the space of real $n \times k$-matrices, and let $\hat{G} = \mathbf{SO}(n) \times \mathbf{SO}(k)$ act on V by $(A, B) \cdot X = AXB^{-1}$, where $(A, B) \in \hat{G}$ and $X \in V$. Then $\hat{G}p = Gp$. Since $(\hat{G}, V)$ is the isotropy representation of the Grassmann manifold $G_k(\mathbf{R}^{n+k})$, we have that Gp is taut. Next, suppose that $k > 1$ is arbitrary and let $q = (v_1, \ldots, v_k) \in V$ be an arbitrary nonzero point. Then there is a non-singular $k \times k$ matrix M such that right-multiplying q by M gives $qM = (e_1, \ldots, e_l, 0, \ldots, 0) \in V$, where $1 \leq l \leq n$. It follows from the above that $G(qM) = (Gq)M$ is taut. Since a taut submanifold in Euclidean space is tight, and tightness is invariant under linear transformations, Gq is tight. But Gq lies in a sphere, and so it is taut. This completes the proof that ρ is taut. □

3.2. The case $G = \mathbf{SU}(n)$, $n = 3$ or $n \geq 5$. Here ρ_1 is one of the following:

(a) the vector representation on $\mathbf{C}^n$ or its dual;
(b) the adjoint representation on $\mathfrak{su}(n)$;
(c) a real form of the representation of $\mathbf{SU}(8)$ on $\Lambda^4\mathbf{C}^8$;
(d) the representation on the space of skew-symmetric matrices $\Lambda^2\mathbf{C}^{2p+1}$, where $p \geq 2$, or its dual.

In view of Lemma 2, we do not need to consider the dual representations in items (a) and (d). The possibilities (b) (even if $n = 4$) and (c) are ruled out by Examples 1. Consider now the case in which ρ_1 is as in (d). Here, a principal isotropy subgroup H is given by p diagonal blocks, each one isomorphic to $\mathbf{SU}(2)$. Denote the representation spaces of ρ_1 and ρ_2 by V_1 and V_2. Now there exists $v_1 \in V_1$ such that $G_{v_1} = H \cong \mathbf{SU}(2)^p$. For the purpose of proving that ρ is not taut, we can assume that ρ_2 is irreducible, as we do now. Then ρ_2 also falls into cases (a) or (d). If ρ_2 is the vector representation, then we can find $v_2 \in V_2$ such that $Hv_2 \approx S^3 \times \cdots \times S^3$, p factors. In this case, $G(v_1, v_2) \approx \mathbf{SU}(2p+1)$; since the third Betti number of a compact connected simple Lie group is 1 and the third Betti number of Hv_2 is $p \geq 2$, ρ cannot be taut by Proposition 1. On the other hand, if ρ_2 is as in (d), it is not difficult to see that $\rho_2|G_{v_1}$ contains as a summand a representation equivalent to that in Lemma 1(b), and thus ρ cannot be taut by Proposition 1. In any event, this shows that ρ is not taut if ρ_1 is as in (d). Now the case in which ρ_1 is as in (a) is covered by the following proposition.

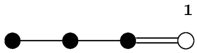
PROPOSITION 5. *Assume that $n \geq 3$ and ρ is the sum of k copies of the vector representation. If $1 \leq k < n$, then ρ is taut. If $k \geq n$, then ρ is not taut.*

Proof. In the case $1 \leq k < n$, we invoke the fact that the isotropy representation of the Grassmann manifold $G_k(\mathbf{C}^{n+k})$ is $\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(k))$ acting on the space of complex $n \times k$ matrices, and it is orbit equivalent to its restriction to the subgroup $\mathbf{SU}(n) \times \mathbf{SU}(k)$ if $k \neq n$ (see [EH99]). It then follows as in Proposition 4 that ρ is taut. In the case $k \geq n$, it is enough to prove nontautness for $k = n$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbf{C}^n$. The isotropy subgroup at e_1 is isomorphic to $\mathbf{SU}(n-1)$, and the slice representation at e_1 decomposes into a sum of trivial representations and $\mathbf{C}^{n-1} \oplus \cdots \oplus \mathbf{C}^{n-1}$, $n-1$ copies. We use Proposition 2 and induction to reduce the proof to the case of $\mathbf{SU}(3)$ acting on $\mathbf{C}^3 \oplus \mathbf{C}^3 \oplus \mathbf{C}^3$. Let $p = (e_1, e_2, e_3)$, and denote by M the $\mathbf{SU}(3)$-orbit through p. Then M is the standard inclusion of $\mathbf{SU}(3)$ into the space $M(3, \mathbf{C})$ of complex 3×3-matrices. The tangent space T_pM is the Lie algebra $\mathfrak{su}(3)$, and the normal space ν_pM is $\mathbf{C}p \oplus i\mathfrak{su}(3)$. By Remark 1(b), it suffices to show that a height function is not perfect. Let $h : M \to \mathbf{R}$ be the height function defined by p. We find the critical points of h. Note that gp, for $g \in \mathbf{SU}(3)$, is a critical point of h if and only if $p \in \nu_{gp}M$, or, what amounts to the same, $g^{-1}p \in \nu_pM$. Now it is easy to see that gp is a critical point of h if and only if $g = \omega I$, where ω is a cubic root of unity and I is the identity matrix, or g is conjugate to a diagonal matrix with entries -1, -1 and 1. It follows that the critical set of h consists of 3 isolated points and a submanifold diffeomorphic to $\mathbf{C}P^2$, whence the sum of its Betti numbers is 6. Since $\mathbf{SU}(3)$ has the homology of $S^3 \times S^5$, h is not perfect in the sense of Bott; see Remark 1(c). Hence, M is not taut. $\square$

3.3. The case $G = \mathbf{Sp}(n)$, $n \geq 3$. Here, ρ_1 is one of the following:

(a) the vector representation on $\mathbf{C}^{2n}$;

(b) the adjoint representation on $\mathfrak{sp}(n)$;

(c) a real form of the 42-dimensional representation  of $\mathbf{Sp}(4)$;

(d) a real form of the representation $\Lambda^2\mathbf{C}^{2n} - \mathbf{C}$.

The possibilities (b) (even if $n = 2$) and (c) are ruled out by Examples 1. Consider the possibility (d). Here, a principal isotropy subgroup H is given by the diagonal embedding of $\mathbf{Sp}(1)^n$ into $\mathbf{Sp}(n)$, so there exists $v_1 \in V_1$ such that $G_{v_1} = H \cong \mathbf{Sp}(1)^n$. To prove that ρ is not taut, we can assume that ρ_2 is irreducible, as we do now, and then ρ_2 is as in (a) or in (d). If ρ_2 is as in (a), the proof follows as in Section 3.2 to deduce that ρ is not taut. If ρ_2 is as in (d), Proposition 6 below implies that ρ is not taut.

PROPOSITION 6. *Let V_n denote a real form of the representation $\Lambda^2\mathbf{C}^{2n} - \mathbf{C}$ of $\mathbf{Sp}(n)$, where $n \geq 3$. Then $(\mathbf{Sp}(n), V_n \oplus V_n)$ is not taut.*

We postpone the proof of Proposition 6 to the end of the paper since the methods used to prove it better belong there. Finally, the case in which ρ_1 is as in (a) is covered by the following proposition.

PROPOSITION 7. *Assume that $n \geq 1$ and ρ is the sum of $k > 1$ copies of the vector representation. Then ρ is taut.*

Proof. The proof is analogous to the proof of Proposition 4. □

3.4. The case G is exceptional. First note that no summand of ρ can be the adjoint representation by Example 1(ii).

If $G = \mathbf{G}_2$, then ρ is the sum of k copies of the 7-dimensional representation. If $k = 2$, ρ is orbit equivalent to $(\mathbf{SO}(7), \mathbf{R}^7 \oplus \mathbf{R}^7)$, which is taut. If $k = 3$, ρ is not taut due to the following proposition.

PROPOSITION 8. *We have that $(\mathbf{G}_2, \mathbf{R}^7 \oplus \mathbf{R}^7 \oplus \mathbf{R}^7)$ is not taut.*

We postpone the proof of Proposition 8 to the end of the paper for the sake of convenience.

If $G = \mathbf{F}_4$, then ρ is the sum of k copies of the 26-dimensional representation. Suppose $k = 2$, $\rho = \rho_1 \oplus \rho_2$. Then there is an isotropy subgroup H of ρ_1 isomorphic to $\mathbf{Spin}(9)$. Now $\rho_2|H$ decomposes as $\mathbf{R} \oplus \mathbf{R}^9 \oplus \mathbf{R}^{16}$, and it is not taut by Proposition 18 below. Hence, ρ is not taut by Proposition 1.

$\mathbf{E}_6$, $\mathbf{E}_7$, and $\mathbf{E}_8$ do not admit representations orbit equivalent to the isotropy representation of a symmetric space.

3.5. The case $G = \mathbf{Spin}(n)$, $n = 3$ or $n \geq 5$. This is case is more involved than the previous ones. Since the case of $\mathbf{SO}(n)$ has already been considered in Section 3.1, we may assume that a summand of ρ is a spin representation in this section. Therefore, the only values of n which need to be considered are 3, 5, 6, 7, 8, 9, 10, and 16.

3.5.1. $G = \mathbf{Spin}(3)$. Here, $G = \mathbf{SU}(2) = \mathbf{Sp}(1)$. By the discussion in Section 3.1, the admissible summands of ρ are the vector representation of $\mathbf{SU}(2)$ on $\mathbf{C}^2$ and the representation on $\mathbf{R}^3$ given by $\mathbf{SU}(2) \to \mathbf{SO}(3)$. The sum of an arbitrary number of copies of $\mathbf{C}^2$ is taut by Proposition 7. On the other hand, $\mathbf{C}^2 \oplus \mathbf{R}^3$ is not taut, because the principal orbit through a point $(a,b) \in \mathbf{C}^2 \oplus \mathbf{R}^3$ with a, $b \neq 0$ is substantial and diffeomorphic to S^3, but as mentioned in the Introduction, a sphere can be taut only in substantial codimension one.

3.5.2. $G = \mathbf{Spin}(5)$. Here, $G = \mathbf{Sp}(2)$. By the discussion in Section 3.1, the admissible summands of ρ are the vector representation of $\mathbf{Sp}(2)$ on $\mathbf{C}^4$ and the representation on $\mathbf{R}^5$ given by $\mathbf{Sp}(2) \to \mathbf{SO}(5)$. The situation in which $\mathbf{R}^5$ is not present is covered by Proposition 7. On the other hand, we have the following proposition.

Proposition 9. *$\mathbf{C}^4 \oplus \mathbf{R}^5$ is not taut.*

Proof. Note that the principal orbits are substantial embeddings of $\mathbf{Sp}(2)$ in S^{12}. We will show that $\mathbf{Sp}(2)$ can admit a taut substantial embedding of codimension 2 in a sphere S^N only if $N = 15$ following an argument which appeared in [Gal93], page 75.

So, suppose that X is diffeomorphic to $\mathbf{Sp}(2)$ and tautly embedded in S^N with $N \geq 12$. Let Y be a sufficiently small tubular neighborhood of X in S^N. X has the homology of $S^3 \times S^7$, so its homology groups vanish except in dimensions 0, 3, 7, and 10. Since $2 \times 3 \neq 7$, it follows as in Proposition 2.2 of [Oza86] that Y is a compact proper Dupin hypersurface. Moreover, a Morse distance function on Y can have critical points of index 0, 3, 7, and 10 only. By the Morse index theorem, the multiplicities of the first three principal curvatures of Y are $m_1 = 3$, $m_2 = 4$, and $m_3 = 3$. According to Theorem C in [GH91], there exists at most 2 different multiplicities k, l, and $g = 2$ or 4 in case $k \neq l$. Therefore, the fourth principal curvature of Y has multiplicity $m_4 = 4$. It follows that $\dim Y = 14$, and hence, $N = 15$. □

3.5.3. $G = \mathbf{Spin}(6)$. Here $G = \mathbf{SU}(4)$. By the discussion in Section 3.1, the admissible summands of ρ are the vector representation of $\mathbf{SU}(4)$ on $\mathbf{C}^4$ and the representation on $\mathbf{R}^6$ given by $\mathbf{SU}(4) \to \mathbf{SO}(6)$. (Note that $(\mathbf{SU}(4), \mathbf{C}^{4*})$ needs not to be considered owing to Lemma 2.) The situation in which $\mathbf{R}^6$ is not present is covered by Proposition 5. Also, $\mathbf{C}^4 \oplus \mathbf{R}^6$ is taut because the singular orbits are round spheres in $\mathbf{C}^4$ and $\mathbf{R}^6$, and the principal orbits are products of those. The following two propositions settle down this case.

Proposition 10. *$\mathbf{C}^4 \oplus \mathbf{R}^6 \oplus \mathbf{R}^6$ is not taut.*

Proof. Let $p \in \mathbf{R}^6$. Then the slice representation at p is $\mathbf{Spin}(5) = \mathbf{Sp}(2)$ acting on $\mathbf{R} \oplus \mathbf{R} \oplus \mathbf{C}^4 \oplus \mathbf{R}^5$. The result follows from Propositions 9 and 2. □

Proposition 11. *$\mathbf{C}^4 \oplus \mathbf{C}^4 \oplus \mathbf{R}^6$ is not taut.*

Proof. We will show that a certain orbit is not taut by finding an explicit height function which is not perfect. We need to have a good parametrization of the orbits. It is useful to use Cayley numbers. Recall that the Cayley algebra can be viewed as $\mathbf{Ca}=\mathbf{H}\oplus\mathbf{H}e$ via the Cayley–Dickson process, where $\mathbf{H}=\mathbf{R}\langle 1,i,j,k\rangle$ is the quaternion algebra (see the Appendix IV.A in [HL82]). Then $\mathbf{Ca}=\mathbf{R}\langle 1,i,j,k,e,ie,je,ke\rangle$. According to [CR98], upon identifying $\mathbf{Ca}\cong\mathbf{R}^8$ and using Cayley multiplication,

$$\begin{aligned}\mathbf{Spin}(8)&=\{(A,B,C)\in\mathbf{SO}(8)\times\mathbf{SO}(8)\times\mathbf{SO}(8)\\ &\qquad A(\xi\eta)=B(\xi)C(\eta),\text{for all } \xi,\ \eta\in\mathbf{Ca}\},\\ \mathbf{Spin}(7)&=\{(A,B,C)\in\mathbf{Spin}(8)A(1)=1\},\\ &=\{(A,B,C)\in\mathbf{Spin}(8)C=\tilde{B}\},\end{aligned}$$

where $\tilde{B}(x)=\overline{B(\bar{x})}$, and

$$\mathbf{Spin}(6)=\{(A,B,\tilde{B})\in\mathbf{Spin}(7):A(i)=i\}.$$

Also, the isomorphism $\mathbf{Spin}(6)\to\mathbf{SU}(4)$ is given by $(A,B,\tilde{B})\mapsto B$, and the projection $\mathbf{Spin}(6)\to\mathbf{SO}(6)$ is given by $(A,B,\tilde{B})\to A$. Therefore, the covering $\varphi:\mathbf{SU}(4)\to\mathbf{SO}(6)$ satisfies $\varphi(g)(x)=g(x)\overline{g(1)}=g(1)\overline{g(\bar{x})}$, where $g\in\mathbf{SU}(4)$ and $x\in\mathbf{R}^6$. Here, we regard $\mathbf{SU}(4)$ as the subgroup of $\mathbf{SO}(8)$ defined by the complex structure in $\mathbf{R}^8$ specified by left multiplication by the element i. This identifies $\mathbf{Ca}\cong\mathbf{C}^4$. Now (note that $i(ke)=je$) $\mathbf{C}^4=\mathbf{C}\langle 1,j,e,ke\rangle$, $\mathbf{R}^6=\mathbf{R}\langle j,k,e,ie,je,ke\rangle$.

Fix the base point $p=(1,j,e)\in V=\mathbf{C}^4\oplus\mathbf{C}^4\oplus\mathbf{R}^6$. Let $G=\mathbf{SU}(4)$ act on V. Then G_p is trivial. Let $M=Gp$, principal orbit diffeomorphic to $\mathbf{SU}(4)$. M can also be parametrized by the Stiefel manifold $St_3(\mathbf{C}^4)$. In fact, given $(z_1,z_2,z_3)\in St_3(\mathbf{C}^4)$, there is a unique $g\in\mathbf{SU}(4)$ such that $g^{-1}(1)=z_1$, $g^{-1}(j)=z_2$, and $g^{-1}(e)=z_3$. Then we get $g^{-1}(1,j,e)=(z_1,z_2,z_3\bar{z}_1)\in M$. View $p=(1,j,e)$ as a vector in ν_pM, and let $h:M\to\mathbf{R}$ be the height function defined by p. We have that $gp\in M$, $g\in\mathbf{SU}(4)$, is a critical point of h if and only if $p\in\nu_{gp}M=g\nu_pM$. It is easy to compute that the normal space to M at $p=(1,j,e)$ is spanned by

$$(1,0,0),(0,j,0),(0,0,e),(j,1,0),(k,-i,0),(je,e,j),(ke,-ie,k).$$

Now the condition that $g^{-1}p\in\nu_pM$ is that there exist A, B, C, D, E, F, $G\in\mathbf{R}$ such that

$$\begin{aligned}(z_1,z_2,z_3\bar{z}_1)=(&A+Dj+Ek+Fje+Gke,\\ &D-Ei+Bj+Fe-Gie,Fj+Gk+Ce).\end{aligned}$$

The relations $(z_i,z_j)=\delta_{ij}$, where $(\cdot,\cdot)$ denotes the Hermitian inner product in $\mathbf{C}^4$, yield the following relations:

$$\begin{aligned}(A+B)(D+Ei)&=0,\\ (F-Gi)(AB+BC+AC-F^2-G^2-D^2-E^2)&=0,\end{aligned}$$

$$\begin{aligned} A^2+D^2+E^2+F^2+G^2&=1,\\ A^2-B^2&=0,\\ C^2+F^2+G^2&=1. \end{aligned}$$

The system admits exactly the following solutions:

- $A=B=-C=\pm 1$, $D=E=F=G=0$;
- $A=B=C=\pm 1$, $D=E=F=G=0$;
- $A=B=C=\pm\frac{1}{2}$, $D=E=0$, $F^2+G^2=\frac{3}{4}$;
- $A=-B$, $C=\pm 1$, $F=G=0$, $A^2+D^2+E^2=1$.

Since $g^{-1}p\mapsto gp$ is a well-defined homeomorphism of M, we deduce that the critical set of h consists of 4 points, 2 circles and 2 spheres. Now the sum of the Betti numbers of the critical manifolds of h is 12. Since $\mathbf{SU}(4)$ has the homology of $S^3\times S^5\times S^7$, M is not taut. □

3.6. $G=\mathbf{Spin}(7)$**.** By the discussion in Section 3.1, the admissible summands of ρ are the vector representation on $\mathbf{R}^7$ and the spin representation on $\mathbf{R}^8$. We first note that $\mathbf{R}^8\oplus\mathbf{R}^7$ is taut because the singular orbits are round spheres in $\mathbf{R}^8$ and $\mathbf{R}^7$, and the principal orbits are products of those. Moreover, $\mathbf{R}^8\oplus\mathbf{R}^8$ and $\mathbf{R}^8\oplus\mathbf{R}^8\oplus\mathbf{R}^8$ are taut because $\mathbf{Spin}(7)$ is transitive on the Stiefel manifolds $St_2(\mathbf{R}^8)$ and $St_3(\mathbf{R}^8)$, so the actions of $\mathbf{Spin}(7)$ on these spaces are orbit equivalent to the actions of $\mathbf{SO}(8)$. We also note that if ρ has 4 summands and $\mathbf{R}^8$ is one of them, say V_1, then the slice representation at a point in V_1 is $\mathbf{G}_2$ acting on $\mathbf{R}^7\oplus\mathbf{R}^7\oplus\mathbf{R}^7$, which is not taut by Proposition 8; hence, ρ is not taut by Proposition 2. We finish the discussion in this case with the following two propositions.

Proposition 12. *$\mathbf{R}^7\oplus\mathbf{R}^7\oplus\mathbf{R}^8$ is taut.*

Proof. We shall use the reduction principle as described in Proposition 3. In order to have a good description of the representation, we resort to Cayley numbers as in the proof of Proposition 11. View $\mathbf{R}^8=\mathbf{R}\langle 1,i,j,k,e,ie,je,ke\rangle$ and $\mathbf{R}^7=\mathbf{R}\langle i,j,k,e,ie,je,ke\rangle$. Let $G=\mathbf{Spin}(7)$, $V=\mathbf{R}^7\oplus\mathbf{R}^7\oplus\mathbf{R}^8$. The action of G on V is given by $(A,B,\tilde{B})\mapsto(A,A,B)$. The isotropy of G at $p=(i,j,1)\in V$ is

$$H=\{(A,A,A)\in\mathbf{Spin}(8):A\in\mathbf{Sp}(2),\ A\text{ fixes }1\}\cong\mathbf{Sp}(1),$$

where we regard $\mathbf{Sp}(2)$ as the subgroup of $\mathbf{SO}(8)$ defined by the complex structures in $\mathbf{R}^8$ given by the left multiplications by the elements i, j. This identifies $\mathbf{R}^8\cong\mathbf{H}\langle 1,e\rangle$.

The description of H shows that the cohomogeneity of (G,V) is 4 and the fixed point subspace

$$V^H=\mathbf{R}\langle i,j,k\rangle\oplus\mathbf{R}\langle i,j,k\rangle\oplus\mathbf{R}\langle 1,i,j,k\rangle\cong\mathbf{R}^{10}.$$

It follows from Theorem 3 that $\dim\bar{N}=6$. The normalizer N of H in G is the same as the stabilizer of V^H in G. Suppose that $(A,B,\tilde{B})\in N$. Then we can write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1,\ A_2 \in \mathbf{SO}(4),\ A_1(1) = 1$, and we view $\mathbf{R}^4 = \mathbf{R}\langle 1, i, j, k\rangle$. Since $\mathbf{Sp}(1) \times \mathbf{Sp}(1) \to \mathbf{SO}(4),\ (p, q) \mapsto l_p r_{\bar{q}}$ (notation as in Lemma 1) is a double covering, we can write $A_2 = l_p r_{\bar{q}}$ for unique (p, q) modulo ± 1. Similarly, $\mathbf{Sp}(1) \to \mathbf{SO}(3),\ s \mapsto l_s r_{\bar{s}}$ is a double covering, so we can write $A_1 = l_s r_{\bar{s}}$ for a unique s modulo ± 1. We deduce that (compare [CR98], Section 2)

$$(1) \qquad (A, B, \tilde{B}) = \left(\begin{pmatrix} l_s r_{\bar{s}} & 0 \\ 0 & l_p r_{\bar{q}} \end{pmatrix},\ \begin{pmatrix} l_s r_{\bar{q}} & 0 \\ 0 & l_p r_{\bar{s}} \end{pmatrix},\ \begin{pmatrix} l_q r_{\bar{s}} & 0 \\ 0 & l_p r_{\bar{s}} \end{pmatrix} \right).$$

Therefore, N consists of the elements of the form (1) for $p,\ q,\ s \in \mathbf{Sp}(1)$, and H consists of the elements with $q = s = 1$. Now

$$\bar{N} = N/H \cong \mathbf{Sp}(1) \times_{\mathbf{Z}_2} \mathbf{Sp}(1) = \{(q, s) \in \mathbf{Sp}(1) \times \mathbf{Sp}(1) : (q, s) \sim (-q, -s)\},$$

the action of $\bar{N}$ on V^H is given by

$$(q, s) \in \bar{N} \mapsto (l_s r_{\bar{s}}, l_s r_{\bar{s}}, l_s r_{\bar{q}}) \in \mathbf{SO}(3) \times \mathbf{SO}(3) \times \mathbf{SO}(4),$$

and thus it is orbit equivalent to the product of the standard action of $\mathbf{SO}(3)$ on $\mathbf{R}^3 \oplus \mathbf{R}^3$ by the standard action of $\mathbf{Sp}(1)$ on $\mathbf{C}^2$. Since these are taut representations, we deduce that $(\bar{N}, V^H)$ is also taut. Now let L be the $\mathbf{Z}_2$-subgroup of H generated by the element (1) with $q = s = 1,\ p = -1$. Then $V^L = V^H$. It follows from Proposition 3 that (G, V) is taut. □

PROPOSITION 13. $\mathbf{R}^7 \oplus \mathbf{R}^8 \oplus \mathbf{R}^8$ *is not taut.*

Proof. We use a method similar to that of the proof of Proposition 12. Let $G = \mathbf{Spin}(7),\ V = \mathbf{R}^7 \oplus \mathbf{R}^8 \oplus \mathbf{R}^8$. The action of G on V is given by $(A, B, \tilde{B}) \mapsto (A, B, B)$. The isotropy of G at $p = (i, 1, j) \in V$ is

$$H = \{(A, A, A) \in \mathbf{Spin}(8) : A \in \mathbf{SU}(4),\ A \text{ fixes } 1,\ j\} \cong \mathbf{SU}(2),$$

and the cohomogeneity of (G, V) is 5. The fixed point subspace

$$V^H = \mathbf{R}\langle i, j, k\rangle \oplus \mathbf{R}\langle 1, i, j, k\rangle \oplus \mathbf{R}\langle 1, i, j, k\rangle \cong \mathbf{R}^{11},$$

and $\dim \bar{N} = 6$. Now N, H and $\bar{N}$ are as in Proposition 12, and the action of $\bar{N}$ on V^H is given by

$$(q, s) \in \bar{N} \mapsto (l_s r_{\bar{s}}, l_s r_{\bar{q}}, l_s r_{\bar{q}}) \in \mathbf{SO}(3) \times \mathbf{SO}(4) \times \mathbf{SO}(4).$$

Let $M = Gp$, and let h denote the height function defined by p on M. It is not difficult to see that the critical set of the restriction $h|M \cap V^H$ coincides with the critical set of h (compare Lemma 3.17 in [GT]). But $M \cap V^H = \bar{N}p$, and a tedious computation shows that the critical set of $h|\bar{N}p$ consists of 8 points and 2 circles, hence its sum of the Betti numbers is 12. If M was taut, it would have to have the homology of $S^5 \times S^6 \times S^7$ by Proposition 1, so the sum of its Betti numbers would have to be 8. It follows that M is not taut. □

3.7. $G = \mathbf{Spin}(8)$**.** By the discussion in Section 3.1, the admissible summands of ρ are the vector representation which we denote by $\mathbf{R}^8_0$, and the half-spin representations, which we denote by $\mathbf{R}^8_+$ and $\mathbf{R}^8_-$. The group of automorphisms of $\mathbf{Spin}(8)$ is isomorphic to the dihedral group of degree 3, and it permutes the representations $\mathbf{R}^8_0$, $\mathbf{R}^8_+$, $\mathbf{R}^8_-$, so this reduces the number of cases to be considered. We now note that $\mathbf{R}^8_0 \oplus \mathbf{R}^8_+$ is taut because the principal orbits are products of spheres; up to permutations, there are no other representations with two summands which need to be considered. Similarly, in the case of three summands, there are only two cases to be considered; see Propositions 14 and 15. In the case of four summands, at least two of them coincide, and we can assume that those are $\mathbf{R}^8_0$. So, there are three cases to be considered: $\mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_+ \oplus \mathbf{R}^8_+$, $\mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_+ \oplus \mathbf{R}^8_-$, and $\mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_+$; the first one of these is not taut since a slice representation contains $(\mathbf{Spin}(7), \mathbf{R}^7 \oplus \mathbf{R}^8 \oplus \mathbf{R}^8)$, which is not taut by Proposition 13, and we can apply Proposition 2; the second one is not taut because it contains as a summand $(\mathbf{Spin}(8), \mathbf{R}^8_0 \oplus \mathbf{R}^8_+ \oplus \mathbf{R}^8_-)$, which is not taut by Proposition 15; and the third one is taut by Proposition 16. In the case of five summands, there is always a slice representation equivalent to $(\mathbf{G}_2, \mathbf{R}^7 \oplus \mathbf{R}^7 \oplus \mathbf{R}^7)$, which is not taut by Proposition 8, and we can apply Proposition 2.

Proposition 14. $\mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_+$ *is taut.*

Proof. We use a method similar to that of the proof of Proposition 12. Let $G = \mathbf{Spin}(8)$, $V = \mathbf{R}^8_0 \oplus \mathbf{R}^8_0 \oplus \mathbf{R}^8_+$. The action of G on V is given by $(A,B,C) \mapsto (A,A,B)$. The isotropy of G at $p = (1,i,1) \in V$ is

$$H = \{(A,A,A) \in \mathbf{Spin}(8) : A \in \mathbf{SU}(4) \text{ fixes } 1\} \cong \mathbf{SU}(3),$$

and the cohomogeneity of (G,V) is 4. The fixed point subspace

$$V^H = \mathbf{R}\langle 1,i\rangle \oplus \mathbf{R}\langle 1,i\rangle \oplus \mathbf{R}\langle 1,i\rangle \cong \mathbf{R}^6,$$

and $\dim \bar{N} = 2$. We now construct two one-parameter subgroups of N which do not lie in H. Let $A \in \mathbf{SO}(8)$ be the rotation by θ on the plane $\mathbf{R}\langle 1,i\rangle$ fixing its orthogonal complement, and let $B(x) = e^{\frac{i\theta}{2}}x$, $C(x) = xe^{\frac{i\theta}{2}}$, for $x \in \mathbf{Ca}$. Then $(A,B,C) \in N$. We denote this transformation by t_θ. Next, let $A \in \mathbf{SO}(8)$ fix 1, i, and let $B \in \mathbf{SU}(4)$ act on $\mathbf{C}\langle 1,j,e,ke\rangle$ by the matrix $\operatorname{diag}(e^{i\varphi}, e^{-i\varphi}, 1, 1)$. Then $(A,B,\tilde{B}) \in N$. We denote this transformation by s_φ. Now

$$\bar{N}^0 = N^0/H \cong S^1 \times S^1 = \{(t_\theta, s_\varphi)\},$$

and the action of $\bar{N}^0$ on V^H is given by

$$(t_\theta, s_\varphi) \in \bar{N}^0 \mapsto \left(e^{i\theta}, e^{i\theta}, e^{i(\frac{\theta}{2}+\varphi)}\right) \in \mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1).$$

This action is clearly taut. Let L be the subgroup of H generated by the diagonal matrices with ± 1 entries. Then $V^L = V^H$, and (G,V) is taut by Proposition 3. $\square$

Proposition 15. $\mathbf{R}^8_0\oplus\mathbf{R}^8_+\oplus\mathbf{R}^8_-$ *is not taut.*

Proof. Here, the action of G on V is given by $(A,B,C)\mapsto(A,B,C)$. The isotropy of G at $p=(1,1,i)\in V$ is

$$H=\{(A,A,A)\in\mathbf{Spin}(8): A\in\mathbf{SU}(4)\text{ fixes }1\}\cong\mathbf{SU}(3),$$

and the cohomogeneity of (G,V) is 4. The fixed point subspace V^H and $\bar{N}^0$ are as in Proposition 14, and the action of $\bar{N}^0$ on V^H is given by

$$(t_\theta,s_\varphi)\in\bar{N}^0\mapsto\left(e^{i\theta},e^{i(\frac{\theta}{2}+\varphi)},e^{i(\frac{\theta}{2}-\varphi)}\right)\in\mathbf{U}(1)\times\mathbf{U}(1)\times\mathbf{U}(1).$$

Since this action is equivalent to that of Lemma 1(a), it is not taut. It follows that (G,V) is not taut by the final argument in the proof of Lemma 6.11 in [GT03]. ☐

Proposition 16. $\mathbf{R}^8_0\oplus\mathbf{R}^8_0\oplus\mathbf{R}^8_0\oplus\mathbf{R}^8_+$ *is taut.*

Proof. Here, the action of G on V is given by $(A,B,C)\mapsto(A,A,A,B)$. The isotropy of G at $p=(1,i,j,1)\in V$ is the same H as in Proposition 12, the cohomogeneity is 7, the fixed point subspace

$$V^H=\mathbf{R}\langle 1,i,j,k\rangle\oplus\mathbf{R}\langle 1,i,j,k\rangle\oplus\mathbf{R}\langle 1,i,j,k\rangle\oplus\mathbf{R}\langle 1,i,j,k\rangle\cong\mathbf{R}^{16},$$

and so $\dim\bar{N}=9$. As in Proposition 12, we compute that

$$N=\left\{\left(\begin{pmatrix}l_sr_{\bar{t}} & 0\\ 0 & l_pr_{\bar{q}}\end{pmatrix},\ \begin{pmatrix}l_sr_{\bar{q}} & 0\\ 0 & l_pr_{\bar{t}}\end{pmatrix},\ \begin{pmatrix}l_qr_{\bar{t}} & 0\\ 0 & l_pr_{\bar{s}}\end{pmatrix}\right): p,q,s,t\in\mathbf{Sp}(1)\right\},$$

and $(\bar{N},V^H)$ is

$$(q,s,t)\in\bar{N}\mapsto(l_sr_{\bar{t}},l_sr_{\bar{t}},l_sr_{\bar{t}},l_sr_{\bar{q}})\in\mathbf{SO}(4)\times\mathbf{SO}(4)\times\mathbf{SO}(4)\times\mathbf{SO}(4).$$

This action is orbit equivalent to the product of $(\mathbf{SO}(4),\mathbf{R}^4\oplus\mathbf{R}^4\oplus\mathbf{R}^4)$ and $(\mathbf{Sp}(1),\mathbf{C}^2)$, hence taut. We take L as in Proposition 12 and we get that (G,V) is taut by Proposition 3. ☐

3.8. $G=\mathbf{Spin}(9)$**.** By the discussion in Section 3.1, the admissible summands of ρ are the vector representation on $\mathbf{R}^9$ and the spin representation on $\mathbf{R}^{16}$. Note that $\mathbf{R}^{16}\oplus\mathbf{R}^{16}\oplus\mathbf{R}^{16}$ is not taut since a slice representation is $(\mathbf{Spin}(7),\mathbf{R}^7\oplus\mathbf{R}^8\oplus\mathbf{R}^7\oplus\mathbf{R}^8)$. The other possibilities are covered by the following two propositions.

Proposition 17. $\mathbf{R}^{16}\oplus\mathbf{R}^{16}$ *is taut.*

Proof. We need to have a good description of the spin representation of $\mathbf{Spin}(9)$. We start by letting $\{e_1,\dots,e_n\}$ be the canonical basis of $\mathbf{R}^n$, and recalling that the Clifford algebra $\mathcal{C}\ell(n)$ (resp. $\mathcal{C}\ell_+(n)$) is the real associative algebra with unit generated by $e_1,\dots,e_n$ subject to the relations $e_ie_j+e_je_i=-2\delta_{ij}$ (resp. $e_ie_j+e_je_i=+2\delta_{ij}$). The group $\mathbf{Spin}(n)$ (resp. $\mathbf{Spin}_+(n)$) is the multiplicative subgroup of $\mathcal{C}\ell(n)$ (resp. $\mathcal{C}\ell_+(n)$) consisting of even products of elements in the unit sphere of $\mathbf{R}^n$. It is clear that there is an isomorphism

$\mathcal{C}\ell(n) \otimes \mathbf{C} \to \mathcal{C}\ell_+(n) \otimes \mathbf{C}$, induced by $e_i \mapsto \sqrt{-1}e_i$, which restricts to an isomorphism $\mathbf{Spin}(n) \to \mathbf{Spin}_+(n)$ (see, e.g., Chapters 13 and 15 in [Pos86]).

Now view

$$\mathbf{R}^9 = \mathbf{R} \oplus \mathbf{Ca}, \qquad \mathbf{R}^{16} = \mathbf{Ca} \oplus \mathbf{Ca},$$

where $\mathbf{Ca} = \mathbf{R}\langle 1, e, i, j, k, ei, ej, ek\rangle$, and write $\{e_0; e_1, \ldots, e_8\}$ for the basis $\{1; 1, \ldots, ek\}$ of $\mathbf{R}^9$. Define

$$\varphi : \mathbf{R}^9 \to M(16, \mathbf{R}), \qquad (r, u) \mapsto \begin{pmatrix} rI_8 & R_u \\ R_{\bar{u}} & -rI_8 \end{pmatrix},$$

where $r \in \mathbf{R}$, $u \in \mathbf{Ca}$, and $R_u : \mathbf{Ca} \to \mathbf{Ca}$ is right Cayley multiplication. Then $\varphi(r, u)^2 = (r^2 + \|u\|^2)I_{16}$. It follows that φ induces a homomorphism $\mathcal{C}\ell_+(9) \to M(16, \mathbf{R})$. Restricting to $\mathbf{Spin}_+(9)$ and identifying $\mathbf{Spin}(9) \cong \mathbf{Spin}_+(9)$, we finally get the spin representation $\Delta_9 : \mathbf{Spin}(9) \to \mathbf{SO}(16)$.

Now consider $G = \mathbf{Spin}(9)$ acting on $V = \mathbf{R}^{16} \oplus \mathbf{R}^{16}$ via $\Delta_9 \oplus \Delta_9$, where $\mathbf{R}^{16} = \mathbf{Ca} \oplus \mathbf{Ca}$. The principal isotropy subgroup H at the point $((1, 0), (e, 1)) \in V$ is isomorphic to $\mathbf{SU}(3)$, and $\Delta_9(H)$ consists of matrices of the form

$$\begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && A & B &&&& \\ && -B & A &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& A & B \\ &&&&&& -B & A \end{pmatrix} \in \mathbf{SO}(16), \tag{2}$$

where $A + iB \in \mathbf{SU}(3)$. Now the cohomogeneity of (G, V) is 4, the fixed point subspace

$$V^H = \mathbf{R}\langle (1,0), (e,0), (0,1), (0,e)\rangle \oplus \mathbf{R}\langle (1,0), (e,0), (0,1), (0,e)\rangle \subset \mathbf{R}^{16} \oplus \mathbf{R}^{16},$$

and $\dim \bar{N} = 4$. Using the above description of Δ_9, one can check that e_0e_1, e_1e_2, e_0e_2 belong to N and generate a subgroup isomorphic to $\mathbf{SU}(2)$. Moreover, $e_3e_4e_5e_6e_7e_8$ centralizes this subgroup and also belongs to N. Hence, $\bar{N}^0 \cong \mathbf{U}(2)$, and $(\bar{N}^0, V^H)$ is $(\mathbf{U}(2), \mathbf{C}^2 \oplus \mathbf{C}^2)$; this representation is taut by an argument similar to one used in the proof of Proposition 5, based on the fact that the isotropy representation of the Grassmann manifold $G_2(\mathbf{C}^2)$ is orbit equivalent to $\mathbf{U}(2) \times \mathbf{U}(2)$ acting on complex 2×2 matrices. Let L be the subgroup of H generated by the elements (2) with A diagonal with ± 1 entries and $B = 0$. Then $V^L = V^H$. Thus, (G, V) is taut by Proposition 3. $\square$

PROPOSITION 18. $\mathbf{R}^9 \oplus \mathbf{R}^{16}$ *is not taut.*

Proof. We use the description of the spin representation given in the proof of Proposition 17. One can check that the principal isotropy subgroup H at $(e_0, (1,1)) \in \mathbf{R}^9 \oplus (\mathbf{Ca} \oplus \mathbf{Ca})$ is isomorphic to $\mathbf{G}_2$, $V^H = \mathbf{R}\langle e_0, e_1\rangle \oplus (\mathbf{R}1 \oplus$

$\mathbf{R}1) \subset \mathbf{R}^9 \oplus (\mathbf{Ca} \oplus \mathbf{Ca})$, the cohomogeneity is 3, and so $\dim \bar{N} = 1$. It then follows that $\theta \mapsto \cos\theta 1 + \sin\theta(e_0e_1)$ defines a one-parameter subgroup in $\bar{N}$ which acts on $(\mathbf{R}1 \oplus \mathbf{R}1)$ as a rotation by an angle of θ, and acts on $\mathbf{R}\langle e_0, e_1\rangle$ as a rotation by an angle of 2θ. Therefore, $(\bar{N}, V^H)$ is not taut. It follows that (G, V) is not taut by the final argument in the proof of Lemma 6.11 in [GT03]. □

3.9. $G = \mathbf{Spin}(10)$**.** By the discussion in Section 3.1, the admissible summands of ρ are the vector representation on $\mathbf{R}^{10}$, and the half-spin representations on $\mathbf{C}^{16}_+$ and $\mathbf{C}^{16}_-$. In view of Lemma 2, $\mathbf{C}^{16}_-$ needs not to be considered. It is clear that the following two propositions cover all possibilities.

PROPOSITION 19. $\mathbf{R}^{10} \oplus \mathbf{C}^{16}_+$ *is not taut.*

Proof. We extend the ideas of Proposition 17. Let $\mathcal{C}\ell^0(n)$ denote the "even" part of $\mathcal{C}\ell(n)$, namely the subalgebra of $\mathcal{C}\ell(n)$ consisting of even products of elements in $\mathbf{R}^n$. Then $\mathbf{Spin}(n)$ is a subgroup of $\mathcal{C}\ell^0(n)$, and an isomorphism $\mathcal{C}\ell^0(n) \cong \mathcal{C}\ell(n-1)$ is given by

$$\begin{cases} e_ie_j \mapsto e_ie_j, & \text{if } i<j<n, \\ e_ie_n \mapsto e_i, & \text{if } i<n. \end{cases}$$

View $\mathbf{R}^9 = \mathbf{R} \oplus \mathbf{Ca}$ and $\mathbf{R}^{16} = \mathbf{Ca} \oplus \mathbf{Ca}$ as in Proposition 17, and define

$$\varphi_\pm : \mathbf{R}^9 \to M(16, \mathbf{C}), \qquad (r, u) \mapsto \pm\sqrt{-1}\begin{pmatrix} rI_8 & R_u \\ R_{\bar{u}} & -rI_8 \end{pmatrix},$$

where $r \in \mathbf{R}$, $u \in \mathbf{Ca}$, and $R_u : \mathbf{Ca} \to \mathbf{Ca}$ is right Cayley multiplication. Then $\varphi_\pm(r,u)^2 = -(r^2 + \|u\|^2)I_{16}$. It follows that $\varphi_\pm$ induce homomorphisms $\mathcal{C}\ell(9) \to M(16, \mathbf{C})$. Now $\mathbf{Spin}(10) \subset \mathcal{C}\ell^0(10) \cong \mathcal{C}\ell(9)$, so these homomorphisms restrict to the half-spin representations $\Delta^\pm_{10} : \mathbf{Spin}(10) \to \mathbf{U}(16)$. Note that $\omega = e_0e_1e_2e_3e_4e_5e_6e_7e_8e_9$ belongs to the center of $\mathbf{Spin}(10)$ and $\Delta^\pm_{10}(\omega) = \pm\sqrt{-1}I_{16}$. It follows that Δ^+_{10} and Δ^-_{10} are not equivalent. It is also clear that $\Delta^\pm_{10}|\mathbf{Spin}(9) = \Delta_9 \oplus \Delta_9$.

Next, consider $G = \mathbf{Spin}(10)$ acting on $V = \mathbf{R}^{10} \oplus \mathbf{C}^{16}_+$. We view $\mathbf{C}^{16}_+ = \mathbf{R}^{16} \oplus \sqrt{-1}\mathbf{R}^{16}$, $\mathbf{Spin}(9)$-invariant decomposition, where $\mathbf{R}^{16} = \mathbf{Ca} \oplus \mathbf{Ca}$. A principal isotropy subgroup can be taken to be the same subgroup H as in Proposition 17, and the fixed subspace

$$\begin{aligned} V^H = {} & \mathbf{R}\langle e_0, e_1, e_2, e_9\rangle \oplus \mathbf{R}\langle (1,0), (e,0), (0,1), (0,e)\rangle \\ & \oplus \mathbf{R}\langle (\varepsilon 1, 0), (\varepsilon e, 0), (0, \varepsilon 1), (0, \varepsilon e)\rangle \subset \mathbf{R}^{10} \oplus \mathbf{R}^{16} \oplus \varepsilon\mathbf{R}^{16}, \end{aligned}$$

where $\varepsilon = \sqrt{-1}$. Now the cohomogeneity of (G, V) is 5 and $\dim \bar{N} = 7$.

It is not difficult to see that $\bar{N}^0$ is locally isomorphic to $\mathbf{U}(1) \times \mathbf{SU}(2)_1 \times \mathbf{SU}(2)_2$, where the $\mathbf{U}(1)$-factor is generated by $e_3e_4e_5e_6e_7e_8$ and the Lie algebras of the $\mathbf{SU}(2)$-factors are respectively spanned by $e_0e_1 + e_2e_9$, $e_0e_2 - e_1e_9$, $e_0e_9 + e_1e_2$, and $e_0e_1 - e_2e_9$, $e_0e_2 + e_1e_9$, $e_0e_9 - e_1e_2$. We want to describe the action of $\bar{N}^0$ on V^H. For that purpose, it is convenient to set $\mathbf{R}^4 = V^H \cap \mathbf{R}^{10}$ and $\mathbf{C}^4 = V^H \cap \mathbf{C}^{16}_+$. Then it can be shown that there is a decomposition $\mathbf{C}^4 = \mathbf{C}^2_1 \oplus \mathbf{C}^2_2$ such that $\mathbf{SU}(2)_1 \times \mathbf{SU}(2)_2$ acts by the product of the standard representations on $\mathbf{C}^2_1 \oplus \mathbf{C}^2_2$ and it acts on $\mathbf{R}^4$ by $\mathbf{SU}(2)_1 \times \mathbf{SU}(2)_2 \to \mathbf{SO}(4)$. Moreover, $\mathbf{U}(1)$ acts scalarly on $\mathbf{C}^2_1$, $\mathbf{C}^2_2$, and trivially on $\mathbf{R}^4$. We finally get that $(\bar{N}^0, V^H)$ is equivalent to

$$\begin{aligned}(e^{j\theta}, p, q) &\in \big(\mathbf{U}(1) \times \mathbf{Sp}(1) \times \mathbf{Sp}(1)\big)/\mathbf{Z}_2 \\ \mapsto \quad &(l_p r_{e^{-j\theta}}, l_q r_{e^{j\theta}}, l_p r_{\bar{q}}) \in \mathbf{SO}(4) \times \mathbf{SO}(4) \times \mathbf{SO}(4),\end{aligned}$$

where we have identified $V^H = \mathbf{H} \oplus \mathbf{H} \oplus \mathbf{H}$. It is also important to note that $\bar{N}$ is not connected, and the element $e_1 e_5 e_7 e_6$ lies in $\bar{N} \setminus \bar{N}^0$.

Finally, consider the $\bar{N}$-orbit of $x = (1,1,1) \in \mathbf{H} \oplus \mathbf{H} \oplus \mathbf{H}$, and let h be the height function defined by x. A careful calculation shows that the sum of the Betti numbers of the critical set of h on $\bar{N}^0 x$ is 12. Therefore, on $\bar{N}x$, this sum is at least 24. The critical set of h on $\bar{N}x$ is the same as its critical set on $M = Gx$. If M is taut, it has the homology of $S^{15} \times S^9 \times S^7 \times S^6$ by Proposition 1, so the sum of the Betti numbers of M has to be 16. Hence, M is not taut. □

PROPOSITION 20. *$\mathbf{C}^{16}_+ \oplus \mathbf{C}^{16}_+$ is not taut.*

Proof. The principal isotropy subgroup of the first summand acts on the second summand by a representation that contains a summand equivalent to $(\mathbf{SU}(4), \mathbf{C}^4 \oplus \mathbf{R}^6 \oplus \mathbf{R}^6 \oplus \mathbf{C}^4)$, which is not taut. Hence, we can apply Proposition 1. □

3.10. $G = \mathbf{Spin}(16)$**.** This case is ruled out because the spin representation on $\mathbf{R}^{128}$ cannot be a summand of a taut representation of $\mathbf{Spin}(16)$ by the argument of Example 1(i).

Proof of Proposition 6. Consider first the representation $(\mathbf{Sp}(n), V_n)$. Let K be $\mathbf{Sp}(1) \times \mathbf{Sp}(n-1)$ diagonally embedded into $\mathbf{Sp}(n)$. Then there exists a point in V_n whose isotropy subgroup is K, and such that its slice representation contains as a summand V_{n-1}. This implies that $V_n \oplus V_n$ admits a slice representation containing $V_{n-1} \oplus V_{n-1}$. By Proposition 2 and induction on n, it is now enough to prove that $(\mathbf{Sp}(3), V_3 \oplus V_3)$ is not taut.

The principal isotropy subgroup of $(\mathbf{Sp}(3), V_3)$ is the diagonal embedding of $\mathbf{Sp}(1)^3$ into $\mathbf{Sp}(3)$; call it K_1. Now V_3, considered as a representation of K_1, decomposes into two copies of the trivial representation and a representation W which, upon identification with $\mathbf{H} \oplus \mathbf{H} \oplus \mathbf{H}$, is orbit equivalent to (notation as in Lemma 1)

$$(p, q, s) \in \mathbf{Sp}(1)^3 \quad \mapsto \quad (l_p r_{\bar{q}}, l_p r_{\bar{s}}, l_q r_{\bar{s}}) \in \mathbf{SO}(4) \times \mathbf{SO}(4) \times \mathbf{SO}(4).$$

By Proposition 1, it is enough to show that (K_1, W) is not taut, and, for that purpose, we will apply the reduction principle described in Proposition 3 to (K_1, W).

The principal isotropy subgroup of (K_1, W) at the point $(1, i, j) \in \mathbf{H} \oplus \mathbf{H} \oplus \mathbf{H}$ is the circle subgroup $H = \{(e^{kt}, e^{kt}, e^{-kt}) : t \in \mathbf{R}\}$ of K_1. Therefore the cohomogeneity of (K_1, W) is 4, the fixed point subspace of H is $W^H = \mathbf{R}\langle 1, k\rangle \oplus \mathbf{R}\langle i, j\rangle \oplus \mathbf{R}\langle i, j\rangle$, and so the dimension of the normalizer N of H in K_1 is 3. It is clear that $N^0 = \{(e^{ka}, e^{kb}, e^{kc}) : a, b, c \in \mathbf{R}\}$. Consider the one-parameter subgroups of N given by $\varphi_a = (e^{ka}, e^{-ka}, 1)$ and $\psi_b = (1, e^{kb}, e^{kb})$. Then φ_a and ψ_b generate $\bar{N}^0$, and $(\bar{N}^0, V^H)$ is $(\varphi_a, \psi_b) \mapsto (e^{k(2a-b)}, e^{k(a+b)}, e^{k(-a+2b)})$, which is not taut by Lemma 1(a). It follows that (K_1, W) is not taut by the final argument in the proof of Lemma 6.11 in [GT03]. $\square$

Proof of Proposition 8. View $\mathbf{R}^7$ as the purely imaginary Cayley numbers, namely, $\mathbf{R}^7 = \langle i, j, k, e, ie, je, ke\rangle$, and set $V = \mathbf{R}^7 \oplus \mathbf{R}^7 \oplus \mathbf{R}^7$. Recall that a triple $(a, b, c) \in V$ is called a special triple if it is orthonormal and $c \perp ab$. Any permutation of the members of a special triple yields a special triple, and there is a bijection between $\mathbf{G}_2$, seen as the group of automorphisms of $\mathbf{Ca}$, and the set of special triples (see [Whi78, Appendix A, Section 5] or [HL82, Appendix IV.A, Lemma A.15]).

Fix the base point $p = (e, i, j) \in V$. Then G_p is trivial and $M = Gp$ is a principal orbit and diffeomorphic to $\mathbf{G}_2$. One easily computes that the normal space $\nu_p M$ is spanned by

$$(e, 0, 0), (0, i, 0), (0, 0, j), (i, e, 0), (0, j, i), (j, 0, e), (k, je, -ie).$$

Let $h : M \to \mathbf{R}$ be the height function defined by p. We have that $gp \in M$, $g \in \mathbf{G}_2$, is a critical point of h if and only if $p \in \nu_{gp}M = g\nu_p M$, or, $g^{-1}p \in \nu_p M$. Note that $g^{-1}p$ is an arbitrary special triple. So, we need to determine when

$$\big(Ae + Di + Fj + Gk, Bi + De + Ej + G(je), Cj + Ei + Fe - G(ie)\big)$$

is a special triple, where A, B, C, D, E, F, $G \in \mathbf{R}$. The conditions are:

$$\begin{aligned}
A^2 + D^2 + F^2 + G^2 &= 1,\\
B^2 + D^2 + E^2 + G^2 &= 1,\\
C^2 + E^2 + F^2 + G^2 &= 1,\\
AD + BD + EF &= 0,\\
BE + DF + EC &= 0,\\
AF + DE + CF &= 0,\\
G(AB + AC + BC - D^2 - E^2 - F^2 - G^2) &= 0.
\end{aligned}$$

In order to solve the system, we consider two cases: $G = 0$ or $G \neq 0$ (in which case $AB + AC + BC = D^2 + E^2 + F^2 + G^2$.

If $G=0$, then

$$X=\begin{pmatrix} A & D & F \\ D & B & E \\ F & E & C \end{pmatrix}$$

is orthogonal and symmetric, thus orthogonally conjugate to one of

$$\mathrm{diag}(1,1,1), \mathrm{diag}(1,1,-1), \mathrm{diag}(1,-1,-1), \mathrm{diag}(-1,-1,-1). \tag{3}$$

This gives already two points and two copies of $\mathbf{R}P^2$ in the critical set of h.

If $G \neq 0$, then $0<|G|<1$ and

$$X=\frac{1}{\sqrt{1-G^2}}\begin{pmatrix} A & D & F \\ D & B & E \\ F & E & C \end{pmatrix}$$

is orthogonally conjugate to one of the diagonal matrices in (3). If $X=\mathrm{diag}(1,1,1)$, then one easily sees that $A=B=C=\frac{1}{2}$, $D=E=F=0$ and $G=\pm\frac{\sqrt{3}}{2}$. If $X=\mathrm{diag}(-1,-1,-1)$, then one easily sees that $A=B=C=-\frac{1}{2}$, $D=E=F=0$ and $G=\pm\frac{\sqrt{3}}{2}$. This adds another four points to the critical set of h. With the above, we conclude that the sum of the Betti numbers of this critical set is at least 12. Since $\mathbf{G}_2$ has the $\mathbf{Z}_2$-homology of $S^3\times S^5\times S^6$, this proves that M is not taut. □

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