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ON THE COMPARISON OF NORMS OF CONVOLUTORS ASSOCIATED WITH NONCOMMUTATIVE DYNAMICS

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ABSTRACT. To any action of a locally compact group G on a pair (A, B) of von Neumann algebras is canonically associated a pair $(\pi_A^\alpha, \pi_B^\alpha)$ of unitary representations of G. The purpose of this paper is to provide results allowing to compare the norms of the operators $\pi_A^\alpha(\mu)$ and $\pi_B^\alpha(\mu)$ for bounded measures μ on G. We have a twofold aim. First, to point out that several known facts in ergodic and representation theory are indeed particular cases of general results about $(\pi_A^\alpha, \pi_B^\alpha)$. Second, under amenability assumptions, to obtain transference of inequalities that will be useful in noncommutative ergodic theory.

1. Introduction

Given a unitary representation π of a locally compact group G on a Hilbert space $\mathcal{H}$ and a bounded measure μ on G, we denote by $\pi(\mu)$ the operator $\int_G \pi(s)\, d\mu(s)$ acting on $\mathcal{H}$. It has long been known that estimates of the spectral radius $r(\pi(\mu))$ or of the norm $\|\pi(\mu)\|$ give useful information.

A first observation (often referred to as the "Herz majorization principle"), asserts that for any locally compact group G, any closed subgroup H of G and any *positive* bounded measure μ on G, one has $\|\lambda_G(\mu)\| \le \|\lambda_{G/H}(\mu)\|$ (and, therefore, $r(\lambda_G(\mu)) \le r(\lambda_{G/H}(\mu))$ as well). Here, λ_G is the left regular representation and $\lambda_{G/H}$ denotes the quasi-regular representation associated with H, that is the unitary representation of G on $L^2(G/H)$ defined by left translations. More generally, given a representation π of H, one has $\|\mathrm{Ind}_H^G \pi(\mu)\| \le \|\lambda_{G/H}(\mu)\|$, where $\mathrm{Ind}_H^G \pi$ is the representation induced of π, from H to G (see [14], [24]).

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For a discrete group and a finitely supported symmetric probability measure μ, the convolution operator $\lambda_G(\mu)$ had been investigated by Kesten [26] in connection with the random walk defined by μ. Kesten had already observed that $r(\lambda_G(\mu)) \leq r(\lambda_{G/H}(\mu))$ when H is a normal subgroup. Moreover in this seminal paper [26], he proved that the normal subgroup H is (what is now called) amenable if and only if there exists an adapted symmetric probability measure μ on G (i.e., the support of μ generates the group G) such that $r(\lambda_{G/H}(\mu)) = r(\lambda_G(\mu))$.

Note that in terms of operator algebras, these results concern the pair

$$L^\infty(G/H) \subset L^\infty(G)$$

of Abelian von Neumann algebras, acted upon by left translations of G.

In [2], to which this paper is a sequel, we considered G-actions on pairs $B \subset A$ of Abelian von Neumann algebras. Our purpose in this paper is to deal more generally with G-actions on any pair $B \subset A$ of von Neumann algebras. In order to state the problems we are interested in, we need first to introduce some notations and definitions. Let A be a von Neumann algebra and α a continuous homomorphism from a locally compact group G into the group $\mathrm{Aut}(A)$ of automorphisms of A. To such a *dynamical system* (A, G, α) is associated a unitary representation π_A^α on the noncommutative L^2-space $L^2(A)$, well defined, up to equivalence (see Section 2). Two particular examples are well known. The first one is when $A = L^\infty(Y, m)$ is an Abelian von Neumann algebra. In this case, α is an action on Y which preserves the class of the measure m and π_A^α is the corresponding unitary representation in $L^2(Y, m)$. When α is the action of G on $A = L^\infty(G/H)$ by translations, we get $\pi_A^\alpha = \lambda_{G/H}$. The second important example concerns the von Neumann algebra $A = \mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ and, for α_s, $s \in G$, the automorphism $T \mapsto \mathrm{Ad}\,\pi(s)(T) = \pi(s)T\pi(s)^*$ of $\mathcal{B}(\mathcal{H})$, where π is a given unitary representation of G on a Hilbert space $\mathcal{H}$. Then the representation π_A^α is equivalent to the tensor product $\pi \otimes \overline{\pi}$ of π with its conjugate $\overline{\pi}$.

By a pair (A, B) of von Neumann algebras, we mean that B is a von Neumann subalgebra of A. An *action α of G on (A, B)* is a dynamical system (A, G, α) such that B is globally G-invariant (we still denote by the same letter the restricted action to B). Our first result is that the "Herz majorization principle" is valid for every pair $(\pi_A^\alpha, \pi_B^\alpha)$. Namely, see the following theorem.

Theorem (Theorem 3.1). *Let α be an action of G on a pair (A, B). For every positive bounded measure μ on G we have*

$$\|\lambda_G(\mu)\| \leq \|\pi_A^\alpha(\mu)\| \leq \|\pi_B^\alpha(\mu)\| \leq \mu(G).$$

The proof is based on the fact that representations of the form π_A^α have enough G-positive vectors, that is vectors ξ such that $\langle \xi, \pi_A^\alpha(s)\xi\rangle \geq 0$ for all $s \in G$. Indeed, every normal positive form[1] φ on A is represented by a well-defined

[1] We shall denote by A_*^+ the cone of such forms.

G-positive vector $\varphi^{1/2}$ in the space $L^2(A)$ of the representation π_A^α and the set $L^2(A)^+$ of such vectors is a self-dual cone in $L^2(A)$. The second ingredient of the proof is the inequality $\langle \varphi^{1/2}, \psi^{1/2} \rangle_{L^2(A)} \leq \langle (\varphi_{|_B})^{1/2}, (\psi_{|_B})^{1/2} \rangle_{L^2(B)}$ for $\varphi, \psi \in A_*^+$ (where $\varphi_{|_B}, \psi_{|_B}$ are the restrictions of φ and ψ to B), resulting from a variational formula due to Kosaki [30] (see Section 2 for notations and details).

In the case where $A = B \otimes M$ is the tensor product of two von Neumann algebras (with tensor product action), we get $\|(\pi_B^\alpha \otimes \pi_M^\alpha)(\mu)\| \leq \|\pi_B^\alpha(\mu)\|$ for every positive bounded measure μ. In fact, this result remains true for any representation π instead of π_M^α and any representation ρ having a separating family of G-positive vectors instead of π_B^α (see Theorem 3.3). In particular, taking $\pi = \lambda_G$ and using Fell's absorption principle, one gets $\|\lambda_G(\mu)\| \leq \|\rho(\mu)\|$. This inequality was proved by Pisier [35] when ρ is of the form $\pi \otimes \overline{\pi}$. Later, Shalom observed in [36] that this inequality holds for any representation having a nonzero G-positive vector.

Whereas the Herz majorization principle involves positivity properties, extensions of Kesten's result express amenability phenomena. We say that the *action α of G on (A, B) is amenable* if there exists a norm one projection[2] E from A onto B such that $\alpha_s \circ E = E \circ \alpha_s$ for $s \in G$. Two special cases are particularly important. When $B = \mathbb{C}$, one says that the *G-action on A is coamenable* (or *amenable in the sense of Greenleaf*). When one considers the pair $(A \otimes L^\infty(G), A)$ with the tensor product action of the action on A by left translations on $L^\infty(G)$, one says that the *G-action on A is amenable* (or *amenable in the sense of Zimmer*). Of course, any action of an amenable group on (A, B) is amenable.

Let us consider an action α of G on (A, B). The existence of a *normal* G-equivariant conditional expectation from A onto B easily implies that π_B^α is a subrepresentation of π_A^α. It is therefore very natural to wonder whether the amenability of the action implies that π_B^α is weakly contained in π_A^α. We believe that this result is true in general, but we can only solve the problem in several particular cases. Recall that a representation π_1 is said to be *weakly contained in a representation* π_2 (and we write $\pi_1 \prec \pi_2$) if for every $f \in L^1(G)$ we have $\|\pi_1(f)\| \leq \|\pi_2(f)\|$ (or, equivalently, if $\|\pi_1(\mu)\| \leq \|\pi_2(\mu)\|$ for every bounded measure μ on G).

Borrowing ideas used by Connes [10] in order to show that injective von Neumann algebras are semidiscrete, we obtain the following theorem.

THEOREM (Theorem 4.6). *Let α be an amenable action of G on a pair (A, B) of von Neumann algebras. We assume that there is a faithful normal invariant state φ on B. Then π_B^α is weakly contained in π_A^α.*

[2] A norm one projection E from A onto B is also called a conditional expectation. It is automatically positive and satisfies $E(b_1 a b_2) = b_1 E(a) b_2$ for $a \in A$ and $b_1, b_2 \in B$.

In presence of "enough" *normal* conditional expectations from A onto B, there is another approach aiming to approximate conditional expectations by normal ones. This gives a stronger weak containment property. For instance, we get the following theorem.

THEOREM (Theorems 4.8, 4.14). *Let α be an amenable action of G on a pair (A,B). We assume either that B is contained in the centre $Z(A)$ of A or that A is a tensor product $B\otimes M$ of von Neumann algebras (with tensor product action). There exists a net (V_i) of isometries from $L^2(B)$ into $L^2(A)$ such that for every $\xi\in L^2(B)$ one has*

$$\lim_i \|\pi_A^\alpha(s)V_i\xi - V_i\pi_B^\alpha(s)\xi\| = 0$$

uniformly on compact subsets of G. In particular, π_B^α is weakly contained in π_A^α.

There are more general statements (see Remark 4.13). However, we are mainly interested in the case $(A\otimes L^\infty(G), A)$. As a consequence of the previous theorem, we see that if an action α on A is amenable, then for every bounded measure μ we have

$$\|\pi_A^\alpha(\mu)\| \le \|\lambda_G(\mu)\|, \tag{1}$$

these inequality being an equality when μ is positive.

Note that (1) is a transference of norm estimates from the regular representation to π_A^α, a classical result when A is Abelian and G amenable (see [8]). For an amenable action of G on an Abelian von Neumann algebra A, this inequality was obtained in [28] when G is discrete and in [2] for any locally compact group. It gives an upper bound for $\|\pi_A^\alpha(\mu)\|$ only depending on μ, particularly useful in ergodic theory (for instance in the study of entropy, see [34, Proposition 4.1]).

As observed in [2], there is no hope for recovering in general the amenability of the action on a pair (A,B) from the weak containment $\pi_B^\alpha \prec \pi_A^\alpha$, even in the commutative setting. Let us consider, for example, the case $A = L^\infty(G)$ and $B = L^\infty(G/H)$. We proved in [2, Proposition 4.2.1, Corollary 4.4.5] the equivalence of the following three conditions:

- the action of G on $(L^\infty(G), L^\infty(G/H))$ is amenable;
- H is amenable;
- $\lambda_{G/H}\prec\lambda_G$ and the trivial representation ι_H of H is weakly contained in the restriction of $\lambda_{G/H}$ to H.

For $H = SL(2,\mathbb{R})$ and $G = SL(3,\mathbb{R})$, one has $\lambda_{G/H}\prec\lambda_G$ although H is not amenable (see [2, Section 4.2]).

However, when $B=\mathbb{C}$, the situation is completely understood. Recall that a bounded measure μ is said to be *adapted* if the closed subgroup generated by the support $\mathrm{Supp}(\mu)$ of μ is G. Observe that for any representation π of G and any probability measure μ on G, it is easily seen that $r(\pi(\mu))=1$ whenever the trivial representation ι_G of G is weakly contained in π. However, the existence of an adapted probability measure μ on G with $r(\pi(\mu))=1$ does not always imply $\iota_G \prec \pi$.

THEOREM (5.3). *Let (A,G,α) be a dynamical system. The following conditions are equivalent:*

(i) *there exists a G-invariant state on A (i.e., the action is coamenable);*
(ii) *the trivial representation ι_G is weakly contained in π_A^α;*
(iii) *there exists an adapted probability measure μ on G with $r(\pi_A^\alpha(\mu))=1$.*

The above theorem is well known when A is an Abelian von Neumann algebra. First, extending Kesten's and Day's results [11], [26], [27], Derriennic and Guivarc'h [12] proved the equivalence between (ii) and (iii) when $A=L^\infty(G)$ (see also [7]). In this case, the equivalence between (i) and (ii) is the Hulanicki–Reiter theorem (see [18, Theorem 3.5.2]). Recall that G is then said to be an amenable group.

Next, the equivalence between (i) and (ii) was obtained by Eymard [15] for G-homogeneous spaces G/H. Later, Guivarc'h [20] proved that the previous theorem holds for any action on an Abelian von Neumann algebra.

Another particular case of the above theorem concerns the von Neumann algebra $A=\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ and $\alpha_s=\mathrm{Ad}\,\pi(s)$, $s\in G$, where π is a given unitary representation of G on $\mathcal{H}$. In this situation the equivalence of (i) and (ii) is due to Bekka [5] and the equivalence of the two last assertions is a recent result of Bekka and Guivarc'h [6].[3]

The equivalence between (i) and (ii) for any dynamical system (A,G,α) is proved by Kirchberg in [29, Sublemma 7.2.1]. As an interesting consequence of this fact, note that the weak containment of the trivial representation ι_G in π_A^α is independent of the topology of G.

This paper is organized as follows. We begin by recalling the basic facts on standard forms of von Neumann algebras. In Section 3, we prove non-commutative Herz majorization theorems. In Section 4, we study the weak containment property $\pi_B^\alpha \prec \pi_A^\alpha$ for an amenable action α of G on (A,B). Finally, in the last section, we consider more specifically the cases of amenable and coamenable actions.

For fundamentals of the theory of von Neumann algebras, we refer to [13], [31]. *In the whole paper, we shall only consider second countable locally compact groups, σ-finite measured spaces and von Neumann algebras with separable preduals*, although these assumptions are not always necessary.

[3] Representations π such that $\iota_G \prec \pi\otimes\overline{\pi}$ are called amenable.

2. Preliminaries on standard forms and noncommutative dynamical systems

2.1. Standard form of a von Neumann algebra. Let M be a von Neumann algebra. A *standard form of* M is a normal faithful representation of M in a Hilbert space $\mathcal{H}_M$ endowed with a conjugate linear isometric involution $J_M : \mathcal{H}_M \to \mathcal{H}_M$ and a self-dual cone $P_M \subset \mathcal{H}_M$ such that:

- $J_M M J_M = M'$ (where M' is the commutant of M in $\mathcal{B}(\mathcal{H}_M)$;
- $J_M c J_M = c^*$ for all $c \in M \cap M'$;
- $J_M \xi = \xi$ for all $\xi \in P_M$;
- $x J_M x J_M (P_M) \subset P_M$ for all $x \in M$.

Such a standard form $(M, \mathcal{H}_M, J_M, P_M)$ exists and is unique, up to isomorphism. We refer to [21] for details about this subject. Given a faithful normal state φ on M, one may take the standard representation to be the Gelfand–Naimark–Segal representation on $L^2(M,\varphi)$. Denoting by ξ_φ the unit of M, viewed in $L^2(M,\varphi)$, then J_M is the antilinear isometry J_φ given by the polar decomposition of the closure of $S_\varphi : x\xi_\varphi \mapsto x^*\xi_\varphi$, $x \in M$. Moreover, P_M is the norm closure P_φ of $\{xJ_\varphi x J_\varphi \xi_\varphi : x \in M\}$.

Usually we shall fix a standard form, denoted by $(M, L^2(M), J_M, P_M)$, or even $(M, L^2(M), J, P)$ for simplicity. The space $L^2(M)$ is ordered by the positive cone P. This cone is self-dual in the sense that

$$P = \{\xi \in L^2(M) : \langle \xi, \eta \rangle \geq 0, \forall \eta \in P\}.$$

Recall also that every element $\xi \in L^2(M)$ can be written in a unique way as $\xi = u|\xi|$ where $|\xi| \in P_M$ and u is a partial isometry in M such that u^*u is the support of ξ. This decomposition is called the polar decomposition of ξ.

The Banach space M_* of all normal forms on M is the predual of M. A crucial fact is that every normal positive form $\phi \in M_*^+$ can be uniquely written as $\phi = \omega_\xi$, with $\xi \in P_M$.[4] It is also very suggestive to denote $\phi^{1/2}$ this vector ξ.

We shall need a concrete description of $(L^2(M), J, P)$. We refer to [22] and [40] for the details concerning the following facts. We fix a concrete representation of M on a Hilbert space $\mathcal{H}$. Let $\sigma : t \mapsto \sigma_t^\psi$ be the modular automorphism group of a normal semifinite faithful weight ψ and let

$$M \rtimes_\sigma \mathbb{R} \subset \mathcal{B}\big(L^2(\mathbb{R}) \otimes \mathcal{H}\big)$$

be the corresponding crossed product. We denote by $\hat{\sigma}$ the dual action of $\mathbb{R}$ on $M \rtimes_\sigma \mathbb{R}$. Recall that $M \rtimes_\sigma \mathbb{R}$ has a canonical normal semifinite trace τ satisfying $\tau \circ \hat{\sigma}_t = e^{-t}\tau$ for all $t \in \mathbb{R}$. Following the point of view of Haagerup, $L^p(M)$ is defined, for $p \geq 1$, as a subspace of the $*$-algebra $\mathcal{M}(M \rtimes_\sigma \mathbb{R})$ formed

[4] Where ω_ξ is the vector state $x \mapsto \langle \xi, x\xi \rangle = \omega_\xi(x)$.

by the closed densely defined operators on $L^2(\mathbb{R}) \otimes \mathcal{H}$, affiliated with $M \rtimes_\sigma \mathbb{R}$, that are measurable with respect to τ (see [40] or [39]). Namely,

$$L^p(M) = \{x \in \mathcal{M}(M \rtimes_\sigma \mathbb{R}) : \hat{\sigma}_t(x) = e^{-t/p}x, \forall t \in \mathbb{R}\}.$$

In this picture, we have a description of $(M, L^2(M), J, P)$ as follows: $L^2(M)$ is defined, as just said, as a space of operators, its positive cone is the cone of all positive operators in $L^2(M)$, J is the adjoint map, and the polar decomposition is the usual one. The spaces $L^1(M)$ and $L^\infty(M)$ are canonically isomorphic to the predual M_* of M and to M, respectively.

Lemma 2.1. *Let $(M, L^2(M), J, P)$ be a standard form of M. Then for $\xi, \eta \in L^2(M)$, we have*

$$|\langle \xi, \eta\rangle|^2 \leq \langle |\xi|, |\eta|\rangle \langle |J\xi|, |J\eta|\rangle.$$

Proof. The proof is exactly that of [13, Lemma 2, page 105]. We use the above description of the standard form. We need to introduce the linear functional

$$tr : h \in L^1(M) \mapsto \varphi_h(1),$$

where φ_h denotes the normal linear form on M associated with h. We recall that if $h \in L^p(M)$ and $k \in L^q(M)$, with $1/p + 1/q = 1$, then $tr(hk) = tr(kh)$. Moreover, for $\xi, \eta \in L^2(M)$ the scalar product is given by $\langle \xi, \eta\rangle = tr(\xi^*\eta)$. The functional tr plays the role of the usual trace on the space of trace-class operators on a Hilbert space.

Let $\xi = u|\xi|$ and $\eta = v|\eta|$ be the polar decompositions of ξ and η, respectively. Then the polar decomposition of $J\xi$ is $J\xi = u^*(u(JuJ)|\xi|)$ since $J\nu = \nu$ for $\nu \in P$, so that $|J\xi| = u|\xi|u^*$. Similarly, we have $|J\eta| = v|\eta|v^*$. Using the tracial property of tr and the Cauchy–Schwarz inequality, we proceed exactly as in [13, Lemma 2, page 105] to get

$$|\langle \xi, \eta\rangle| \leq tr(|\xi||\eta|)^{1/2} tr(|J\eta||J\xi|)^{1/2}.$$

□

2.2. Standard form of a pair of von Neumann algebras. Now let B be a von Neumann subalgebra of A and let us examine some results relating the standard form $(B, L^2(B), J_B, P_B)$ of B to that of A.

Let us consider first a pair (A, B) such that there exists a normal faithful conditional expectation E from A onto B. Let us choose a faithful normal state ψ on B and set $\varphi = \psi \circ E$. The Hilbert space $L^2(B, \psi)$ is canonically embedded into $L^2(A, \varphi)$ and the standard form of B is obtained from that of A by restriction to $L^2(B, \psi)$. Indeed, one checks that $L^2(B, \psi)$ is stable under J_φ and that J_ψ is the restriction of J_φ to $L^2(B, \psi)$. Moreover, $P_\psi = P_\varphi \cap L^2(B, \psi)$ and the standard representation of B into $L^2(B, \psi)$ is the restriction of the standard representation of A into $L^2(A, \varphi)$ (see, for instance [37, page

130]). For every $\xi,\eta \in L^2(B,\psi)$ and $a\in A$, one has $\langle \xi, E(a)\eta\rangle = \langle \xi, a\eta\rangle$. More generally, we shall need the following lemma.

Lemma 2.2. *Let E be a normal conditional expectation from A onto B. There exists a unique positive isometry q_E (i.e., sending $L^2(B)^+$ into $L^2(A)^+$) from $L^2(B)$ into $L^2(A)$ such that $\langle \xi, E(a)\eta\rangle = \langle q_E(\xi), aq_E(\eta)\rangle$ for all $a\in A$.*

Proof. The uniqueness of q_E is immediate since for $\xi \in L^2(B)^+$, the normal form $\omega_\xi \circ E$ is uniquely implemented by $q_E(\xi)\in L^2(A)^+$. To prove the existence of q_E we introduce the support e of E, that is the smallest projection in A with $E(1-e)=0$. We have $e\in A\cap B'$ and $\theta : b\mapsto be$ is an isomorphism from B onto Be. Through this isomorphism, we may view the standard form of B as represented into $L^2(Be)$ which is embedded in $L^2(eAe)$ by the previous remarks applied to the faithful normal conditional expectation $a\mapsto eE(a)$ from eAe onto eB. Now $L^2(eAe)$ is obviously included into $L^2(A)$. The composition of all these isometries give the required $q_E : L^2(B)\to L^2(A)$. $\square$

In the other direction, we shall need the map p from $L^2(A)^+$ into $L^2(B)^+$ defined by

$$\langle \xi, b\xi\rangle = \langle p(\xi), bp(\xi)\rangle$$

for all $b\in B$. In other terms, for $\varphi \in A_*^+$, we have $p(\varphi^{1/2}) = (\varphi_{|_B})^{1/2}$.

Example 2.3. (a) Assume that B is a von Neumann subalgebra of $Z(A)$. We write $B = L^\infty(X,m)$ and $A = \int_X^\oplus A(x)\,dm(x)$, so that $L^2(A) = \int_X^\oplus L^2(A(x))\,dm(x)$ (see [38]). Let $\xi = \int_X^\oplus \xi(x)\,dm(x)$ be an element of $L^2(A)^+$. Then $p(\xi)$ is the function $x\mapsto \|\xi(x)\|_2$, belonging to $L^2(X,m)^+$.

(b) Assume that A is a finite von Neumann algebra, equipped with a faithful normal trace τ and let E be the faithful normal conditional expectation from A onto B such that $\tau\circ E = \tau$. Then one has $A^+ \subset L^2(A,\tau)^+$ and $B^+ \subset L^2(B,\tau_{|_B})^+$. For $a\in A^+$, one checks that $p(a) = E(a^2)^{1/2}$.

(c) Take $A = \mathcal{B}(\mathcal{H})$ and $B=\mathbb{C}$. Let $\varphi = Tr(h\cdot) \in \mathcal{B}(H)_*^+$ where h is a positive trace-class operator on $\mathcal{H}$. Then $p(\varphi^{1/2}) = p(h^{1/2}) = \|h^{1/2}\|_2$.

Lemma 2.4. *The map $p\colon L^2(A)^+ \to L^2(B)^+$ is norm preserving and we have*

$$\langle \xi,\eta\rangle \le \langle p(\xi), p(\eta)\rangle$$

for all $\xi,\eta\in L^2(A)^+$.

Proof. Let $\varphi,\psi\in L^1(A)^+$ and let $\Delta_{\varphi,\psi}$ be the unique positive self-adjoint operator on $L^2(A)$ such that $J\Delta_{\varphi,\psi}^{1/2} a\psi^{1/2} = a^*\varphi^{1/2}$ for all $a\in A$. Using the formula

$$\sqrt{\lambda} = \frac{1}{\pi}\int_0^{+\infty} \frac{\lambda}{\lambda+t}\frac{dt}{\sqrt{t}},$$

we get

$$\langle \varphi^{1/2}, \psi^{1/2}\rangle = \langle \Delta_{\varphi,\psi}^{1/2}\psi^{1/2}, \psi^{1/2}\rangle$$

$$= \frac{1}{\pi}\int_0^{+\infty} \langle \Delta_{\varphi,\psi}(\Delta_{\varphi,\psi}+t)^{-1}\psi^{1/2}, \psi^{1/2}\rangle \frac{dt}{\sqrt{t}}.$$

A quadratic interpolation method gives

$$\langle \Delta_{\varphi,\psi}(\Delta_{\varphi,\psi}+t)^{-1}\psi^{1/2}, \psi^{1/2}\rangle = \inf\{\varphi(yy^*)/t + \psi(z^*z)\}, \tag{2}$$

where the infimum is taken on the pairs $(y,z)\in A^2$ such that $y+z=1$ (see [30, Lemma 2.1] and the proof of the Wigner–Yanase–Dyson–Lieb theorem [30, Theorem 5.2]). We conclude by observing that the expression (2) obviously increases when φ and ψ are replaced by their restriction to B. □

Remark 2.5. In Examples 2.3(a) and (c), the above lemma is immediately obtained by Cauchy–Schwarz inequality. Applied to Example 2.3(b), this lemma gives the following inequality:

$$\forall x,y\in A^+, \quad \tau(xy)\le \tau(E(x^2)^{1/2}E(y^2)^{1/2}).$$

2.3. Unitary implementation of automorphisms. Finally, let us recall a very important property of standard forms. Let $\mathrm{Aut}(M)$ be the automorphism group of the von Neumann algebra M and let $(M, L^2(M), J, P)$ be a fixed standard form. For every $\gamma\in\mathrm{Aut}(M)$, there is a unique $u(\gamma)$ in the unitary group $\mathcal{U}(L^2(M))$ such that $u(\gamma)(P)\subset P$, $Ju(\gamma)=u(\gamma)J$ and $\gamma(x)=u(\gamma)xu(\gamma)^*$ for all $x\in M$. This unitary is called the *canonical implementation* of γ (see [21, Theorem 3.2]).

The group $\mathrm{Aut}(M)$ acts on M_* by $(\gamma,\varphi)\mapsto\varphi\circ\gamma^{-1}$. We equip it with the topology of pointwise norm convergence on M_*. Then the map $\gamma\mapsto u(\gamma)$ is a continuous homomorphism from $\mathrm{Aut}(M)$ into the unitary group $\mathcal{U}(L^2(M))$ equipped with the strong operator topology [21, Proposition 3.6].

Lemma 2.6. *Let (A,B) be a pair of von Neumann algebras and $\gamma\in\mathrm{Aut}(A)$ such that $\gamma(B)=B$. We fix standard forms of A and B and denote by $u_A(\gamma)$ and $u_B(\gamma)$ the unitary implementations of γ and of its restriction to B, respectively:*

(i) *We have $p\circ u_A(\gamma)(\xi)=u_B(\gamma)\circ p(\xi)$ for every $\xi\in L^2(A)^+$.*

(ii) *Let E be a normal conditional expectation from A onto B and set $\gamma\cdot E=\gamma\circ E\circ\gamma^{-1}$. Then $q_{\gamma\cdot E}=u_A(\gamma)\circ q_E\circ u_B(\gamma)^*$.*

Proof. Immediate. □

2.4. Representations defined by noncommutative dynamical systems. An *action of a locally compact group G on a von Neumann algebra M* is a continuous homomorphism $s\mapsto\alpha_s$ from G into $\mathrm{Aut}(M)$. We also say that (M,G,α) is a *dynamical system*. We fix a standard form of M and for $s\in G$, we denote by $\pi_M^\alpha(s)$ the canonical unitary $u(\alpha_s)$ implementing α_s. Then π_M^α is a unitary representation of G, that is, a continuous homomorphism from G into $\mathcal{U}(L^2(M))$. Note that π_M^α is well defined, up to equivalence.

Observe that every $\xi \in P_M$ is *G-positive* in the sense that for all $s \in G$ we have $\langle \xi, \pi_M^\alpha(s)\xi\rangle \geq 0$.

Note also that the set of representations of the form π_M^α is stable under direct sums and tensor products. Furthermore, each representation π_M^α is equivalent to its conjugate.

EXAMPLE 2.7. (a) Let X be a standard Borel space with a Borel left G-action $(s,x) \in G \times X \mapsto sx \in X$. When equipped with a G-quasi-invariant measure m, we say that (X,G,m) is a (nonsingular) *measured G-space.* To such a measured G-space is associated the dynamical system $(L^\infty(X,m),G,\alpha)$ where $\alpha_s(f)(x) = f(s^{-1}x)$ for $f \in L^\infty(X,m)$, $s \in G$, $x \in X$.

We denote by r (or r_X in case of ambiguity) the *Radon–Nikodým derivative* defined by

$$\forall s \in G, \forall f \in L^1(X,m), \quad \int_X f(s^{-1}x)r(x,s)\,dm(x) = \int_X f(x)\,dm(x).$$

Recall that $(L^\infty(X,m), L^2(X,m), J, L^2(X,m)^+)$ is a standard form of $L^\infty(X,m)$ (where J is the complex conjugation and $L^2(X,m)^+$ is the cone of nonnegative functions in $L^2(X,m)$). The unitary representation $\pi^\alpha_{L^\infty(X,m)}$ (rather denoted π_X) associated with the dynamical system $(L^\infty(X,m),G,\alpha)$ is defined by

$$\pi_X(s)\xi(x) = \sqrt{r(x,s)}\,\xi(s^{-1}x)$$

for $\xi \in L^2(X,m)$ and $(s,x) \in G \times X$.

(b) Let π be a representation of a locally compact group G in a Hilbert space $\mathcal{H}$. We consider the dynamical system (M,G,α) where M is the von Neumann algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ and α is the action such that $\alpha_s(T) = \pi(s)T\pi(s)^*$ for $T \in \mathcal{B}(\mathcal{H})$ and $s \in G$. A standard form for $\mathcal{B}(\mathcal{H})$ is $(\mathcal{B}(\mathcal{H}), \mathcal{H}\otimes\overline{\mathcal{H}}, J, (\mathcal{H}\otimes\overline{\mathcal{H}})^+)$ where $\mathcal{H}\otimes\overline{\mathcal{H}}$ is canonically identified with the Hilbert space of Hilbert–Schmidt operators, J is the adjoint operator (also described as $J : \xi\otimes\overline{\eta} \mapsto \eta\otimes\overline{\xi}$), and $(\mathcal{H}\otimes\overline{\mathcal{H}})^+$ is the cone of nonnegative Hilbert–Schmidt operators. The canonical representation $\pi^\alpha_{\mathcal{B}(\mathcal{H})}$ canonically associated with (M,G,α) is $\pi\otimes\overline{\pi}$. This representation acts on the space of Hilbert–Schmidt operators by

$$\pi\otimes\overline{\pi}(s)(T) = \pi(s)T\pi(s)^*.$$

(c) More generally, let M be a von Neumann algebra and $s \mapsto \pi(s)$ be a continuous representation on $L^2(M)$ with $\pi(s) \in \mathcal{U}(M)$ for all $s \in G$. Denote by α the corresponding action on M by inner automorphisms, that is $\alpha_s = \mathrm{Ad}\,\pi(s)$ for $s \in G$. Then $\pi_M^\alpha(s) = \pi(s)J_M\pi(s)J_M$.

In the particular case where $M = L(G)$ is the group von Neumann algebra and $\pi = \lambda_G$ is the left regular representation, π_M^α is the conjugation representation γ_G.

3. A noncommutative "Herz majorization principle"

An *action of a locally compact group G on the pair (A,B) of von Neumann algebras* is a dynamical system (A,G,α) such that the von Neumann subalgebra B is left globally invariant under the action. The restricted action of G on B will still be denoted by α.

THEOREM 3.1. *Let α be an action of the locally compact group G on (A,B). Then for every probability measure μ on G, we have*

$$\|\pi_A^\alpha(\mu)\| \le \|\pi_B^\alpha(\mu)\|. \tag{3}$$

The proof uses the following well-known way of computing norms.

LEMMA 3.2. *Let $T\in\mathcal{B}(\mathcal{H})$ be a positive operator on a Hilbert space $\mathcal{H}$ and let $\mathcal{S}\subset\mathcal{H}$ be a separating family of norm one vectors (i.e., for $S\in\mathcal{B}(\mathcal{H})$, $S\xi=0$ for all $\xi\in\mathcal{S}$ implies $S=0$). Then*

$$\|T\| = \lim_{n\to\infty}\sup_{\xi\in\mathcal{S}}\omega_\xi(T^n)^{1/n}.$$

Proof. Let μ_ξ be the spectral measure of T on the spectrum $\sigma(T)\subset[0,\|T\|]$, associated with ξ. Since the family $\mathcal{S}$ is separating, the union of the supports of μ_ξ, $\xi\in\mathcal{S}$, is dense into $\sigma(T)$. Given $\epsilon>0$, let $\xi_0\in\mathcal{S}$ be such that $\mu_{\xi_0}([\|T\|-\epsilon,\|T\|])>0$. We have

$$\|T\|\ge\sup_{\xi\in\mathcal{S}}\omega_\xi(T^n)^{1/n}\ge(\|T\|-\epsilon)\mu_{\xi_0}([\|T\|-\epsilon,\|T\|])^{1/n},$$

and the conclusion follows immediately. □

Proof of Theorem 3.1. We shall apply the previous lemma with $T=\pi_A^\alpha(\nu)$, $\nu=\check{\mu}*\mu$, and for $\mathcal{S}$ we take the set of norm one vectors in $L^2(A)^+$. For every n, we have

$$\|\pi_A^\alpha(\mu)\|^2=\lim_{n\to\infty}\sup_{\xi\in\mathcal{S}}\omega_\xi(\pi_A^\alpha(\nu^{*n}))^{1/n}$$

(where $\nu^{*n}=\nu*\cdots*\nu$ is the n-fold convolution product) and

$$\begin{aligned}\omega_\xi(\pi_A^\alpha(\nu^{*n})) &= \int\langle\xi,\pi_A^\alpha(t)\xi\rangle\,d\nu^{*n}(t)\\ &\le\int\langle p(\xi),p(\pi_A^\alpha(t)\xi)\rangle\,d\nu^{*n}(t)\\ &=\int\langle p(\xi),\pi_B^\alpha(t)(p(\xi))\rangle\,d\nu^{*n}(t)\\ &=\omega_{p(\xi)}(\pi_B^\alpha(\nu^{*n}))\le\|\pi_B^\alpha(\mu)\|^{2n}\end{aligned}$$

by Lemmas 2.4 and 2.6. The inequality (3) is then an immediate consequence of Lemma 3.2. □

When A is the tensor product of B by another von Neumann algebra C, with a tensor product action $\alpha = \beta \otimes \gamma$, it follows from Theorem 3.1 that

$$\|\pi_B^\beta \otimes \pi_C^\gamma(\mu)\| \le \min\{\|\pi_B^\beta(\mu)\|, \|\pi_C^\gamma(\mu)\|\}$$

for every probability measure μ. In fact, this is also a particular case of the following more general result.

THEOREM 3.3. *Let G be a locally compact group, ρ a representation having a separating set $\mathcal{P}$ of norm one G-positive vectors and π any representation. Then for every probability measure μ on G, we have*

$$\|(\rho \otimes \pi)(\mu)\| \le \|\rho(\mu)\|.$$

Proof. We use again Lemma 3.2 with $T = (\rho \otimes \pi)(\check{\mu} * \mu)$ and

$$\mathcal{S} = \{\xi \otimes \eta : \xi \in \mathcal{P}, \eta \in \mathcal{H}(\pi), \|\eta\| = 1\}$$

where $\mathcal{H}(\pi)$ is the Hilbert space of the representation π. We have

$$\|(\rho \otimes \pi)(\mu)\|^2 = \lim_{n\to\infty} \sup_{\xi \in \mathcal{S}} \big\|\omega_\xi\big((\rho \otimes \pi)(\nu^{*n})\big)\big\|^{1/n}.$$

Observe that for $\xi \otimes \eta \in \mathcal{S}$,

$$\begin{aligned}
\omega_{\xi\otimes\eta}\big((\rho \otimes \pi)(\nu^{*n})\big) &= \int \omega_\xi(\rho(t))\omega_\eta(\pi(t))\, d\nu^{*n}(t) \\
&\le \int \omega_\xi(\rho(t))\, d\nu^{*n}(t) \\
&= \omega_\xi(\rho(\nu^{*n})) \le \|\rho(\mu)\|^{2n},
\end{aligned}$$

due to the positivity of $\omega_\xi(\rho(t))$. The conclusion follows immediately. □

COROLLARY 3.4 ([36, Lemma 2.3]). *Let ρ be a representation of G having a nonzero G-positive vector. Then for every probability measure μ on G, we have*

$$\|\lambda_G(\mu)\| \le \|\rho(\mu)\|.$$

Proof. Let $\xi \in \mathcal{H}(\rho)$ be a norm one G-positive vector for ρ. Then $\mathcal{P} = \rho(G)\xi$ is a set of G-positive vectors. Let K be the Hilbert subspace of $\mathcal{H}(\rho)$ generated by $\mathcal{P}$. It is G-invariant and we denote by $\rho_{|_K}$ the representation of G obtained by restriction. We apply Theorem 3.3 with $\rho_{|_K}$ instead of ρ and $\pi = \lambda_G$. We have

$$\|(\rho_{|_K} \otimes \lambda_G)(\mu)\| \le \|\rho_{|_K}(\mu)\| \le \|\rho(\mu)\|.$$

Moreover, a well-known observation of J. M. G. Fell [16] says that the regular representation absorbs any other representation. In particular, $\rho_{|_K} \otimes \lambda_G$ is equivalent to a multiple of λ_G and, therefore, $\|(\rho_{|_K} \otimes \lambda_G)(\mu)\| = \|\lambda_G(\mu)\|$. □

COROLLARY 3.5 ([32], [35]). *Let π be a representation of G. Then for every probability measure μ on G, we have $\|\lambda_G(\mu)\| \le \|(\pi \otimes \overline{\pi})(\mu)\|$. Moreover, if π has a separating set of G-positive vectors, then $\|(\pi \otimes \overline{\pi})(\mu)\| \le \|\pi(\mu)\|$.*

Proof. The first inequality follows from Corollary 3.4 and the second from Theorem 3.3. □

REMARK 3.6. Let $U_1, \dots, U_n$ be n unitary operators in a Hilbert space and let $c_1, \dots, c_n, d_1, \dots, d_n$ be $2n$ nonnegative real numbers. Let us denote by $g_1, \dots, g_n$ the generators of the free group $\mathbb{F}_n$. As a particular case of the previous corollary, one finds Pisier's inequality [35]:

$$\Big\| \sum_{i=1}^{n} \big(c_i \lambda_{\mathbb{F}_n}(g_i) + d_i \lambda_{\mathbb{F}_n}(g_i)^* \big) \Big\| \le \Big\| \sum_{i=1}^{n} (c_i U_i \otimes \overline{U}_i + d_i U_i^* \otimes \overline{U}_i^*) \Big\|.$$

The left-hand side of this inequality is $2\sqrt{2n-1}$ when $c_i = d_i = 1$ for every i (due to Kesten [26]). It is equal to $2\sqrt{n-1}$ when $c_i = 1$ and $d_i = 0$ for every i (due to Akemann and Ostrand [4]).

To conclude this section, let us explain how the inequality $\|\pi_A^\alpha(\mu)\| \le \|\pi_B^\alpha(\mu)\|$, when μ is a probability measure and $B \subset Z(A)$, is a particular case of the classical "Herz majorization principle".

First, we need to recall some definitions. Let (X, m) be a measured space and let $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$ be a m-measurable field of Hilbert spaces on X (see [13, Chapter II]). We denote by $L^2(\mathcal{H}) = \int_X^\oplus \mathcal{H}(x)\, dm(x)$ the direct integral Hilbert space. For $x, y \in X$, the set of bounded linear maps from $\mathcal{H}(x)$ to $\mathcal{H}(y)$ will be denoted by $\mathcal{B}(\mathcal{H}(x), \mathcal{H}(y))$ and $\mathrm{Iso}(\mathcal{H}(x), \mathcal{H}(y))$ will be its subset of Hilbert space isomorphisms.

DEFINITION 3.7. Let (X, G, m) be a measured G-space and $\mathcal{H}$ as above. A *(unitary) cocycle representation* of (X, G, m), acting on the measurable field $\mathcal{H}$, is a map

$$U : (x, s) \in X \times G \mapsto U(x, s) \in \mathcal{B}(\mathcal{H}(s^{-1}x), \mathcal{H}(x))$$

such that:

(a) for each $s \in G$, $U(x, s) \in \mathrm{Iso}(\mathcal{H}(s^{-1}x), \mathcal{H}(x))$ for m-almost every x;
(b) for each $(s, t) \in G \times G$, $U(x, st) = U(x, s)U(s^{-1}x, t)$ for m-almost every $x \in X$;
(c) for every pair of measurable sections ξ, η of $\mathcal{H}$, and every $s \in G$ the map $x \mapsto \langle \eta(x), U(x, s)\xi(s^{-1}x) \rangle$ is measurable.

To every cocycle representation U of (X, G, m) is associated a representation of G, called the *induced representation* and denoted $\mathrm{Ind}\, U$ (or $\mathrm{Ind}_X U$ in case of ambiguity). Let us recall its definition. If U acts on $\mathcal{H}$, $\mathrm{Ind}\, U$ is the representation into $L^2(\mathcal{H})$ defined by

$$(\mathrm{Ind}\, U(s)\xi)(x) = \sqrt{r(x, s)}\, U(x, s)\xi(s^{-1}x)$$

for $\xi \in L^2(\mathcal{H})$ and $(x, s) \in X \times G$. This extends the classical construction of the representation induced by a representation of a closed subgroup H, which amounts to consider the left action of G on G/H.

The fact that $\|\mathrm{Ind}\, U(\mu)\| \le \|\pi_X(\mu)\|$ when μ is a probability measure is very easy to prove (see [2, Proposition 2.3.1] for instance).

Now let α be an action of G on a von Neumann algebra A, preserving a subalgebra $B = L^\infty(X,m)$ of $Z(A)$. By disintegrating A with respect to B, we get $A = \int_X A(x)\, dm(x)$ and $L^2(A) = \int_X L^2(A(x))\, dm(x)$ (see [38]). We know by [33, Theorem 1] that the action of G on $L^\infty(X,m)$ has a point realization. Therefore, we may write π_B^α as

$$\pi_B^\alpha(s)\xi(x) = \sqrt{r(x,s)}\xi(s^{-1}x)$$

for $\xi \in L^2(X,m)$ and $(x,s) \in X \times G$. Thus, we have $\pi_B^\alpha = \pi_X$. Moreover, by [19, Proposition 1], there is a cocycle representation $U_A : (x,s) \in X \times G \mapsto U_A(x,s) \in \mathcal{B}(L^2(A(s^{-1}x)), L^2(A(x)))$ such that $\pi_A^\alpha = \mathrm{Ind}\, U_A$.

4. Representations associated with amenable pairs

Let (A,G,α) be a dynamical system. It is easily checked that the representation π_A^α has a nonzero invariant vector if and only if there is a normal invariant state on A. More generally, we have, in one direction the following proposition.

Proposition 4.1. *Let α be an action of G on a pair (A,B) of von Neumann algebras. Assume that there exists a normal G-equivariant conditional expectation E from A onto B. Then π_B^α is a subrepresentation of π_A^α.*

Proof. This follows immediately from Lemmas 2.2 and 2.6. Indeed, for an equivariant normal conditional expectation E, the isometry q_E of Lemma 2.2 intertwines the representations π_A^α and π_B^α. □

When the conditional expectation is not required to be normal, we are led to the following definition, due to Zimmer [41] for pairs of Abelian von Neumann algebras.

Definition 4.2. We say that *an action of G on a pair (A,B) of von Neumann algebras is amenable* if there exists an equivariant conditional expectation from A onto B.

Let us also recall the definitions of the two following important particular cases.

Definition 4.3. Let (A,G,α) be a dynamical system:

(i) We say that the action is *coamenable* if there is a G-invariant state on A.
(ii) We say that the action is *amenable* if there is a G-equivariant conditional expectation from $A \otimes L^\infty(G)$ (with its usual tensor product action) onto A.

In particular, one of the definitions of amenability for a locally compact group G is the coamenability of the action on $L^\infty(G)$ by left translations.

Let α be an action of G on a pair (A,B). Note that if the action on B is amenable then, by [1, Proposition 2.5], the action on the pair (A,B) is amenable whenever there exists a conditional expectation from A onto B.

In this section, we are interested in the following problem:

- *does the amenability of the action α on (A,B) imply that π_B^α is weakly contained in π_A^α?*

We shall only give partial answers. First, we introduce some notations. For $f \in L^1(G)$, $a \in A$ and $\varphi \in A_*$ we set

$$f * a = \int_G f(s)\alpha_s(a)\,ds, \qquad (\varphi * f)(a) = \varphi(f * a).$$

The left and right translated $s \cdot f$, $f \cdot s$ of f are defined by

$$(s \cdot f)(t) = f(s^{-1}t), \qquad (f \cdot s)(t) = f(ts^{-1})\Delta(s)^{-1},$$

where Δ is the modular function of G. Note that $f * (\alpha_s(a)) = (f \cdot s) * a$ and $\alpha_s(f * a) = (s \cdot f) * a$ for $a \in A$.

Let us now recall a useful equivalent definition of amenability using a notion of invariant conditional expectation which is stronger than equivariance.

Definition 4.4. Let us consider an action α of G on a pair (A,B) of von Neumann algebras. A *topologically invariant conditional expectation* is a conditional expectation $E : A \to B$ such that $E(f * a) = f * (E(a))$ for every $f \in L^1(G)$ and every $a \in A$.

The following result is well known (see [1] for instance).

Proposition 4.5. *Let G act on (A,B). This action is amenable if and only if there exists a topologically invariant conditional expectation $E : A \to B$.*

4.1. Let us state a first answer to our problem.

Theorem 4.6. *Let α be an amenable action of G on a pair (A,B) of von Neumann algebras. We assume that there is a faithful normal invariant state φ on B. Then π_B^α is weakly contained in π_A^α.*

The proof uses the following key lemma inspired by the very simple proof given by Connes [10] to show that an injective von Neumann algebra is semi-discrete.

Lemma 4.7. *We keep the assumptions of the previous theorem. Given any compact subset K of G and $\varepsilon > 0$, there exists a normal state ψ on A such that*

$$\|\psi_{|_B} - \varphi\| \le \varepsilon, \qquad \sup_{s \in K} \|\psi \circ \alpha_s - \psi\| \le \varepsilon.$$

Proof. We follow the proof of [10, Lemma 2]. We introduce the weakly compact convex set

$$\mathcal{C} = \{x - \varphi(x)1 : x \in B, \|x\| \le \varepsilon^{-1}\}.$$

Let E be a topologically invariant conditional expectation from A onto B. The (nonnormal) state $\varphi \circ E$ belongs to polar set of the convex hull $\mathrm{co}(\mathcal{C} \cup A^+)$, which is weakly closed. Therefore, using the bipolar theorem, we see that there is a net (ψ_i) of normal states on A such that $\lim_i \psi_i = \varphi \circ E$ in the weak*-topology and $\|\psi_{i|_B} - \varphi\| \leq \varepsilon$ for every i.

Now, we use the classical Day–Namioka convexity argument. We denote by $\mathcal{C}'$ the convex set of normal states ψ on A such that $\|\psi_{|B} - \varphi\| \leq \varepsilon$. Let $h_1, \dots, h_k$ be fixed elements in $L^1(G)^+$ with $\int_G h_j(t)\, dt = 1,\ 1 \leq j \leq k$. For $\psi \in \mathcal{C}'$, we set

$$b_j(\psi) = \psi * h_j - \psi.$$

Let us denote by $\mathcal{C}''$ the range of $\mathcal{C}'$ in the product A_*^k by the map

$$\psi \mapsto (b_1(\psi), \dots, b_k(\psi)).$$

Since E is topologically invariant and φ is invariant, we have $(\varphi \circ E)(h_j * a) = (\varphi \circ E)(a)$ for $a \in A$. Therefore, we know that $(0, \dots, 0)$ belongs to the closure of $\mathcal{C}''$ in A_*^k equipped with the product topology, where we consider the weak topology on A_*. Since $\mathcal{C}''$ is convex, we may replace this latter topology by the norm topology, using the Hahn–Banach separation theorem. It follows that there exists a net (ψ_i) in $\mathcal{C}'$ such that for every $h \in L^1(G)^+$ with $\int_G h(t)\, dt = 1$ we have

$$\lim_i \|\psi_i - \psi_i * h\|_{A_*} = 0.$$

Now, we fix $h \in L^1(G)^+$ such that $\int_G h(t)\, dt = 1$. Given $\eta > 0$, we choose a neighborhood V of e in G such that $\|h \cdot s - h\|_1 \leq \eta$ for $s \in V$. Then we can find a finite number of elements $s_1, \dots, s_n$ in G such that $K \subset \bigcup_{i=1}^n V s_i$. We set $s_0 = e$ and we choose $\psi \in \mathcal{C}'$ satisfying

$$\|\psi - \psi * (h \cdot s_i)\|_{A_*} \leq \eta$$

for $0 \leq i \leq n$.

Let $s \in K$ and choose i such that $s \in V s_i$. We have

$$\begin{aligned}
&\|\psi * h - (\psi * h) \circ \alpha_s\|_{A_*} \\
&\quad \leq \|\psi * h - \psi\|_{A_*} + \|\psi - \psi * (h \cdot s)\|_{A_*} \\
&\quad \leq \eta + \|\psi - \psi * (h \cdot s_i)\|_{A_*} + \|\psi * (h \cdot s_i) - \psi * (h \cdot s)\|_{A_*} \\
&\quad \leq 2\eta + \|h - h \cdot (s s_i^{-1})\|_{A_*} \leq 3\eta.
\end{aligned}$$

To conclude, it suffices to take $\eta = \varepsilon/3$ and to replace ψ by $\psi * h$. $\square$

Proof of Theorem 4.6. We fix $\varepsilon > 0$ and a compact subset K of G. Let ψ be a normal state on A as in Lemma 4.7. We set $\xi_\varphi = \varphi^{1/2} \in L^2(B)^+$ and $\xi_\psi = \psi^{1/2} \in L^2(A)^+$. Note that $\xi_\varphi = \pi_B^\alpha(s)\xi_\varphi$ for every $s \in G$ since φ is G-invariant. On the other hand, $\psi \circ \alpha_s$ corresponds to $\pi_A^\alpha(s)\xi_\psi$ in $L^2(A)^+$. It follows from the Powers–Størmer inequality [21, Lemma 2.10] that

$$\|\xi_\psi - \pi_A^\alpha(s)\xi_\psi\|_2^2 \leq \|\psi - \psi \circ \alpha_s\|_{A_*}.$$

Using the facts that $\|\xi_\psi - \pi_A^\alpha(s)\xi_\psi\|_2 \le \sqrt{\varepsilon}$ and $\|\psi_{|_B} - \varphi\| \le \varepsilon$, we get for $s \in K$ and $b \in B$,

$$\begin{aligned}
&|\langle b\xi_\psi, \pi_A^\alpha(s) b\xi_\psi\rangle - \langle b\xi_\varphi, \pi_B^\alpha(s) b\xi_\varphi\rangle| \\
&\quad = |\langle \xi_\psi, b^*\alpha_s(b)\pi_A^\alpha(s)\xi_\psi\rangle - \langle \xi_\varphi, b^*\alpha_s(b)\xi_\varphi\rangle| \\
&\quad \le |\langle \xi_\psi, b^*\alpha_s(b)\xi_\psi\rangle - \langle \xi_\varphi, b^*\alpha_s(b)\xi_\varphi\rangle| + \|b\|^2\sqrt{\varepsilon} \\
&\quad \le \|b\|^2\big(\varepsilon + \sqrt{\varepsilon}\big).
\end{aligned}$$

Finally, we note that $B\xi_\varphi$ is dense in $L^2(B)$ since φ is faithful. It follows from the above observations that every coefficient of the representation π_B^α is the limit, uniformly on compact subsets of G, of a net of coefficients of π_A^α. Therefore, we have $\pi_B^\alpha \prec \pi_A^\alpha$. □

4.2. We now turn to situations where we can take advantage of the existence of sufficiently many normal conditional expectations.

THEOREM 4.8. *Let α be an amenable action of G on a pair (A, B) of von Neumann algebras. We assume that B is contained in the center $Z(A)$ of A. Then π_B^α is weakly contained in π_A^α. More precisely, there exists a net (V_i) of isometries from $L^2(B)$ into $L^2(A)$ such that for every $\xi \in L^2(B)$ one has*

$$\lim_i \|\pi_A^\alpha(s)V_i\xi - V_i\pi_B^\alpha(s)\xi\| = 0$$

uniformly on compact subsets of G.

This last condition implies the weak containment property, as it is easily seen in the following lemma.

LEMMA 4.9. *Let π (resp. ρ) be a representation on $\mathcal{H}(\pi)$ (resp. $\mathcal{H}(\rho)$). Assume the existence of a net V_i of isometries from $\mathcal{H}(\rho)$ into $\mathcal{H}(\pi)$ such that for every $\xi \in \mathcal{H}(\rho)$*

$$\lim_i \|\pi(s)V_i\xi - V_i\rho(s)\xi\| = 0$$

uniformly on compact subsets of G. Then ρ is weakly contained in π.

Proof. Let $\xi \in \mathcal{H}(\rho)$ and $f \in L^1(G)$. Since V_i is an isometry, we have

$$\begin{aligned}
\|\rho(f)\xi\| &= \|V_i\rho(f)\xi\| \\
&\le \|V_i\rho(f)\xi - \pi(f)V_i\xi\| + \|\pi(f)V_i\xi\| \\
&\le \int_G |f(s)|\|V_i\rho(s)\xi - \pi(s)V_i\xi\|\, ds + \|\pi(f)\|\|\xi\|.
\end{aligned}$$

Since $\lim_i \int_G |f(s)|\|V_i\rho(s)\xi - \pi(s)V_i\xi\|\, ds = 0$ it follows that $\|\rho(f)\xi\| \le \|\pi(f)\|\|\xi\|$. We conclude that ρ is weakly contained in π. □

In order to prove Theorem 4.8, we need some preliminaries. We shall denote by $\mathcal{B}_B(A, B)$ the Banach space of bounded maps F from A into B that are B-linear in the sense that $F(ba) = bF(a)$ for $a \in A$ and $b \in B$. This space is the dual of the quotient $A \widehat{\otimes}_B B_*$ of the projective tensor product $A \widehat{\otimes} B_*$

by the vector subspace generated by $\{a \otimes b\varphi - ab \otimes \varphi : a \in A, b \in B, \varphi \in B_*\}$. We denote by $\mathcal{B}_B(A,B)_1^+$ the weak*-closed convex subset of positive elements $F \in \mathcal{B}_B(A,B)$ with $F(1) \le 1$. We want to introduce a weak*-dense convex subset $\mathcal{C}$ in $\mathcal{B}_B(A,B)_1^+$, consisting of normal maps. To that purpose, we disintegrate A with respect to $B = L^\infty(X,m)$, that is, we write $A = \int_X^\oplus A(x)\, dm(x)$ and $L^2(A) = \int_X^\oplus L^2(A(x))\, dm(x)$. Given a measurable section $\xi : x \mapsto \xi(x) \in L^2(A(x))$ with $\|\xi(x)\|_2 \le 1$ almost everywhere, we denote by ϖ_ξ the element of $\mathcal{B}_B(A,B)_1^+$ such that

$$\varpi_\xi(a)(x) = \langle \xi(x), a(x)\xi(x)\rangle_{L^2(A(x))}$$

for $a \in A$. Obviously, ϖ_ξ is a normal element in $\mathcal{B}_B(A,B)_1^+$ and we denote by $\mathcal{C}$ the convex set of such maps ϖ_ξ.

Lemma 4.10. (i) *The set $\mathcal{C}$ is weak*-dense in $\mathcal{B}_B(A,B)_1^+$.*

(ii) *Every conditional expectation from A onto B is the weak*-limit of a net of normal conditional expectations belonging to $\mathcal{C}$.*

Proof. Assume that there is an element $F \in \mathcal{B}_B(A,B)_1^+$ that is not in the weak*-closure of $\mathcal{C}$. Using the Hahn–Banach separation theorem, we find an element $\Phi = \sum_i a_i \otimes \varphi_i$ in $A_{sa} \hat{\otimes}_B B_{*sa}$ (where the sum is finite) and $r \in \mathbb{R}$ with

$$\langle F, \Phi\rangle > r \quad \text{and,} \quad \forall \varpi_\xi \in \mathcal{C}, \quad \langle \varpi_\xi, \Phi\rangle \le r.$$

By polar decomposition, we may assume that $\varphi_i \ge 0$ for all i. Moreover, setting $\varphi = \sum_i \varphi_i$, thanks to the Radon–Nikodým theorem we easily put Φ in the form $a \otimes \varphi$ with $a \in A_{sa}$. Let e be a spectral projection of a with $ae = a^+$, the positive part of a. Obviously, we have $\langle F(a^+), \varphi\rangle \ge \langle F(a), \varphi\rangle = \langle F, \Phi\rangle > r$.

On the other hand, observe that for every $\varpi_\xi \in \mathcal{C}$ the map $b \mapsto \varpi_\xi(eb) = \varpi_{\xi e}(b)$ still belongs to $\mathcal{C}$. Therefore, writing φ as $h \in L^1(X,m)^+$, we have

$$\int_X h(x)\langle \xi(x), a^+(x)\xi(x)\rangle\, dm(x) = \langle \varpi_{\xi e}, \Phi\rangle \le r.$$

Let (ξ_n) be a sequence of measurable sections such that $\{\xi_n(x) : n \in \mathbb{N}\}$ is dense in the unit ball of $L^2(A(x))$ for almost every $x \in X$. Let us fix $\varepsilon > 0$. We may find a measurable partition $(X_{n,\varepsilon})_n$ of X such that for every $x \in X_{n,\varepsilon}$ we have $\langle \xi_n(x), a^+(x)\xi_n(x)\rangle \ge \|a^+(x)\|(1-\varepsilon)$. For $x \in X_{n,\varepsilon}$, we set $\xi(x) = \xi_n(x)$. Then we have

$$r \ge \int_X h(x)\langle \xi(x), a^+(x)\xi(x)\rangle\, dm(x) \ge (1-\varepsilon)\int_X h(x)\|a^+(x)\|\, dm(x).$$

By letting ε go to 0 we get $r \ge \int_X h(x)\|a^+(x)\|\, dm(x)$.

Since F is positive, B-linear with $F(1) \le 1$, we get $F(a^+)(x) \le \|a^+(x)\|$ a.e. and, therefore,

$$r < \langle F(a^+), \varphi\rangle = \int_X h(x) F(a^+)(x)\, dm(x) \le \int_X h(x)\|a^+(x)\|\, dm(x) \le r.$$

The contradiction thus obtained concludes the proof of (i).

If $F(1)=1$, it is easy to see that we may approximate F by elements ϖ_ξ with $\|\xi(x)\|_2=1$ a.e. □

We shall denote by $\mathcal{E}(A,B)\subset\mathcal{B}_B(A,B)_1^+$ the subset of all normal conditional expectations from A onto B.

Lemma 4.11. *Let α be an action of G on (A,B) where $B\subset Z(A)$. The following conditions are equivalent:*

(i) *There exists a G-equivariant conditional expectation from A onto B.*

(ii) *There exists a net (Φ_i) of normal conditional expectations from A onto B such that for $\varphi\in B_*$, $f\in L^1(G)$ and $a\in A$ we have*

$$\lim_i\langle\varphi, f*(\Phi_i(a))-\Phi_i(f*a)\rangle=0$$

Proof. (i) ⇒ (ii). Let E be a topologically invariant conditional expectation and let (Φ_i) be a net in $\mathcal{E}(A,B)$ such that $\lim_i\Phi_i=E$ in the weak*-topology, whose existence was proved in Lemma 4.10. Assertion (ii) follows immediately from the invariance of E.

The converse is also obvious. □

To go further, let us introduce some more notations. For $s\in G$ and $F\in\mathcal{B}_B(A,B)$ we set $s\cdot F=\alpha_s\circ F\circ\alpha_{s^{-1}}$. Note that $s\cdot F\in\mathcal{E}(A,B)$ whenever $F\in\mathcal{E}(A,B)$. Finally, for $F\in\mathcal{E}(A,B)$ and $f\in L^1(G)$, we define $f*F\in\mathcal{E}(A,B)$ by

$$\forall a\in A,\quad (f*F)(a)=\int_G f(s)(s\cdot F)(a)\,ds.$$

Lemma 4.12. *Let α be an action on a pair (A,B). The following conditions are equivalent:*

(i) *There exists a net (Φ_i) in $\mathcal{E}(A,B)$ such that for every $\varphi\in B_*$, $f\in L^1(G)$ and $a\in A$ we have*

$$\lim_i\langle\varphi, f*(\Phi_i(a))-\Phi_i(f*a)\rangle=0.$$

(ii) *There exists a net (Φ_i) in $\mathcal{E}(A,B)$ such that for every $\varphi\in B_*$ and every $f\in L^1(G)$ with $\int_G f(s)\,ds=1$ we have*

$$\lim_i\|\varphi\circ(f*\Phi_i-\Phi_i)\|_{B_*}=0.$$

(iii) *For every compact subset K of G, every finite subset $\mathcal{F}$ of $L^2(B)$ and every $\varepsilon>0$, there exists $\Phi\in\mathcal{E}(A,B)$ such that*

$$\sup_{(s,\xi)\in K\times\mathcal{F}}\|\pi_A^\alpha(s)q_\Phi\xi-q_\Phi\pi_B^\alpha(s)\xi\|_2\le\varepsilon.$$

(iv) *There exists a net* (Φ_i) *in* $\mathcal{E}(A,B)$ *such that for every* $f\in L^1(G)$ *and* $\xi\in L^2(B)$ *we have*

$$\lim_i \int f(s)\|\pi_A^\alpha(s)q_{\Phi_i}\xi - q_{\Phi_i}\pi_B^\alpha(s)\xi\|_2\, ds = 0.$$

Proof. (i) $\Rightarrow$ (ii) Let (Φ_i) as in the statement of (i). We have

$$\begin{aligned}\langle\varphi, f*(\Phi_i(a)) - \Phi_i(f*a)\rangle &= \int_G f(s)\langle\varphi, \alpha_s\circ\Phi_i(a) - \Phi_i\circ\alpha_s(a)\rangle\, ds\\ &= \int_G f(s)\langle\varphi\circ\alpha_s, \Phi_i(a) - \alpha_{s^{-1}}\circ\Phi_i\circ\alpha_s(a)\rangle\, ds.\end{aligned}$$

Note that $s\mapsto f(s)\varphi\circ\alpha_s$ is in $L^1(G,B_*)$ and that elements of this form generate $L^1(G,B_*)$. It follows that

$$\lim_i \int_G \langle h(s), \Phi_i(a) - \alpha_{s^{-1}}\circ\Phi_i\circ\alpha_s(a)\rangle\, ds = 0$$

for every $h\in L^1(G,B_*)$ and $a\in A$.

Now, we use again the Day–Namioka convexity argument. Let $h_1,\dots,h_k$ be fixed elements in $L^1(G,B_*)$. For $1\le j\le k$ and $\Phi\in\mathcal{E}(A,B)$, we set

$$b_j(\Phi) = \int_G h_j(s)\circ(\Phi - s\cdot\Phi)\, ds \in A_*.$$

Let us denote by $\mathcal{C}'$ the range of $\mathcal{E}(A,B)$ in the product A_*^k by the map $\Phi\mapsto(b_1(\Phi),\dots,b_k(\Phi))$. We know that $(0,\dots,0)$ belongs to the closure of $\mathcal{C}'$ in A_*^k equipped with the product topology, where we consider the weak topology on A_*. Since $\mathcal{C}'$ is convex, we may replace this latter topology by the norm topology. Therefore, there exists a net (Φ_i) in $\mathcal{E}(A,B)$ such that for every $f\in L^1(G)$ with $\int_G f(s)\, ds = 1$ and every $\varphi\in B_*$, we have

$$\lim_i \|\varphi\circ(\Phi_i - f*\Phi_i)\|_{A_*} = \lim_i \left\| \int_G f(s)\varphi\circ(\Phi_i - s\cdot\Phi_i)\, ds\right\|_{A_*} = 0.$$

(ii) $\Rightarrow$ (iii) Let K be a compact subset of G and $\mathcal{F}$ a finite subset of $L^2(B)^+$. Let $f\in L^1(G)^+$ such that $\int_G f(s)\, ds = 1$. We argue as in the proof of Lemma 4.7 to show that given $\eta>0$, there exists $\Psi\in\mathcal{E}(A,B)$ such that

$$\sup_{(s,\xi)\in K\times\mathcal{F}} \left\|\omega_\xi\circ\big(s\cdot(f*\Psi) - f*\Psi\big)\right\|_{A_*} \le \eta.$$

We take $\Phi = f*\Psi$. Using the Powers–Størmer inequality and Lemma 2.6 we get for $s\in K$ and $\xi\in\mathcal{F}$,

$$\begin{aligned}\|\pi_A^\alpha(s^{-1})q_\Phi\xi - q_\Phi\pi_B^\alpha(s^{-1})\xi\|_2^2 &= \|q_\Phi\xi - \pi_A^\alpha(s)q_\Phi\pi_B^\alpha(s^{-1})\xi\|_2^2\\ &\le \|\omega_\xi\circ(\Phi - s\cdot\Phi)\|_{A_*}.\end{aligned}$$

To conclude, it suffices to replace K by K^{-1} and to take $\eta = \varepsilon^2$.

(iii) $\Rightarrow$ (iv) is obvious. It is not difficult to show that (iv) implies (i) and we skip the proof. $\square$

REMARK 4.13. An inspection of the above proof shows that when α is an amenable action on (A,B) with $B \subset Z(A)$, one may take the Φ_i's in the convex set $\mathcal{C}$ of Lemma 4.10. Using the cocycle representation U_A introduced at the end of Section 3, and taking $\Phi_i = \varpi_{\xi_i}$ we get, for $\eta \in L^2(B) = L^2(X,m)$,

$$\begin{aligned}
&\|q_{\Phi_i}\pi_B^\alpha(s^{-1})\eta - \pi_A^\alpha(s^{-1})q_{\Phi_i}\eta\|_2^2 \\
&\quad = \|\pi_A^\alpha(s)q_{\Phi_i}\pi_B^\alpha(s^{-1})\eta - q_{\Phi_i}\eta\|_2^2 \\
&\quad = \int_X \|U_A(x,s)\xi_i(s^{-1}x)\eta(x) - \xi_i(x)\eta(x)\|_2^2 \, dm(x) \\
&\quad = \int_X |\eta(x)|^2 \|U_A(x,s)\xi_i(s^{-1}x) - \xi_i(x)\|_2^2 \, dm(x).
\end{aligned}$$

Now by Lemma 4.12(iv), we see that α is amenable if and only if there exists a sequence (ξ_n) of sections of the Hilbert bundle $(L^2(A(x))_{x\in X}$ with $\|\xi_n(x)\|_2 = 1$ almost everywhere, such that

$$\lim_n \int_{X\times G} f(x,s)\|U_A(x,s)\xi_n(s^{-1}x) - \xi_n(x)\|_2^2 \, dm(x)\, ds = 0$$

for every $f \in L^1(X \times G)$. Expressed in term of groupoid, this is equivalent to the fact that the representation U_A of the measured groupoid $X \rtimes G$ weakly contains the trivial representation (see [3] for details).

Proof of Theorem 4.8. Immediate consequence of Lemmas 4.11 and 4.12. $\square$

THEOREM 4.14. *Let α be a tensor product action on $A = B \otimes M$. Assume that there is an equivariant conditional expectation from A onto B. Then the conclusions of Theorem 4.8 hold. In particular for every probability measure μ on G we have $\|\pi_B^\alpha(\mu)\| = \|(\pi_B^\alpha \otimes \pi_M^\alpha)(\mu)\|$.*

Proof. We first observe that by restriction there is an equivariant conditional expectation from $Z(B) \otimes M$ onto $Z(B)$. We write $Z(B)$ as $L^\infty(X,m)$ and we disintegrate the representation π_B^α so that for $\xi = \int_X^\oplus \xi(x)\,dm(x) \in \int_X^\oplus L^2(B(x))\,dm(x)$ we have (see the end of Section 3),

$$\pi_B^\alpha(s)\xi(x) = \sqrt{r(x,s)}U_B(x,s)\xi(s^{-1}x).$$

Using Theorem 4.8 and Remark 4.13, we get a net (Φ_i) of conditional expectations from $Z(B)\otimes M$ onto $Z(B)$, of the form $\Phi_i = \varpi_{\xi_i}$ (where $\xi_i : X \to L^2(M)$ is a measurable map with $\|\xi_i(x)\|_2 = 1$ almost everywhere), such that for every $\eta \in L^2(Z(B))^+$ we have

$$\lim_i \left\|\pi_{Z(B)\otimes M}^\alpha(s)q_{\Phi_i}\eta - q_{\Phi_i}\pi_{Z(B)}^\alpha(s)\eta\right\|_2 = 0$$

uniformly on compact subsets of G.

Each linear isometry $q_{\Phi_i} : L^2(Z(B)) \to L^2(Z(B)) \otimes L^2(M)$ extends to an isometry $q_{\Phi_i} : L^2(B) \to L^2(B) \otimes L^2(M)$ by setting

$$q_{\Phi_i}\eta(x) = \eta(x) \otimes \xi_i(x)$$

for $\eta = \int_X^\oplus \eta(x)\, dm(x) \in L^2(B)$.

A straightforward computation shows that for $\eta \in L^2(B)$,

$$\|\pi^\alpha_{B\otimes M}(s)q_{\Phi_i}\eta - q_{\Phi_i}\pi^\alpha_B(s)\eta\|_2 = \left\|\pi^\alpha_{Z(B)\otimes M}(s)q_{\Phi_i}|\eta| - q_{\Phi_i}\pi^\alpha_{Z(B)}(s)|\eta|\right\|_2$$

where we denote by $|\eta|$ the element $x \mapsto \|\eta(x)\|_2$ of $L^2(Z(B))$. This ends the proof. $\square$

Remark 4.15. We may more generally use the same kind of techniques for any action α on $B \otimes M$ leaving $B \otimes 1$ invariant even if the action is not a tensor product action. We may even deal with an action on a pair $(B \otimes M, B \otimes N)$ where $N \subset Z(M)$, such that $B \otimes N$ is globally invariant under α. As a consequence, using the structure theory of type I von Neumann algebras, one gets the following result.

Theorem 4.16. *Let α be an amenable action of G on a pair (A, B) where B is a type I von Neumann algebra such that $Z(B) \subset Z(A)$. Then there exists a net (V_i) of isometries from $L^2(B)$ into $L^2(A)$ such that for every $\xi \in L^2(B)$ one has*

$$\lim_i \|\pi^\alpha_A V_i\xi - V_i\pi^\alpha_B(s)\xi\| = 0$$

uniformly on compact subsets of G. In particular π^α_B is weakly contained in π^α_A.

Since we are mainly interested in amenable and coamenable actions, we shall not give the rather tedious proof. In fact, one would be more interested in deciding whether the above theorem is true when α is an amenable action on (A, B), under the assumption that there are "enough" normal conditional expectations from A onto B.

Recall that a group G is said to have *property T* if every of its representations that weakly contains the trivial representation ι_G actually contains ι_G as a subrepresentation, that is has a nonzero G-invariant vector. It follows that for such groups, every dynamical system having an invariant state, i.e., $\iota_G \prec \pi^\alpha_A$, has a normal invariant state, i.e., $\iota_G \le \pi^\alpha_A$. More generally, we have the following theorem.

Theorem 4.17. *Let G be a locally compact group having property T and let α be an amenable action on a pair (A, B) of von Neumann algebras. We assume that B is contained in the centre of A and that the action on B is ergodic and leaves invariant a normal faithful state. Then there exists an equivariant normal conditional expectation from A onto B. In particular, π^α_B is a subrepresentation of π^α_A.*

Proof. We write $B = L^\infty(X, m)$ where m is an invariant probability measure. Since G has property T and preserves the finite measure m, one knows that the measured groupoid $X \rtimes G$ has property T (see [3, Corollary 5.16] for instance). By Remark 4.13, its representation U_A weakly contains the trivial one. By definition of property T for a measured groupoid, the trivial representation is actually contained in U_A. This means that there exists a section $\xi : x \mapsto \xi(x) \in L^2(A(x))$ with $\|\xi(x)\|_2 = 1$ almost everywhere, such that $U_A(x, s)\xi(s^{-1}x) = \xi(x)$ almost everywhere on $X \times G$. Then $E = \varpi_\xi$ is a normal equivariant conditional expectation from A onto B. □

4.3. We now mention another positive answer to the problem considered in this section. Let B be a von Neumann algebra, G a locally compact group and α the G-action on B associated to a representation of $\pi : G \to \mathcal{U}(B)$ (see Example 2.7(c)). This action extends to the action $\alpha : s \mapsto \mathrm{Ad}\,\pi(s)$ on $A = \mathcal{B}(L^2(B))$. Observe that the amenability of the G-action on (A, B) is equivalent to the injectivity of B, that is to the existence of a norm one projection from $\mathcal{B}(L^2(B))$ onto B.

Proposition 4.18. *Let α be the G-action on (A, B) defined above. Assume that B is injective. Then we have $\pi^\alpha_B \prec \pi^\alpha_A$, that is $\pi J_B \pi J_B \prec \pi \otimes \overline{\pi}$.*

Proof. By the result asserting that an injective von Neumann algebra is semi-discrete (see [9], [10]) we have, for every $a_i, b_i \in B$, $i = 1, \dots, n$,

$$\Big\| \sum_{i=1}^n a_i J_B b_i J_B \Big\| \le \Big\| \sum_{i=1}^n a_i \otimes \overline{b_i} \Big\|.$$

In particular, for f_i, g_i in $L^1(G)$ we get

$$\Big\| \sum_{i=1}^n \pi(f_i) J_B \pi(g_i) J_B \Big\| \le \Big\| \sum_{i=1}^n \pi(f_i) \otimes \overline{\pi}(g_i) \Big\|.$$

It follows that

$$\Big\| \int_{G\times G} h(s,t)\pi(s) J_B \pi(t) J_B \, ds\, dt \Big\| \le \Big\| \int_{G\times G} h(s,t)\pi(s) \otimes \overline{\pi}(t)\, ds\, dt \Big\| \tag{4}$$

for $h \in L^1(G \times G)$. Let μ be a bounded measure on G and denote by ν its image by the diagonal map $s \mapsto (s, s)$. Moreover, let us consider an approximate unit (φ_i) of $L^1(G)$. Applying the inequality (4) to $h = (\varphi_j \otimes \varphi_j) * \nu$, we get

$$\begin{aligned} &\Big\| \pi(\varphi_j) J_B \pi(\varphi_j) J_B \int_G \pi(s) J_B \pi(s) J_B \, d\mu(s) \Big\| \\ &\le \Big\| \pi(\varphi_j) \otimes \overline{\pi}(\varphi_j) \int_G \pi(s) \otimes \overline{\pi}(s)\, d\mu(s) \Big\| \end{aligned}$$

from which we easily get $\|(\pi J_B \pi J_B)(\mu)\| \le \|(\pi \otimes \overline{\pi})(\mu)\|$. □

Remark 4.19. If we apply this proposition to the group von Neumann algebra $B = L(G)$ and to the representation $\pi = \lambda_G$, we get that whenever $L(G)$ is injective (for instance when G is almost connected by [9, Corollary 6.7]), then the conjugation representation γ_G of G is weakly contained in $\lambda_G \otimes \overline{\lambda}_G$ and, therefore, in λ_G. Note that when the reduced C^*-algebra of G is nuclear, it has been proved by Kaniuth [25] that γ_G is weakly contained in the direct sum of the representations $\pi \otimes \overline{\pi}$ where π ranges over the reduced dual of G.

5. Amenable and coamenable actions

Proposition 5.1. *Let (A, G, α) be an amenable dynamical system. We have*

$$\pi^\alpha_{Z(A)} \prec \pi^\alpha_A \prec \lambda_G.$$

In particular for every probability measure μ on G we have

$$\|\lambda_G(\mu)\| = \|\pi^\alpha_A(\mu)\| = \left\|\pi^\alpha_{Z(A)}(\mu)\right\|.$$

Proof. By [1, Corollary 3.6], we know that the action of G on $Z(A)$ is amenable. Moreover, it follows from [1, Proposition 2.5] that the action of G on the pair $(A, Z(A))$ is amenable. By Theorems 4.8 and 4.14, we have respectively $\pi^\alpha_{Z(A)} \prec \pi^\alpha_A$ and $\pi^\alpha_A \prec \pi^\alpha_A \otimes \lambda_G$. Then the conclusion follows from Fell's absorption principle.

The second part of the proposition follows from Theorem 3.1. □

Remark 5.2. The property $\pi^\alpha_A \prec \lambda_G$ means that for every bounded measure μ on G we have $\|\pi^\alpha_A(\mu)\| \le \|\lambda_G(\mu)\|$. It is a transference property of norm estimates, in the style of the ones that prove to be so useful in classical harmonic analysis and ergodic theory (see [8]). In the noncommutative setting, one can also establish L^p-transference inequalities and apply them to prove ergodic theorems. This is the subject of a forthcoming paper.

Theorem 5.3. *Let (A, G, α) be a dynamical system. The following conditions are equivalent:*

(i) *there exists a G-invariant state on A (i.e., the action is coamenable);*
(ii) *the trivial representation ι_G is weakly contained in π^α_A;*
(iii) *there exists an adapted probability measure μ on G with $r(\pi^\alpha_A(\mu)) = 1$.*

Proof. (i) ⇒ (ii) is a particular case of Theorem 4.8 where we take $B = \mathbb{C}$ (in this case, all technical difficulties disappear and the proof is indeed straightforward by usual convexity arguments).

(ii) ⇒ (iii) is obvious. In fact, if (ii) holds, one easily sees that 1 is an approximate eigenvalue of $\pi^\alpha_A(\mu)$ for any probability measure μ on G.

To show (iii) ⇒ (i) we follow the lines of the proof of Theorem 1 in [6] (or of [12, Theorem]), that we reproduce for the reader's convenience. First, since $\pi^\alpha_A(\mu)$ is a contraction of spectral radius 1, there exist a complex number c with $|c| = 1$ and a sequence (ξ_n) of unit vectors in $L^2(A)$ such that

$\lim_n \|\pi_A^\alpha(\mu)\xi_n - c\xi_n\| = 1$ (see the proof of [12, Theorem 1]). Using Lemma 2.1 and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\left|\int_G \langle \xi_n, \pi_A^\alpha(s)\xi_n\rangle\, d\mu(s)\right| &\le \int_G |\langle \xi_n, \pi_A^\alpha(s)\xi_n\rangle|\, d\mu(s)\\ &\le \int_G \langle |\xi_n|, \pi_A^\alpha(s)|\xi_n|\rangle^{1/2} \langle |J\xi_n|, \pi_A^\alpha(s)|J\xi_n|\rangle^{1/2}\, d\mu(s)\\ &\le \left(\int_G \langle |\xi_n|, \pi_A^\alpha(s)|\xi_n|\rangle\, d\mu(s)\right)^{1/2}\\ &\quad \times \left(\int_G \langle |J\xi_n|, \pi_A^\alpha(s)|J\xi_n|\rangle\, d\mu(s)\right)^{1/2}.\end{aligned}$$

Since

$$\int_G \langle |\xi_n|, \pi_A^\alpha(s)|\xi_n|\rangle\, d\mu(s) \le 1 \quad \text{and} \quad \int_G \langle |J\xi_n|, \pi_A^\alpha(s)|J\xi_n|\rangle\, d\mu(s) \le 1$$

and since $\lim_n |\int_G \langle \xi_n, \pi_A^\alpha(s)\xi_n\rangle\, d\mu(s)| = 1$, we infer that

$$\lim_n \int_G \langle |\xi_n|, \pi_A^\alpha(s)|\xi_n|\rangle\, d\mu(s) = 1.$$

It follows that there exists a subsequence (η_n) of $(|\xi_n|)$ such that

$$\lim_n \langle \eta_n, \pi_A^\alpha(s)\eta_n\rangle = 1$$

and, therefore,

$$\lim_n \|\pi_A^\alpha(s)\eta_n - \eta_n\| = 0$$

for all s in a subset S of G whose complement has μ-measure zero.

Let us denote by A^c the C^*-subalgebra of all $x \in A$ such that $s \mapsto \alpha_s(x)$ is norm continuous, and for $n \in \mathbb{N}$, denote by φ_n the state $x \mapsto \langle \eta_n, x\eta_n\rangle$ defined on A^c. Let φ be a weak*-limit point of (φ_n) in the dual space of A^c. The set F of elements $s \in G$ such that $\varphi \circ \alpha_s = \varphi$ is a closed subgroup containing S. Therefore, we have $\mu(G \setminus F) = 0$. It follows that $F = G$ since μ is an adapted probability measure. To conclude we use the well-known fact that the existence of a G-invariant state on A is equivalent to the existence of a G-invariant state on A^c) (see [1, Lemma 2.1] for instance). □

Remark 5.4. As a consequence of the previous theorem, we see that $\iota_G \prec \pi_A^\alpha$ if and only if $\iota_G \prec \pi_A^\alpha \otimes \overline{\pi_A^\alpha}$. Indeed, whenever this last condition holds, the implication (ii) ⇒ (i) gives the existence of a state φ on $\mathcal{B}(L^2(A))$ such that $\varphi \circ \operatorname{Ad} \pi_A^\alpha(s)$ for every $s \in G$ and therefore the existence of a G-invariant state on A by restriction. Applied to the regular representation λ_G, one recovers a result of Fell [17] saying that G is amenable whenever λ_G weakly contains a finite dimensional representation.

REMARK 5.5. As said in the Introduction, one cannot expect in general that $\pi_B^\alpha \prec \pi_A^\alpha$ implies that the action is amenable. This is already not true when A is Abelian. In [2], we studied some particular cases where this fact holds in the Abelian setting. Let us recall below another important particular case, due to Connes [9], where this fact holds in the noncommutative setting. Here, we take $A = \mathcal{B}(L^2(B))$ and G is the unitary group $\mathcal{U}(B)$ of B. This group is not countable, but we observe that the only result used in the sequel, namely Theorem 5.3, does not require any separability assumption. We let G act on (A, B) by $\alpha_U = \mathrm{Ad}\, U$ for $U \in G$. The amenability of the G-action on (A, B) is, by definition, the injectivity of B. Recall that an hypertrace for B is a G-invariant state on A.

THEOREM 5.6 ([9]). *We keep the above assumptions. The following conditions are equivalent:*

(i) *There exists an hypertrace for B;*

(ii) *For every finite subsets $\{U_1, \dots, U_n\}$ of $\mathcal{U}(B)$ and $\{c_1, \dots, c_n\}$ of $\mathbb{C}$, we have*

$$\Big|\sum_{i=1}^n c_i\Big| \le \Big\|\sum_{i=1}^n c_i U_i \otimes \overline{U}_i\Big\|.$$

(iii) *For every finite subset $\{U_1, \dots, U_n\}$ of $\mathcal{U}(B)$ we have*

$$\Big\|\sum_{i=1}^n U_i \otimes \overline{U}_i\Big\| = n.$$

Moreover, if B is a factor, these conditions are equivalent to

(iv) *B is a finite injective factor.*

In particular, when B is a finite factor, we see that B is injective if and only if $\pi_B^\alpha \prec \pi_A^\alpha$.

Proof. Assertion (ii) means that $\iota_G \prec \pi_A^\alpha$. Therefore, (i) $\Rightarrow$ (ii) is a particular case of (i) $\Rightarrow$ (ii) in Theorem 5.3.

(ii) $\Rightarrow$ (iii) is obvious. Assuming that (iii) holds, let us prove (i). Thanks to (iii) $\Rightarrow$ (i) in Theorem 5.3, for every finite subset $\mathcal{F}$ of $\mathcal{U}(B)$, we get a state $\psi_{\mathcal{F}}$ on A, invariant by $\mathrm{Ad}\, U$, $U \in \mathcal{F}$. Taking a limit point of $(\psi_{\mathcal{F}})$ along the filter of finite subsets of $\mathcal{U}(B)$, we obtain an hypertrace for B.

(iv) $\Rightarrow$ (i) is obvious. Indeed, if τ is the tracial state of B and E is a conditional expectation from $\mathcal{B}(L^2(B))$ onto B, then $\tau \circ E$ is an hypertrace.

Let us sketch the proof of (i) $\Rightarrow$ (iv) whenever B is a factor. Let ψ be an hypertrace for B. Its restriction to B is a trace, and therefore B is finite. Denote by τ its trace. For $a \in A^+$ and $b \in B$, we set $\psi_a(b) = \psi(ab)$. Then ψ_a is a positive state on B with $\psi_a \le \|a\|\tau$. We denote by $E(a)$ the Radon–Nikodým derivative of ψ_a with respect to τ. Then it is easy to check that E extends into a conditional expectation from A onto B (e.g., see [23, Lemma 2.2]).

Assume now that B is a finite factor. If B is injective, we have $\pi_B^\alpha \prec \pi_A^\alpha$ by Theorem 4.6. Conversely, assume that $\pi_B^\alpha \prec \pi_A^\alpha$. Since the tracial state of B is G-invariant, we have $\iota_G \leq \pi_B^\alpha$ and, therefore, (ii) holds. It follows that B injective by (iv). $\square$

The following proposition extends the equivalence between (i) and (ii) stated in Theorem 5.3

Proposition 5.7. *Let α be an action of a locally compact group G on (A, B). We assume that B is a finite factor and that $\alpha(G)$ contains the group $\{\mathrm{Ad}\, U : U \in \mathcal{U}(B)\}$. Then the action is amenable if and only if $\pi_B^\alpha \prec \pi_A^\alpha$.*

Proof. Assume that $\pi_B^\alpha \prec \pi_A^\alpha$. Since B is a finite factor, we have $\iota_G \leq \pi_B^\alpha$ and therefore $\iota_G \prec \pi_A^\alpha$. By Theorem 5.3, there exists a G-invariant state ψ on A. We have $\psi(ab) = \psi(ba)$ for $a \in A$ and $b \in \mathcal{U}(B)$. As in the previous theorem, this state gives rise to a conditional expectation from A onto B, easily seen to be G-invariant. $\square$

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