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RIESZ TRANSFORMS ASSOCIATED TO BESSEL OPERATORS

MICHAËL VILLANI

ABSTRACT. For $\nu > 0$, we consider the Bessel operator S_ν defined on $L^2(\mathbb{R}^+, x^{2\nu}\,dx)$ by $S_\nu = -\frac{d^2}{dx^2} - \frac{2\nu}{x}\frac{d}{dx}$. We prove, in a simple way, that the Riesz transform associated to S_ν is bounded on $L^p(\mathbb{R}^+, x^{2\nu}\,dx)$, $1 < p < \infty$, with a constant only depending on p. We also give a weighted version and estimate the constant.

1. Introduction

1.1. Motivation: the classical case. Let Δ_n be the standard Laplacian defined on $\mathbb{R}^n$ by $\Delta_n := -\sum_j \frac{\partial^2}{\partial X_j^2}$.
Then we have the classical result (see [8, Theorem 3]) for the associated Riesz transforms $R_j := \frac{\partial}{\partial X_j}\Delta_n^{-1/2}$, $(1 \le j \le n)$:

THEOREM 1. *For every $p \in]1;\infty[$, there exists a constant $C_p > 0$ only depending on p, such that*

$$C_{p'}^{-1}\|f\|_{L^p(\mathbb{R}^n,dX)} \le \left\| \left(\sum_j |R_j(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n,dX)} \le C_p \|f\|_{L^p(\mathbb{R}^n,dX)}.$$

The restriction of Δ_n to radial functions (i.e., $f(X) = f((\sum X_j^2)^{1/2}) = f(x)$), is

$$S_n = -\frac{d^2}{dx^2} - \frac{n-1}{x}\frac{d}{dx} \tag{1}$$

and Theorem 1 becomes:

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COROLLARY 2. *For every $p\in]1;\infty[$, there exists a constant $C_p>0$, such that*

$$C_{p'}^{-1}\|f\|_{L^p(\mathbb{R}^+,x^{n-1}\,dx)}\le\left\|\frac{d}{dx}S_n^{-1/2}(f)\right\|_{L^p(\mathbb{R}^+,x^{n-1}\,dx)}\le C_p\|f\|_{L^p(\mathbb{R}^+,x^{n-1}\,dx)}.$$

Indeed, noting that $\frac{\partial x}{\partial X_j}=\frac{\partial(\sum X_k^2)^{1/2}}{\partial X_j}=\frac{X_j}{x}$, we get

$$\begin{aligned}\sum_j|R_j(f)|^2&=\sum_j\left|\frac{\partial}{\partial X_j}\Delta_n^{-1/2}(f)\right|^2\\&=\sum_j\left|\frac{X_j}{x}\frac{d}{dx}S_n^{-1/2}(f)\right|^2\\&=\left|\frac{d}{dx}S_n^{-1/2}(f)\right|^2.\end{aligned}$$

1.2. Bessel operators S_ν. Now we do not assume n to be an integer. More precisely, for $\nu>0$, we define

$$S_\nu:f\longmapsto-\frac{d^2}{dx^2}f-\frac{2\nu}{x}\frac{d}{dx}f.$$

Then $S_\nu=D^*D$ where D^* is the adjoint operator of $D=\frac{d}{dx}$ in $L^2(\mathbb{R}^+,x^{2\nu}\,dx)$. Our aim is to show in a simple way that Corollary 2 extends to operator S_ν. From now on, we define the measure $d\nu(x):=x^{2\nu}\,dx$.

We consider $R_\nu:=DS_\nu^{-1/2}$ the Riesz transform associated to S_ν, which verifies $R_\nu^*R_\nu=Id$ on $L^2(\mathbb{R}^+,\,d\nu(x))$.

1.3. Results. Our two main results are the following theorems.

THEOREM 3. *For every $p\in]1;\infty[$, there exists a constant $K_p>0$ only depending on p, such that, for $\nu>0$,*

$$K_{p'}^{-1}\|f\|_{L^p(\mathbb{R}^+,d\nu(x))}\le\|R_\nu(f)\|_{L^p(\mathbb{R}^+,d\nu(x))}\le K_p\|f\|_{L^p(\mathbb{R}^+,d\nu(x))}.$$

THEOREM 4. *Let $p\in]1;\infty[$ and $\nu>0$, then for $\alpha\in]-2\nu;2\nu(p-1)[$, there is a constant $K_{p,\nu,\alpha}>0$ such that:*

$$K_{p',\nu,\alpha}^{-1}\|f\|_{L^p(\mathbb{R}^+,x^\alpha\,d\nu(x))}\le\|R_\nu(f)\|_{L^p(\mathbb{R}^+,x^\alpha\,d\nu(x))}\le K_{p,\nu,\alpha}\|f\|_{L^p(\mathbb{R}^+,x^\alpha\,d\nu(x))}.$$

These results are due to Muckenhoupt and Stein (see [5]) with a non-explicit constant. Indeed, their definition of the conjugate function (using harmonic extensions) coincides in L^2 with the definition of R_ν. However, the proof offered here is simpler and proves the independence of constants on the parameter ν. This proof is based on a method due to Pisier (see [6]), using transference of the Hilbert transform on $\mathbb{R}$.

Here, the idea is that $R_\nu(f) = \frac{2}{\sqrt{\pi}} \frac{\partial}{\partial x} \int_0^\infty \exp(-t^2 S_\nu)(f)\, dt$ appears as the restriction to radial functions on $\mathbb{R}^2$ of an operator defined for $F \in S(\mathbb{R}^2)$ by $F \rightsquigarrow \frac{\partial}{\partial X_1} \int_0^\infty (\int\int F(Y)\phi_\nu(Y - \frac{X}{t})\, dY)\, dt$ (see (13)).

In our setting, we do not need Muckenhoupt's weight's theory.

Remark 1. Theorem 4 implies a result of [1] given in part 5, related to $\widetilde{S}_\nu := -\frac{d^2}{dx^2} + \frac{\nu^2-\nu}{x^2}$.

We can give a quantitative version of Theorem 4.

Proposition 5. *Let $p \in]1;\infty[$ and $\alpha \in \mathbb{R}$, then with the notation of Theorem 4,*

(i) *if $\alpha < 0$,*

$$K_{p,\nu,\alpha} \sim \frac{\sqrt{\pi}}{2} \gamma_{p'} \|H^*\|_{p\to p}, \quad \nu \to \infty.$$

(ii) *If $\alpha > 0$,*

$$K_{p,\nu,\alpha} \sim 2^{\frac{1}{p'}} \left(\sqrt{\pi}\right)^{\frac{1}{p}} \|H^*\|_{p\to p} \nu^{\frac{1}{2p}}, \quad \nu \to \infty.$$

2. Preliminaries

2.1. Notation. If $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$, then we denote $x := (\sum X_j^2)^{1/2}$.

2.2. Constants. We define two normalization constants:

$$I_\nu := \int_0^\infty e^{-y^2/4} y^{2\nu}\, dy = \int_0^\infty e^{-s}(4s)^{\nu-\frac{1}{2}} 2\, ds = 2^{2\nu}\Gamma\left(\nu + \frac{1}{2}\right), \tag{2}$$

$$C_\nu := \int_0^\pi (\sin\theta)^{2\nu-1}\, d\theta.$$

2.3. Riesz transforms. We express R_ν by the formula:

$$R_\nu(f)(x) = \frac{2}{\sqrt{\pi}} \frac{d}{dx} \int_0^\infty \exp(-t^2 S_\nu)(f)(x)\, dt, \tag{3}$$

2.4. Hilbert transform. Let $\varphi \in L^p(\mathbb{R})$, we define

$$H\varphi(s) := p.v.\frac{1}{\pi} \int_{\mathbb{R}} \varphi(s-t) \frac{dt}{t},$$

$$H_\varepsilon\varphi(s) := \frac{1}{\pi} \int_{\varepsilon \le |t| \le \frac{1}{\varepsilon}} \varphi(s-t) \frac{dt}{t},$$

$$H^*\varphi(s) := \sup_{\varepsilon > 0} |H_\varepsilon \varphi(s)|.$$

Then (see, for example, [3]):

Proposition 6. (i) *$\forall p \in]1;\infty[$, H is a bounded operator on $L^p(\mathbb{R})$.*

(ii) *$\forall p \in]1;\infty[$, there is a constant $C_p > 0$ such that:*

$$\|H^*\varphi\|_{L^p(\mathbb{R})} \le C_p \|\varphi\|_{L^p(\mathbb{R})} \quad \forall \varphi \in L^p(\mathbb{R}).$$

We denote $\|H\|_{p\to p} := \|H\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}$.

2.5. Although we will not use this result, we give an inspiring expression of $\exp(-t^2 S_n)$ [see (21) part 6]. The usual heat kernel on $\mathbb{R}^n$ has a well-known expression giving for $f \in S(\mathbb{R}^+)$,

$$\exp(-t^2 S_n)(f)(x) = \frac{1}{C}\int_0^\infty \int_0^\pi f(|x - yte^{i\theta}|)e^{-\frac{y^2}{4}}(\sin\theta)^{n-2}s^{n-1}\,d\theta\,ds,$$

in such a way that the right-hand side equals 1 when $f \equiv 1$.

Proof. Making successively (for $Y \in \mathbb{R}^n$ and $y = |Y|$) the changes of variables:

$$(*) \qquad Y = yY', \qquad dY = y^{n-1}\,dy\,d\sigma(Y')$$

$$(**) \qquad Y' = \cos\theta\frac{X}{x} + \sin\theta\widetilde{Y}, \quad \textit{where } |\widetilde{Y}| = 1 \textit{ and } (X, \widetilde{Y}) = \frac{\pi}{2},$$

whence $d\sigma(Y') = (\sin\theta)^{n-2}\,d\theta\,d\sigma(\widetilde{Y})$ *and* $|X - yY'| = |x - ye^{i\theta}|$

$$(***) \qquad s = \frac{y}{t}$$

we get

$$\begin{aligned}
\exp(-t^2 S_n)(f)(x) &= \frac{1}{(2\sqrt{\pi}t)^n}\int_{\mathbb{R}^n} f(|X - Y|)e^{-\frac{|Y|^2}{4t^2}}\,dY \\
&= \frac{1}{(2\sqrt{\pi}t)^n}\int_{\mathbb{R}^+}\int_{S^{n-1}} f(|X - yY'|) \\
&\quad \times e^{-\frac{y^2}{4t^2}}y^{n-1}\,dy\,d\sigma(Y') \quad (*) \\
&= \frac{1}{(2\sqrt{\pi}t)^n}\int_{\mathbb{R}^+}\int_0^\pi\int_{S^{n-2}} f(|x - ye^{i\theta}|) \\
&\quad \times e^{-\frac{y^2}{4t^2}}y^{n-1}(\sin\theta)^{n-2}\,d\sigma(\widetilde{Y})\,d\theta\,dy \quad (**) \\
&= \frac{Vol(S^{n-2})}{(2\sqrt{\pi}t)^n}\int_{\mathbb{R}^+}\int_0^\pi f(|x - ye^{i\theta}|) \\
&\quad \times e^{-\frac{y^2}{4t^2}}y^{n-1}(\sin\theta)^{n-2}\,d\theta\,dy \\
&= \frac{Vol(S^{n-2})}{(2\sqrt{\pi})^n}\int_{\mathbb{R}^+}\int_0^\pi f(|x - yte^{i\theta}|) \\
&\quad \times e^{-\frac{y^2}{4}}y^{n-1}(\sin\theta)^{n-2}\,d\theta\,dy \quad (***).
\end{aligned}$$

□

3. Tools for the proof

Eigenvectors of S_ν. Let T be the bounded function defined on $\mathbb{R}^+$ by

$$T(x) := Z_\nu\,\frac{J_{\nu-1/2}(x)}{x^{\nu-1/2}},$$

where $J_{\nu-\frac{1}{2}}$ is a Bessel function and Z_ν a constant such that $T(0)=1$. Then T has the following properties (see [2, pages 27–29, 35] and [7, pages 34–37]):

$\forall \xi \in \mathbb{R}^+$

(9) $x \mapsto T(\xi x) = T_\xi(x)$ *is an eigenvector of* S_ν *for the eigenvalue* ξ^2,

$$\frac{1}{I_\nu}\int_0^\infty T_s(t)e^{-\frac{s^2}{4}}\,d\nu(s) = e^{-t^2}, \tag{10}$$

$$T_\xi(x)T_\xi(y) = \frac{1}{C_\nu}\int_0^\pi T_\xi(|x-e^{i\theta}y|)(\sin\theta)^{2\nu-1}\,d\theta. \tag{11}$$

4. Proof of Theorem 3

4.1.

Proof. Let $f \in S(\mathbb{R}^+)$.

Step 1: expression of $\exp(-t^2S_\nu)(f)$. Let $x>0$, $y>0$ and $\theta \in [0;\pi]$. Let $(\aleph)$ be the change of variable defined by $X=(x,0)$ and $Y=(Y_1,Y_2)=(y\cos\theta, y\sin\theta)$, so that $|x-e^{i\theta}y| = |X-Y|$.

By (9), the kernel $p_{t^2}^{(\nu)}$ of $\exp(-t^2S_\nu)$ w.r. to $d\nu$ can be expressed by (see [4, page 1335]):

$$p_{t^2}^{(\nu)}(x,y) = \frac{2^{2\nu+1}}{(I_\nu)^2}\int_0^\infty e^{-t^2s^2}T_s(x)\;T_s(y)\,d\nu(s). \tag{12}$$

So, taking $F(X)=f(|X|)$ and $T_s(X)=T_s(|X|)$, we get

$$\begin{aligned}
&\exp(-t^2S_\nu)(f)(x) \\
&= \int_0^\infty f(y)p_{t^2}^{(\nu)}(x,y)\,d\nu(y) \\
&= \frac{2^{2\nu+1}}{(I_\nu)^2}\int_0^\infty\int_0^\infty f(y)e^{-t^2s^2}T_s(x)\;T_s(y)\,d\nu(s)\,d\nu(y) \quad \text{by (12)} \\
&= \frac{2^{2\nu+1}}{C_\nu(I_\nu)^2}\int_0^\infty\int_0^\infty\int_0^\pi f(y)e^{-t^2s^2}T_s(|x-e^{i\theta}y|) \\
&\quad \times(\sin\theta)^{2\nu-1}\,d\theta\,d\nu(s)\,d\nu(y) \quad \text{by (11)} \\
&= \frac{2^{2\nu+1}}{C_\nu(I_\nu)^2}\int_{\mathbb{R}\times\mathbb{R}^+}\int_0^\infty F(Y)\mathbb{T}_s(X-Y)Y_2^{2\nu-1}e^{-t^2s^2}\,d\nu(s)\,dY \quad \text{by } (\aleph) \\
&= \frac{1}{C_\nu\cdot I_\nu}\int_{\mathbb{R}\times\mathbb{R}^+}F(Y)e^{-\frac{|X-Y|^2}{4t^2}}Y_2^{2\nu-1}t^{-2\nu-1}\,dY \quad \text{by (10)} \\
&= \frac{1}{C_\nu\cdot I_\nu}\int_{\mathbb{R}\times\mathbb{R}^+}F(tY)e^{-\frac{|X/t-Y|^2}{4}}Y_2^{2\nu-1}\,dY \\
&= \int_{\mathbb{R}\times\mathbb{R}^+}F(tY)\phi_\nu\left(Y-\frac{X}{t}\right)dY,
\end{aligned} \tag{13}$$

where $\phi_\nu(Y)\,dY = \frac{e^{-|Y|^2/4} Y_2^{2\nu-1}}{C_\nu \cdot I_\nu}\,dY_1\,dY_2$ is a probability measure.

Step 2: expression of R_ν.

$$\begin{aligned}
(14)\quad & \frac{\sqrt{\pi}}{2} R_\nu(f)(x) \\
&= \frac{d}{dx}\int_0^\infty \exp(-t^2 S_\nu)(f)(x)\,dt \quad \text{by (3)} \\
&= \frac{\partial}{\partial X_1}\int_0^\infty \left[\int_{\mathbb{R}\times\mathbb{R}^+} F(tY)\phi_\nu\left(Y - \frac{X}{t}\right)dY\right]dt \quad \text{by (13)} \\
&= \frac{\partial}{\partial X_1}\int_0^\infty \left[\int_{\mathbb{R}\times\mathbb{R}^+} F(X-tY)\phi_\nu(Y)\,dY\right]dt \quad (*)\text{(see below)} \\
&= \int_0^\infty \left[\int_{\mathbb{R}\times\mathbb{R}^+} \frac{\partial F}{\partial X_1}(X-tY)\phi_\nu(Y)\,dY\right]dt \quad \big(f \in S(\mathbb{R}^+)\big) \\
&= \lim_{\varepsilon\to 0^+}\int_\varepsilon^{\frac{1}{\varepsilon}} \left[\int_{\mathbb{R}\times\mathbb{R}^+} \frac{\partial F}{\partial X_1}(X-tY)\phi_\nu(Y)\,dY\right]dt \quad \big(f \in S(\mathbb{R}^+)\big) \\
&= -\lim_{\varepsilon\to 0^+}\int_\varepsilon^{\frac{1}{\varepsilon}} \left[\int_{\mathbb{R}\times\mathbb{R}^+} \frac{\partial F}{\partial Y_1}(X-tY)\phi_\nu(Y)\,dY\right]\frac{dt}{t} \\
&= \lim_{\varepsilon\to 0^+}\int_\varepsilon^{\frac{1}{\varepsilon}} \left[\int_{\mathbb{R}\times\mathbb{R}^+} F(X-tY)\frac{\partial \phi_\nu}{\partial Y_1}(Y)\,dY\right]\frac{dt}{t} \\
&= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}\times\mathbb{R}^+}\int_\varepsilon^{\frac{1}{\varepsilon}} F(X-tY)\frac{dt}{t}\frac{\partial \phi_\nu}{\partial Y_1}(Y)\,dY \quad \text{(Fubini)} \\
&= \frac{1}{2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}\times\mathbb{R}^+}\int_{\varepsilon\le |t|\le \frac{1}{\varepsilon}} F(X-tY)\frac{dt}{t}\frac{\partial \phi_\nu}{\partial Y_1}(Y)\,dY \quad (**).
\end{aligned}$$

Let us verify equalities $(*)$ *and* $(**)$. Noting that F is even w.r. to the second coordinate, ϕ_ν is even w.r. to the first one and then $\frac{\partial \phi_\nu}{\partial Y_1}$ is odd w.r. to the first one, the change of variable $Y = (Y_1, Y_2) \longmapsto W = (\frac{x}{t} - Y_1, Y_2)$ gives

$$\begin{aligned}
&\int_{-\infty}^\infty\int_0^\infty F(tY)\phi_\nu\left(Y - \frac{X}{t}\right)dY_2\,dY_1 \\
&\quad = \int_{-\infty}^\infty\int_0^\infty F(tY_1, -tY_2)\phi_\nu\left(\frac{x}{t} - Y_1, Y_2\right)dY_2\,dY_1 \\
&\quad = \int_{-\infty}^\infty\int_0^\infty F(X-tW)\phi_\nu(W)\,dW,
\end{aligned}$$

and the change of variable $(Y_1, t) \to (-Y_1, -t)$ gives

$$\int_{-\infty}^{+\infty}\int_\varepsilon^{\frac{1}{\varepsilon}} F(X-tY)\frac{dt}{t}\frac{\partial \phi_\nu}{\partial Y_1}(Y)\,dY_1 = \int_{+\infty}^{-\infty}\int_{-\varepsilon}^{-\frac{1}{\varepsilon}} F(X-tY)\frac{dt}{t}\frac{\partial \phi_\nu}{\partial Y_1}(Y)\,dY_1.$$

Step 3: upper bound for $|R_\nu(f)(x)|$. We denote, for $\theta \in [0,\pi]$:

(i) $H_Y^* F(X) := \sup_{\varepsilon>0} \left| \int_{\varepsilon \le |t| \le \frac{1}{\varepsilon}} F(X - tY) \frac{dt}{t} \right|,$

(ii) $$\sup_{\varepsilon>0} \left| \int_{\varepsilon \le |t| \le \frac{1}{\varepsilon}} f(|x - te^{i\theta} y|) \frac{dt}{t} \right| = \sup_{\varepsilon>0} \left| \int_{\varepsilon \le |t| \le \frac{1}{\varepsilon}} f(|x - te^{i\theta}|) \frac{dt}{t} \right| = H_\theta^* f(x),$$

(iii) $\gamma_p := \|Y_1\|_{L^p(\frac{e^{-Y_1^2/4}}{2\sqrt{\pi}}\, dY_1)}$.

By definition, $\phi_\nu(Y)\, dY$ is a probability measure; since $\frac{e^{-Y_1^2/4}}{2\sqrt{\pi}}\, dY_1$ is also a probability measure, so is $2\sqrt{\pi}\frac{e^{-Y_2^2/4} Y_2^{2\nu-1}}{C_\nu \cdot I_\nu}\, dY_2$. So, step 2 and Hölder inequality imply, with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$
\begin{aligned}
(15)\qquad & 2\sqrt{\pi}|R_\nu(f)(x)| \\
& \le 2 \int_{\mathbb{R}\times\mathbb{R}^+} H_Y^* F(X) \left| \frac{\partial \phi_\nu}{\partial Y_1}(Y) \right| dY \quad \textit{by } (14) \\
& = \int_{\mathbb{R}\times\mathbb{R}^+} H_Y^* F(X)\ |Y_1| \phi_\nu(Y)\, dY \\
& \le \|Y_1\|_{L^{p'}(\frac{e^{-Y_1^2/4}}{2\sqrt{\pi}}\, dY_1)} \\
& \quad \times \left\| 2\sqrt{\pi} \int_0^\infty H_Y^* F(X) \times \frac{e^{-Y_2^2/4} Y_2^{2\nu-1}}{C_\nu \cdot I_\nu}\, dY_2 \right\|_{L^p(\frac{e^{-Y_1^2/4}}{2\sqrt{\pi}}\, dY_1)} \\
& \le \gamma_{p'} \|H_Y^* F(X)\|_{L^p(\phi_\nu(Y)\, dY)} \\
& = \gamma_{p'} \|1\|_{L^p(e^{-y^2/4}\frac{d\nu(y)}{I_\nu})} \|H_\theta^* f(x)\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\, d\theta}{C_\nu})} \\
& = \gamma_{p'} \|H_\theta^* f(x)\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\, d\theta}{C_\nu})}.
\end{aligned}
$$

Step 4: the method of rotation. Let $X, Y \in \mathbb{R}^2$ and let $N \in \mathbb{R}^2$ be such that: $|N| = 1$ and $(N, Y) = \frac{\pi}{2}$, so, denoting $|Y| = y$, we get $X = x\cos\theta\ \frac{Y}{y} + x\sin\theta\ N$, $\theta \in [0,\pi]$ and $|X - t\frac{Y}{y}| = |x - te^{i\theta}|$.

LEMMA 7. *Let* $p \in]1;\infty[$. *Then* (a) *for* $F \in S(\mathbb{R}^2)$ *and fixed* $Y \in \mathbb{R}^2$

$$
\begin{aligned}
& \left\| p.v. \frac{1}{\pi} \int_{\mathbb{R}} F(X - tY) \frac{dt}{t} \right\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\, d\theta}{C_\nu}\, d\nu(x))} \\
& \quad = \left\| p.v. \frac{1}{\pi} \int_{\mathbb{R}} F\left(X - t\frac{Y}{y} \right) \frac{dt}{t} \right\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\, d\theta}{C_\nu}\ \ d\nu(x))} \\
& \quad \le \|H\|_{p\to p}\ \|F\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\, d\theta}{C_\nu}\, d\nu(x))}.
\end{aligned}
$$

(b) *In particular if F is radial (i.e., $F(X)=f(x)$), we get:*

$$\text{(i)}\qquad \left\|p.v.\frac{1}{\pi}\int_{\mathbb{R}} f(|x-te^{i\theta}|)\frac{dt}{t}\right\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu}\,d\nu(x))} \leq \|H\|_{p\to p}\|f\|_{L^p(\,d\nu(x))}$$

$$\text{(ii)}\qquad \|H^*_\theta f(x)\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu}\,d\nu(x))} \leq \|H^*\|_{p\to p}\pi\|f\|_{L^p(\,d\nu(x))}. \tag{16}$$

Proof. We will use the method of rotation in $\mathbb{R}^2$ (see, for example, [3]). Denoting $X=x\cos\theta\ \frac{Y}{y}+x\sin\theta\ N=s\ \frac{Y}{y}+w\ N$, noting that $x\,dx\,d\theta=dw\,ds$, we get

$$\begin{aligned}
&\left\|p.v.\frac{1}{\pi}\int_{\mathbb{R}} F\left(X-t\frac{Y}{y}\right)\frac{dt}{t}\right\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu}\,d\nu(x))}\\
&\quad=\left\|p.v.\frac{1}{\pi}\int_{\mathbb{R}} F\left((s-t)\frac{Y}{y}+wN\right)\frac{dt}{t}\right\|_{L^p(\frac{w^{2\nu-1}\,dw}{C_\nu}\,ds)}\\
&\quad=\|\|H\varphi_w\|_{L^p(\,ds)}\|_{L^p(\frac{w^{2\nu-1}\,dw}{C_\nu})}\qquad \text{where } \varphi_w : s\longmapsto F\left(s\frac{Y}{y}+wN\right)\\
&\quad\leq \|H\|_{p\to p}\|\varphi_w\|_{L^p(\frac{w^{2\nu-1}\,dw}{C_\nu}\,ds)}\qquad \text{(Proposition 6(i))}\\
&\quad=\ \|H\|_{p\to p}\cdot\|F\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu}\,d\nu(x))}.
\end{aligned}$$

If F is radial, then $F(X-t\frac{Y}{y})=f(|x-te^{i\theta}|)$ and

$$\|F\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu}\,d\nu(x))}=\|f\|_{L^p(d\nu(x))}$$

if $\|1\|_{L^1(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu})}$ is defined, i.e., if and only if $\nu>0$.

Similarly, by using H^* and Proposition 6(ii), we prove (16). ☐

Step 5: conclusion. By steps 3 and 4 (see (16) and (15)),

$$\|R_\nu(f)\|_{L^p(\,d\nu(x))}\leq K_p\|f\|_{L^p(\,d\nu(x))}, \tag{17}$$

where $K_p\leq\gamma_{p'}\|H^*\|_{p\to p}\frac{\sqrt{\pi}}{2}$.

Step 6: The lower estimate. As usual, the lower estimate can be deduced from the upper one by duality. Indeed, since $R^*_\nu R_\nu=Id$, we get

$$\begin{aligned}
\|f\|_{L^p(\,d\nu(x))}&=\sup_{\|h\|_{L^{p'}(\,d\nu(x))}=1}\left\{\int_0^\infty (R^*_\nu R_\nu f)(x)\cdot h(x)\,d\nu(x)\right\}\\
&=\sup_{\|h\|_{L^{p'}(\,d\nu(x))}=1}\left\{\int_0^\infty (R_\nu f)(x)\cdot (R_\nu h)(x)\,d\nu(x)\right\}\\
&\leq K_{p'}\|R_\nu f\|_{L^p(\,d\nu(x))}\quad \text{by Hölder and (17).}
\end{aligned}$$

☐

4.2.

REMARK 2. Actually, in step 2 we have, $d\nu(x)$-ae.,

$$\sqrt{\pi}R_\nu(f)(x) = \int_0^\infty \int_{\mathbb{R}} \left[p.v. \int_{\mathbb{R}} F(X - tY)\frac{dt}{t} \right] \frac{\partial \phi_\nu}{\partial Y_1}(Y)\, dY.$$

We then get $K_p \leq \frac{\gamma_{p'} \|H\|_{p\to p}\sqrt{\pi}}{2}$.

Proof. It suffices to show that we may apply Lebesgue theorem at the end of step 2 (for almost every x). Hence, it suffices to show that

$$E(x) = \left\| H_Y^* F(X) \frac{\partial \phi_\nu}{\partial Y_1}(Y) \right\|_{L^1(\,dY)} < \infty, \quad d\nu(x)\text{-ae.}$$

Since

$$E(x) \leq \left\| H_Y^* F(X) e^{-\frac{Y_1^2+Y_2^2}{16}} Y_2^{\frac{2\nu-1}{2}} \right\|_{L^2(\,dY)} \left\| e^{-3\frac{Y_1^2+Y_2^2}{16}} Y_1 Y_2^{\frac{2\nu-1}{2}} \right\|_{L^2(\,dY)}$$

it suffices to show that $\|H_Y^* F(X)\|_{L^2(Y_2^{2\nu-1} e^{-(Y_1^2+Y_2^2)/8}\, dY)} < \infty$, $d\nu(x)$-ae. This in turn follows from step 3 and (16) since

$$\begin{aligned}
&\|H_Y^* F\|_{L^2(\frac{Y_2^{2\nu-1}}{C_\nu} e^{-(Y_1^2+Y_2^2)/8}\, dY \ \ d\nu(x))} \\
&\quad = \|1\|_{L^2(e^{-y^2/8}\, d\nu(y))} \|H_\theta^* f\|_{L^2(\frac{(\sin\theta)^{2\nu-1}\, d\theta}{C_\nu}\, d\nu(x))} \\
&\quad \leq \|1\|_{L^2(e^{-y^2/8}\, d\nu(y))} \|H^*\|_{2\to 2}\ \|f\|_{L^2(\,d\nu(x))}.
\end{aligned}$$

□

5. Weighted norm inequalities

5.1. Theorem 4.

THEOREM. *Let* $p \in]1;\infty[$ *and* $\nu > 0$*; then for every* $\alpha \in\ ]-2\nu; 2\nu(p-1)[$*, there is a constant* $K_{p,\nu,\alpha} > 0$ *such that*

$$K_{p',\nu,\alpha}^{-1} \|f\|_{L^p(\mathbb{R}^+, x^\alpha\, d\nu(x))} \leq \|R_\nu(f)\|_{L^p(\mathbb{R}^+, x^\alpha\, d\nu(x))} \leq K_{p,\nu,\alpha} \|f\|_{L^p(\mathbb{R}^+, x^\alpha\, d\nu(x))}.$$

Proof. We will proceed as in the proof of Theorem 3, except for step 3.

$$\begin{aligned}
&\sqrt{\pi}|R_\nu(f)(x)| \\
&\quad \leq \int_{\mathbb{R}\times\mathbb{R}^+} H_Y^* F(X) \left| \frac{\partial \phi_\nu}{\partial Y_1}(Y) \right| dY \quad \text{(step 3)} \\
&\quad = \|y\|_{L^1(e^{-y^2/4}\frac{d\nu(y)}{I_\nu})} \|H_\theta^* f(x) \cos\theta\|_{L^1(\frac{(\sin\theta)^{2\nu-1}\, d\theta}{C_\nu})} \quad (\theta \in [0,\pi]) \\
&\quad \leq \|y\|_{L^1(e^{-y^2/4}\frac{d\nu(y)}{I_\nu})} \|\cos\theta (\sin\theta)^{\frac{-\alpha}{p}}\|_{L^{p'}(\frac{(\sin\theta)^{2\nu-1}\, d\theta}{C_\nu})} \\
&\qquad \times \|H_\theta^* f(x) (\sin\theta)^{\frac{\alpha}{p}}\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\, d\theta}{C_\nu})} \quad \text{(Hölder)} \\
&\quad \leq \|y\|_{L^1(e^{-y^2/4}\frac{d\nu(y)}{I_\nu})} \||\cos\theta|^{\frac{1}{p'}} (\sin\theta)^{\frac{-\alpha}{p}}\|_{L^{p'}(\frac{(\sin\theta)^{2\nu-1}\, d\theta}{C_\nu})}
\end{aligned}$$

$$\times \|H^*_\theta f(x)(\sin\theta)^{\frac{\alpha}{p}}\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu})} \quad (\clubsuit)$$
$$= K'_{p,\nu,\alpha}\ \|H^*_\theta f(x)(\sin\theta)^{\frac{\alpha}{p}}\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu})}.$$

Let us note that $K'_{p,\nu,\alpha}$ is finite if and only if

$$\||\cos\theta|^{\frac{1}{p'}}(\sin\theta)^{\frac{-\alpha}{p}}\|_{L^{p'}(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu})} < \infty,$$

which holds if and only if $-\frac{\alpha p'}{p} + 2\nu - 1 > -1$ i.e. $\alpha < 2\nu(p-1)$. We then get

$$\begin{aligned}
&2\sqrt{\pi}\|R_\nu(f)\|_{L^p(x^\alpha\,d\nu(x))}\\
&\quad\le K'_{p,\nu,\alpha}\|H^*_\theta f(x)(\sin\theta)^{\frac{\alpha}{p}}\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu}x^\alpha\,d\nu(x))}\\
&\quad= K'_{p,\nu,\alpha}\left(\frac{C_{\nu+\alpha/2}}{C_\nu}\right)^{\frac{1}{p}}\|H^*_\theta f(x)\|_{L^p(\frac{(\sin\theta)^{2\nu-1+\alpha}\,d\theta}{C_{\nu+\alpha/2}}x^\alpha\,d\nu(x))}\\
&\quad\le K'_{p,\nu,\alpha}\ \|H^*\|_{p\to p}\pi\left(\frac{C_{\nu+\alpha/2}}{C_\nu}\right)^{\frac{1}{p}}\ \|f\|_{L^p(x^\alpha\,d\nu(x))}.
\end{aligned}$$

Indeed, in the last inequality, we apply Lemma 7 with $2\nu+\alpha$ instead of 2ν, which is allowed if and only if $C_{\nu+\frac{\alpha}{2}}$ is finite i.e., $\alpha > -2\nu$. Theorem 4 follows. □

Remark 3. Step ($\clubsuit$) is not necessary, but it will allow to estimate easily the constant $K_{p,\nu,\alpha}$ in the next paragraph.

Remark 4. When α is negative there is a simpler proof which gives a simpler constant $\widetilde{K}_{p,\nu,\alpha}$: indeed, modifying step 3 in the proof of Theorem 3

$$\begin{aligned}
&2\sqrt{\pi}\|R_\nu(f)\|_{L^p(x^\alpha\,d\nu(x))}\\
&\quad\le \gamma_{p'}\|H^*_\theta f(x)\|_{L^p(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu}x^\alpha\,d\nu(x))} \quad \text{(step 3)}\\
&\quad= \gamma_{p'}\|H^*_\theta f(x)(\sin\theta)^{-\frac{\alpha}{p}}\|_{L^p(\frac{(\sin\theta)^{2\nu-1+\alpha}\,d\theta}{C_\nu}x^\alpha\,d\nu(x))}\\
&\quad\le \gamma_{p'}\|H^*_\theta f(x)\|_{L^p(\frac{(\sin\theta)^{2\nu-1+\alpha}\,d\theta}{C_\nu}x^\alpha\,d\nu(x))} \quad \text{(because } -\alpha \text{ positive)}\\
&\quad\le \gamma_{p'}\|H^*\|_{p\to p}\pi\left(\frac{C_{\nu+\frac{\alpha}{2}}}{C_\nu}\right)^{\frac{1}{p}}\|f\|_{L^p(x^\alpha\,d\nu(x))} \quad \text{(Lemma 7 with } 2\nu+\alpha).
\end{aligned}$$

5.2. Application: the operator of [1]. In this article, the operator $\widetilde{S}_\nu$ is defined on $L^2(\mathbb{R}^+, dx)$ by

$$\widetilde{S}_\nu := -\frac{d^2}{dx^2} + \frac{\nu^2-\nu}{x^2} = A^*_\nu A_\nu, \quad \text{where } A_\nu = x^\nu\left[\frac{d}{dx}\right]x^{-\nu}.$$

Let $\theta_\nu : L^2(\mathbb{R}^+, dx) \longrightarrow L^2(\mathbb{R}^+, d\nu(x))$ be the multiplication by $x^{-\nu}$. Then

$$\widetilde{S}_\nu = \theta_\nu^{-1} S_\nu \theta_\nu. \tag{18}$$

So, Theorem 4 immediately implies the following theorem.

THEOREM 8. *Let $\widetilde{R}_\nu$ be the Riesz transform associated to $\widetilde{S}_\nu$, namely $\widetilde{R}_\nu := A_\nu \widetilde{S}_\nu^{-1/2}$. Then for every $p \in]1;+\infty[$, there exists a constant $K_{p,\nu} > 0$ such that*

$$K_{p',\nu}^{-1} \|f\|_{L^p(\mathbb{R}^+,dx)} \le \|\widetilde{R}_\nu(f)\|_{L^p(\mathbb{R}^+,\,dx)} \le K_{p,\nu}\|f\|_{L^p(\mathbb{R}^+,dx)}.$$

Proof. Theorem 4 with $\alpha = \nu(p-2) \in]-2\nu; 2\nu(p-1)[$ implies

$$\begin{aligned}
\|\widetilde{R}_\nu(f)\|_{L^p(\mathbb{R}^+,\,dx)} &= \|x^\nu R_\nu(x^{-\nu} f)\|_{L^p(\mathbb{R}^+,\,dx)} \quad \text{by (18)} \\
&= \|R_\nu(x^{-\nu} f)\|_{L^p(\mathbb{R}^+, x^{p\nu}\,dx)} \\
&\le K_{p,\nu,\nu(p-2)} \|x^{-\nu} f\|_{L^p(\mathbb{R}^+, x^{p\nu}\,dx)} \\
&= K_{p,\nu,\nu(p-2)} \|f\|_{L^p(\mathbb{R}^+,\,dx)}.
\end{aligned}$$

The left inequality follows from the right one as in step 6. □

5.3. Another version of Theorem 4. In Theorem 4, ν is fixed and the weight α varies. Taking the converse point of view, we will get Proposition 5.

Proof of Theorem 4. We have to estimate the constants appearing in Theorem 4.

The Gamma function. For every $x, y > 0$, let

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\,dt \quad \text{and} \quad B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt;$$

we then have

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma\left(x + \frac{1}{2}\right) \sim \sqrt{x}\,\Gamma(x), \quad x \to \infty \tag{19}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{20}$$

Computation of C_ν.

$$\begin{aligned}
C_\nu &= \int_0^\pi (\sin\theta)^{2\nu-1}\,d\theta \\
&= 2\int_0^1 (1-u^2)^{\nu-1}\,du \quad (u = \cos\theta) \\
&= \int_0^1 (1-t)^{\nu-1} \frac{dt}{\sqrt{t}} \quad (t = u^2) \\
&= B\left(\frac{1}{2}, \nu\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\nu)}{\Gamma(\nu + \frac{1}{2})} = \frac{\sqrt{\pi}\,\Gamma(\nu)}{\Gamma(\nu+\frac{1}{2})} \sim \sqrt{\frac{\pi}{\nu}} \quad \text{by (19) and (20).}
\end{aligned}$$

Estimations.

(i) $$\|y\|_{L^1(e^{-y^2/4}\frac{d\nu(y)}{I_\nu})} = \frac{1}{I_\nu}\int_0^\infty e^{-y^2/4} y^{2\nu+1}\,dy$$
$$= \frac{1}{I_\nu}\int_0^\infty e^{-s} 2^{2\nu+1} s^\nu\,ds \quad (s = y^2/4)$$
$$= \frac{2^{2\nu+1}\Gamma(\nu+1)}{I_\nu} = 2\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})} \quad \text{by (19) and (20).}$$

(ii) $$\left\| |\cos\theta|^{\frac{1}{p'}} (\sin\theta)^{\frac{-\alpha}{p}} \right\|^{p'}_{L^{p'}(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu})}$$
$$= \frac{1}{C_\nu}\int_0^\pi |\cos\theta|(\sin\theta)^{-\alpha\frac{p'}{p}+2\nu-1}\,d\theta$$
$$= \frac{2}{C_\nu}\int_0^1 u^{-\alpha\frac{p'}{p}+2\nu-1} du \quad (u = \sin\theta)$$
$$= \frac{2}{C_\nu}\frac{1}{\frac{-\alpha}{p-1}+2\nu}.$$

(iii) $$K_{p,\nu,\alpha}$$
$$= \|y\|_{L^1(e^{-y^2/4}\frac{d\nu(y)}{I_\nu})} \left\| |\cos\theta|^{\frac{1}{p'}} (\sin\theta)^{\frac{-\alpha}{p}} \right\|_{L^{p'}(\frac{(\sin\theta)^{2\nu-1}\,d\theta}{C_\nu})}$$
$$\times \|H^*\|_{p\to p}\ \frac{\sqrt{\pi}}{2}\left(\frac{C_{\nu+\alpha/2}}{C_\nu}\right)^{\frac{1}{p}}$$
$$= 2^{\frac{1}{p'}}\left(\sqrt{\pi}\right)^{\frac{1}{p}}\|H^*\|_{p\to p}\left(\frac{\nu}{\frac{-\alpha}{p-1}+2\nu}\right)^{\frac{1}{p'}}\left(\frac{\nu\Gamma(\nu+\frac{\alpha}{2})}{\Gamma(\nu+\frac{\alpha}{2}+\frac{1}{2})}\right)^{\frac{1}{p}}$$
(i) and (ii)
$$\sim 2^{\frac{1}{p'}}(\sqrt{\pi})^{\frac{1}{p}}\|H^*\|_{p\to p}\nu^{\frac{1}{2p}}, \quad \nu\to\infty \quad \text{by (19) and (20).}$$

(iv) $$\widetilde{K}_{p,\nu,\alpha} = \frac{\sqrt{\pi}}{2}\gamma_{p'}\|H^*\|_{p\to p}\left(\frac{C_{\nu+\frac{\alpha}{2}}}{C_\nu}\right)^{\frac{1}{p}}$$
$$\sim \frac{\sqrt{\pi}}{2}\gamma_{p'}\ \|H^*\|_{p\to p}, \quad \nu\to\infty \quad (\text{by the estimation of } C_\nu). \quad \square$$

6. One last remark

It can be useful to see $(e^{-t^2 S_\nu})_{t>0}$ as the compression of a one parameter group of isometries of an L^p space, more precisely: using the change of variable $Y \mapsto W$ as in step 2, (13) can be rewritten as

(21)
$$\exp(-t^2 S_\nu)(f)(x) = \frac{1}{C_\nu \cdot I_\nu}\int_0^\infty\int_0^\pi f(|x - te^{i\theta}y|)e^{-\frac{y^2}{4}}(\sin\theta)^{2\nu-1}\,d\theta\,d\nu(y).$$

Noting that $|(X_1 - ty, X_2)| = |x - te^{i\theta}y|$, we have

$$e^{-t^2 S_\nu}(f) = J_1^* J^* U_t J J_1(f),$$

where J_1 is the canonical embedding

$$J_1 : E_1 = L^p(\mathbb{R}^+, d\nu(x)) \hookrightarrow E_2 = L^p\left(\mathbb{R}^+ \times [0;\pi], d\nu(x) \otimes \frac{(\sin\theta)^{2\nu-1}}{C_\nu}\, d\theta\right),$$

E_2 is identified to $E_3 = L^p(\mathbb{R} \times \mathbb{R}^+, dX_1 \otimes \frac{X_2^{2\nu-1}}{C_\nu}\, dX_2)$ via polar coordinates $(x,\theta) \mapsto (X_1, X_2) = (x\cos\theta, x\sin\theta)$, J is the canonical embedding

$$J : E_3 \hookrightarrow E_4 = L^p\left(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, dX_1 \otimes \frac{X_2^{2\nu-1}}{C_\nu}\, dX_2 \otimes \frac{e^{-y^2/4}}{I_\nu}\, d\nu(y)\right),$$

and, for real t, $U_t : G(X_1, X_2, y) \mapsto G(X_1 - ty, X_2, y)$. Indeed, U_t is an isometry of E_4 and J_1^* (resp. J^*) is the integration w.r.t. to θ (resp. y).

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Michaël Villani, Département de Mathématiques, Université de Cergy-Pontoise Site de Saint Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France

E-mail address: mickvil@yahoo.fr