Cable algebras and rings of $\mathbb{G}_a$-invariants

Gene Freudenburg and Shigeru Kuroda

Abstract For a field k, the ring of invariants of an action of the unipotent k-group $\mathbb{G}_a$ on an affine k-variety is quasiaffine, but not generally affine. Cable algebras are introduced as a framework for studying these invariant rings. It is shown that the ring of invariants for the $\mathbb{G}_a$-action on $\mathbb{A}_k^5$ constructed by Daigle and Freudenburg is a monogenetic cable algebra. A generating cable is constructed for this ring, and a complete set of relations is given as a prime ideal in the infinite polynomial ring over k. In addition, it is shown that the ring of invariants for the well-known $\mathbb{G}_a$-action on $\mathbb{A}_k^7$ due to Roberts is a cable algebra.

1. Introduction

We introduce cable algebras to describe the structure of rings of invariants for algebraic actions of the unipotent group $\mathbb{G}_a$ on affine varieties over a ground field k. Winkelmann [14] has shown that such rings are always quasiaffine over k, but they are not generally affine. Roberts [12] gave the first example of a nonaffine invariant ring for a $\mathbb{G}_a$-action on an affine space. Specifically, Roberts's example involved an action of $\mathbb{G}_a$ on the affine space $\mathbb{A}_k^7$, where k is of characteristic zero. Subsequent examples of $\mathbb{G}_a$-actions of nonfinite type were constructed by Freudenburg [4] and by Daigle and Freudenburg [2], for $\mathbb{A}_k^6$ and $\mathbb{A}_k^5$, respectively. These examples are counterexamples to Hilbert's fourteenth problem.

Kuroda [8] used subalgebra analogue to Groebner bases for ideals (SAGBI) basis techniques to show that an infinite system of invariants constructed by Roberts for the action on $\mathbb{A}_k^7$ generates the invariant ring as a k-algebra. Tanimoto [13] used the same techniques to identify generating sets for the actions on $\mathbb{A}_k^6$ and $\mathbb{A}_k^5$. Our results show that Tanimoto's generating sets are not minimal (see Section 9.1). From the point of view of classical invariant theory, a structural description of a ring of invariants involves the determination of a minimal set of generators of the ring as a k-algebra, together with a minimal set of generators for the ideal of their relations. However, for an infinite set of generators, or even a large finite set of generators, such a description can be complicated, and the choice of generating set can seem arbitrary.

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When k is of characteristic zero, $\mathbb{G}_a$-actions on an affine k-variety X are equivalent to locally nilpotent derivations of the coordinate ring $k[X]$, and the invariant ring $k[X]^{\mathbb{G}_a}$ equals the kernel of the derivation. In many cases, $k[X]^{\mathbb{G}_a}$ admits a nonzero locally nilpotent derivation, and this gives additional structure to exploit.

For a commutative k-domain B, a locally nilpotent derivation D of B induces a directed tree structure on B. A *D-cable* is any complete linear subtree rooted in the kernel of D. The condition for B to be a cable algebra is a finiteness condition: B is a *cable algebra* if (for some D) $D \neq 0$ and B is generated by a finite number of D-cables over the kernel of D. Then B is a *simple cable algebra* if it is generated by one D-cable over k. Elements in the ideal of relations in the infinite polynomial ring for the generating cables are *cable relations*.

To illustrate this, consider a nilpotent linear operator N on a finite-dimensional k-vector space V. Choose a basis $\{x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n_i\}$ of V so that the effect of N for fixed i is

$$x_{i,n_i} \to x_{i,n_i-1} \to \cdots \to x_{i,2} \to x_{i,1} \to 0.$$

This defines the Jordan form of N, which in turn gives a cable structure on the symmetric algebra $S(V)$. In particular, N induces a locally nilpotent derivation D on $S(V)$, and each sequence $x_{i,j}$ for fixed i is a D-cable $\hat{x}_i$, where $S(V) = k[\hat{x}_1, \ldots, \hat{x}_m]$. In this sense, the cable algebra structure induced by a locally nilpotent derivation can be viewed as a generalization of Jordan block form for a nilpotent linear operator.

For rings of nonfinite type over k, the ring $S = k[x, xv, xv^2, \ldots]$ is a prototype, where $k[x, v]$ is the polynomial ring in two variables over k. The partial derivative $\partial/\partial v$ restricts to a locally nilpotent derivation D of S, and the infinite sequence $\frac{1}{n!}xv^n$ defines a D-cable $\hat{s}$ for which $S = k[\hat{s}]$. So S is a simple cable algebra. Although S is not quasiaffine, it plays an important role in our investigation. For example, one of our main objects of interest is the ring A of invariants for the $\mathbb{G}_a$-action on $\mathbb{A}^5$ constructed by Daigle and Freudenburg, and we show that A admits a mapping onto S.

1.1. Description of main results

We assume throughout that k is a field of characteristic zero. On the polynomial ring $B = k[a, v, x, y, z] = k^{[5]}$, define the locally nilpotent derivation D of B by

$$D = a^3 \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} + a^2 \frac{\partial}{\partial v}.$$

For the corresponding $\mathbb{G}_a$-action on $X = \mathbb{A}^5_k$, the ring of invariants $k[X]^{\mathbb{G}_a}$ is not finitely generated over k (see [2]).

If $A = \ker D$, the kernel of D, then the partial derivative $\frac{\partial}{\partial v}$ restricts to A, and ∂ denotes the restriction of $\frac{\partial}{\partial v}$ to A. We give a complete description of the ring A as a cable algebra relative to ∂, including its relations as a cable ideal in the infinite polynomial ring $\Omega = k[x_0, x_1, x_2, \ldots]$. Moreover, we construct a specific ∂-cable $\hat{\sigma} = (\sigma_n)$ from these relations, wherein σ_{n+1} is expressed as an explicit rational function in $\sigma_0, \dots, \sigma_n$. Our proofs do not use SAGBI bases, relying instead on properties of the down operator Δ on Ω, a k-derivation defined by

$$\Delta x_i = x_{i-1} \quad (i \geq 1) \qquad \text{and} \qquad \Delta x_0 = 0.$$

Let $\Omega[t] = \Omega^{[1]}$, and extend Δ to $\tilde{\Delta}$ on $\Omega[t]$ by $\tilde{\Delta} t = 0$.

Generators. Theorem 5.1: There exists an infinite homogeneous ∂-cable $\hat{s}$ rooted at a, and for any such ∂-cable we have $A = k[h, \hat{s}]$ for $h \in \ker \partial$. Moreover, this is a minimal generating set for A over k.

Relations. Theorem 7.1: There exists an ideal $\mathcal{I} = (\hat{\Theta}_4, \hat{\Theta}_6, \hat{\Theta}_8, \dots)$ in $\Omega[t]$ generated by quadratic homogeneous $\tilde{\Delta}$-cables $\hat{\Theta}_n$ such that $A \cong \Omega[t]/\mathcal{I}$.

Constructs. Theorem 7.6: Let A_a be the localization of A at a, and define a sequence $\sigma_n \in A_a$ by $\sigma_0 = a$ and

$$\sigma_1 = av - x, \qquad \sigma_2 = \frac{1}{2}(av^2 - 2xv + 2a^2 y),$$

$$\sigma_3 = \frac{1}{6}(av^3 - 3xv^2 + 6a^2 yv - 6a^4 z).$$

Given $n \geq 4$, let $e \geq 1$ be such that $-2 \leq n - 6e \leq 3$. If $\sigma_0, \dots, \sigma_{n-1} \in A_a$ are known, define $\sigma_n \in A_a$ implicitly as follows.

(i) If $n = 6e - 2$ or $n = 6e + 2$, then $\sum_{i=0}^{n} (-1)^i \sigma_i \sigma_{n-i} = 0$.
(ii) If $n = 6e - 1$ or $n = 6e + 3$, then $\sum_{i=1}^{n} (-1)^i i \sigma_i \sigma_{n-i} = 0$.
(iii) If $n = 6e$, then $\sum_{i=0}^{n+2} (-1)^i (3i(i-1) - n(n+2)) \sigma_i \sigma_{n+2-i} = 0$.
(iv) If $n = 6e+1$, then $\sum_{i=1}^{n+3} (-1)^{i+1} ((i-1)(i-2) - n(n+2)) i \sigma_i \sigma_{n+3-i} = 0$.

Then $\sigma_n \in A$ for each $n \geq 0$ and $\hat{\sigma} = (\sigma_n)$ is a ∂-cable rooted at a.

As seen in these results, quadratic relations in Ω are especially important. A basis for the vector space of quadratic forms in $\ker \Delta$ is given by $\{\theta_n^{(0)} \mid n \in 2\mathbb{N}\}$, where

$$\theta_n^{(0)} = \sum_{i=0}^{n} (-1)^i x_i x_{n-i}.$$

If $\{\hat{\theta}_n\}$ is any system of quadratic Δ-cables with $\hat{\theta}_n$ rooted at $\theta_n^{(0)}$, then the vertices of these cables form a basis for Ω_2, the space of quadratic forms in Ω (see Lemma 3.8). Moreover, the quadratic ideals

$$\mathcal{Q}_n = (\hat{\theta}_n, \hat{\theta}_{n+2}, \hat{\theta}_{n+4}, \dots), \quad n \in 2\mathbb{N},$$

are independent of the system of cables chosen (see Theorem 3.12). These ideals, called *fundamental Q-ideals*, are intrinsically important to the theory at hand. Compare this to the linear case. The only linear form in $\ker \Delta$ is x_0, up to a constant, and if $\hat{L} = (L_n)$ is any homogeneous Δ-cable rooted at x_0, then the linear forms L_n, $n \geq 0$, form a basis of the space of linear forms Ω_1 and we have equality of Ω-ideals:

$$(\hat{L}) = (L_0, L_1, L_2, \dots) = (x_0, x_1, x_2, \dots).$$

Therefore, $\Omega/(\hat{L}) = k$ and $(\hat{L})$ is a maximal ideal of Ω.

We show the following. We have that $\mathcal{Q}_2$ is a prime ideal of Ω and $\Omega/\mathcal{Q}_2 \cong_k S$, where $S \subset k[x,v] = k^{[2]}$ is the simple cable algebra of nonfinite type and of transcendence degree 2 over k defined by $S = k[x, xv, xv^2, \ldots]$ (see Theorem 3.21). We have that $\mathcal{Q}_4$ is a prime ideal of Ω and $\Omega/\mathcal{Q}_4 \cong_k A/hA$, which is a simple cable algebra of nonfinite type and of transcendence degree 3 over k (see Theorem 6.1).

Finally, we show that the ring of invariants for the Roberts action in dimension 7 is a cable algebra. On the polynomial ring $k[X,Y,Z,S,T,U,V]$, define the locally nilpotent derivation

$$\mathcal{D}_2 = X^3 \frac{\partial}{\partial S} + Y^3 \frac{\partial}{\partial T} + Z^3 \frac{\partial}{\partial U} + (XYZ)^2 \frac{\partial}{\partial V},$$

where $\mathcal{D}_2$ commutes with the 3-cycle α defined by $\alpha(X,Y,Z,S,T,U,V) = (Z,X,Y,U,S,T,V)$. The partial derivative $\partial/\partial V$ restricts to the kernel $\mathcal{A}_2$ of $\mathcal{D}_2$, and δ_2 denotes the restricted derivation. There exists a δ_2-cable $\hat{P}$ in $\mathcal{A}_2$ rooted at X, and for any such δ_2-cable,

$$\mathcal{A}_2 = k[H_2, \alpha H_2, \alpha^2 H_2, \hat{P}, \alpha\hat{P}, \alpha^2\hat{P}],$$

where $H_2 \in \ker\delta_2$ (see Theorem 8.2).

1.2. Additional background

Let K be any field. For $n \leq 3$, the ring of invariants of a $\mathbb{G}_a$-action on $\mathbb{A}^n_K$ is of finite type, due to a fundamental theorem of Zariski. It is not known if the ring of invariants of a $\mathbb{G}_a$-action on $\mathbb{A}^4_K$ is always of finite type (see Section 9.4). According to the classical Mauer–Weitzenböck theorem, if the characteristic of K is zero, then $K[\mathbb{A}^n_K]^{\mathbb{G}_a}$ is of finite type when $\mathbb{G}_a$ acts on $\mathbb{A}^n_K$ by linear transformations. However, it is not known if this is true for all fields. To date, there is no known example of a field K of positive characteristic and a $\mathbb{G}_a$-action on $\mathbb{A}^n_K$ for which $K[\mathbb{A}^n_K]^{\mathbb{G}_a}$ is of nonfinite type.

2. Locally nilpotent derivations

Let k be a field of characteristic zero, and let B be a commutative k-domain. A *locally nilpotent derivation* of B is a derivation $D: B \to B$ such that, for each $b \in B$, there exists $n \in \mathbb{N}$ (depending on b) such that $D^n b = 0$. Let $\ker D$ denote the kernel of D. The set of locally nilpotent derivations of B is denoted by $\mathrm{LND}(B)$. Note that $k \subset \ker D$ for any $D \in \mathrm{LND}(B)$ (cf. [5, Principle 1]).

It is well known that the study of $\mathbb{G}_a$-actions on an affine k-variety X is equivalent to the study of locally nilpotent derivations on the corresponding coordinate ring $k[X]$. In particular, the action induced by $D \in \mathrm{LND}(B)$ is given by the exponential map $\exp(tD)$, $t \in \mathbb{G}_a$, and $k[X]^{\mathbb{G}_a} = \ker D$.

In this section, we give some of the basic properties for rings with locally nilpotent derivations. The reader is referred to [5] for further details on the subject.

2.1. Basic definitions and properties

Given $D \in \mathrm{LND}(B)$, if $A = \ker D$, then A is filtered by the *image ideals*

$$I_n := A \cap D^n B \quad (n \geq 0) \qquad \text{and} \qquad I_\infty := \bigcap_{n \geq 0} I_n.$$

Note that $I_0 = A$ and $I_{n+1} \subset I_n$ for $n \geq 0$. We call I_1 the *plinth ideal* for D, and we call I_∞ the *core ideal* for D.

A *slice* for D is any $s \in B$ such that $Ds = 1$. Note that D has a slice if and only if $D : B \to B$ is surjective.

A *local slice* for D is any $s \in B$ such that $D^2 s = 0$ but $Ds \neq 0$. For a local slice $s \in B$ of D, let B_{Ds} and A_{Ds} denote the localizations of B and A at Ds, respectively. Then $B_{Ds} = A_{Ds}[s]$, where s is transcendental over A_{Ds}. Given $b \in B$, $\deg_D b$ is the degree of b as a polynomial in s, which is independent of the choice of local slice s. The corresponding *Dixmier map* $\pi_s : B_{Ds} \to A_{Ds}$ is the algebra map defined by

$$\pi_s(f) = \sum_{i \geq 0} \frac{(-1)^i}{i!} D^i f \cdot \left(\frac{s}{Ds}\right)^i \quad \text{for all } f \in B_{Ds}.$$

If E is any k-derivation of B which commutes with D, then it is immediate from this definition that

$$E\pi_s(f) = \pi_s(Ef) - \pi_s(Df)E(s/Ds) \quad \text{for all } f \in B_{Ds}. \tag{1}$$

Let $S \subset B$ be a nonempty subset, and let $k \subset R \subset A$ be a subring. Define the subring

$$R[S, D] = R[D^i s \mid s \in S, i \geq 0].$$

Note that D restricts to $R[S, D]$, and note that $R[S, D]$ is the smallest subring of B containing R and S to which D restricts.

2.2. The down operator

Let $\Omega = k[x_0, x_1, x_2, \ldots]$ be the infinite polynomial ring, and let Ω_+ be the ideal of Ω defined by

$$\Omega_+ = \sum_{n \geq 0} x_n \cdot \Omega.$$

Let $\Delta \in \mathrm{LND}(\Omega)$ denote the *down operator* on Ω

$$\Delta x_n = x_{n-1} \quad (n \geq 1) \qquad \text{and} \qquad \Delta x_0 = 0.$$

Then $\Delta : \Omega_+ \to \Omega_+$ is surjective (see [6, Theorem 3.1]).

The ring Ω has a $\mathbb{Z}^2$-grading defined by $\deg x_i = (1, i)$, where each x_i is homogeneous ($i \geq 0$). For this grading, Δ is homogeneous and $\deg \Delta = (0, -1)$. Given $r, s \geq 0$, let $\Omega_{(r,s)}$ denote the vector space of homogeneous elements of Ω of degree (r, s), and let $\Omega_r = \sum_s \Omega_{(r,s)}$. Then $\Delta : \Omega_{(r,s)} \to \Omega_{(r,s-1)}$ is surjective for each $r, s \geq 1$.

2.3. Tree structure induced by an LND

Let B be a commutative k-domain. To any $D \in \mathrm{LND}(B)$ we associate the rooted tree $\mathrm{Tr}(B,D)$ whose vertex set is B and whose (directed) edge set consists of pairs (f, Df), where $f \neq 0$. Equivalently, $\mathrm{Tr}(B,D)$ is the tree defined by the partial order on B defined by $a \leq b$ if and only if $D^n b = a$ for some $n \geq 0$.

Let $A = \ker D$.

(i) Given $a, b \in B$ with $b \neq 0$, b is a *predecessor* of a if and only if a is a *successor* of b if and only if $a < b$. Similarly, b is an *immediate predecessor* of a if and only if a is an *immediate successor* of b if and only if $Db = a$.

(ii) The *terminal vertices* of $\mathrm{Tr}(B,D)$ are those without predecessors, that is, elements of $B \setminus DB$. If D has a slice, that is, $DB = B$, then $\mathrm{Tr}(B,D)$ has no terminal vertices.

(iii) Every subtree X of $\mathrm{Tr}(B,D)$ has a unique root, denoted $\mathrm{rt}(X)$.

(iv) A subtree X of $\mathrm{Tr}(B,D)$ is *complete* if every vertex of X which is not terminal in $\mathrm{Tr}(B,D)$ has at least one predecessor in X.

(v) A subtree X of $\mathrm{Tr}(B,D)$ is *linear* if every vertex of X has at most one immediate predecessor in X.

(vi) If B is graded by an abelian group, then any homogeneous $b \in B$ is a *homogeneous vertex* of $\mathrm{Tr}(B,D)$. A subtree X of $\mathrm{Tr}(B,D)$ is *homogeneous* if every $b \in \mathrm{vert}(X)$ is homogeneous. If D is homogeneous, then the *full homogeneous subtree* is the subtree of $\mathrm{Tr}(B,D)$ spanned by the homogeneous vertices.

3. Cables and cable algebras

3.1. D-cables

DEFINITION 3.1

Let B be a commutative k-domain, and let $D \in \mathrm{LND}(B)$. A *D-cable* is a complete linear subtree $\hat{s}$ of $\mathrm{Tr}(B,D)$ rooted at a nonzero element of $\ker D$. Then $\hat{s}$ is a *terminal* D-cable if it contains a terminal vertex, and $\hat{s}$ is an *infinite* D-cable if it is not terminal.

We make several remarks and further definitions, assuming that B is a commutative k-domain, $D \in \mathrm{LND}(B)$, $I_n = \ker D \cap D^n B$ $(n \geq 0)$, and $I_\infty = \bigcap_{n\geq 0} I_n$.

(i) If $\hat{s}$ is a D-cable, then $\hat{s}$ is terminal if and only if its vertex set is finite, and $\hat{s}$ is infinite if and only if $\hat{s} \subset DB$.

(ii) A D-cable is denoted by $\hat{s} = (s_j)$, where $s_j \in B$ for $j \geq 0$ and $Ds_j = s_{j-1}$ for $j \geq 1$. It is rooted at $s_0 \in \ker D$, which is nonzero. For multiple D-cables $\hat{s}_1, \ldots, \hat{s}_n$, we will write $\hat{s}_i = (s_i^{(j)})$ for $1 \leq i \leq n$ and $j \geq 0$.

(iii) The *length* of a D-cable $\hat{s}$ is the number of its edges (possibly infinite), denoted $\mathrm{length}(\hat{s})$. If $\hat{s} = (s_n)$ and $N = \mathrm{length}(\hat{s})$, then $s_0 \in I_N$, and if $\hat{s}$ is terminal, then s_N is its terminal vertex.

(iv) Every $b \in \ker D \setminus DB$ is a terminal vertex of $\mathrm{Tr}(B,D)$ and defines a terminal D-cable of length zero.

(v) If B is graded by an abelian group, then a D-cable is *homogeneous* if it is a homogeneous subtree of $\mathrm{Tr}(B,D)$.

(vi) Every nonzero vertex $b\in B$ belongs to a D-cable. If two D-cables $\hat{s}=(s_n)$ and $\hat{t}=(t_n)$ have $s_m=t_n$ for some $m,n\geq 0$, then $m=n$ and $s_i=t_i$ for all $i\leq m$. If $\hat{s}$ and $\hat{t}$ share an infinite number of vertices, then $\hat{s}=\hat{t}$.

(vii) Suppose that $B'\subset B$ is a subset with $DB'\subset B'$. If $\hat{s}\subset B$ is a D-cable such that either $\hat{s}\cap B'$ is infinite, or $\hat{s}$ is terminal of length N and $s_N\in B'$, then $\hat{s}\subset B'$.

(viii) If $P\in\Omega$ is a polynomial in $x_0,\dots,x_n$ and $\hat{s}$ is a D-cable of length at least n, then $P(\hat{s})$ means $P(s_0,\dots,s_n)$.

(ix) Given D-cables $\hat{s}_1,\dots,\hat{s}_n$ for $n\geq 0$, the notation $k[\hat{s}_1,\dots,\hat{s}_n]$ (resp., $(\hat{s}_1,\dots,\hat{s}_n)$) indicates the k-subalgebra of B (resp., ideal of B) generated by the vertices of $\hat{s}_i$ for $1\leq i\leq n$.

(x) Let $\hat{s}=(s_n)$ be a D-cable of length N. If $\hat{s}$ is terminal, define the map $\phi_{\hat{s}}:k^{[N+1]}\to B$ by $\phi_{\hat{s}}(x_i)=s_i$ for $0\leq i\leq N$. If $\hat{s}$ is infinite, define $\phi_{\hat{s}}:\Omega\to B$ by $\phi_{\hat{s}}(x_i)=s_i$ for all $i\geq 0$. Elements of $\ker\phi_{\hat{s}}$ are the *cable relations* associated to $\hat{s}$. Note that $D\phi_{\hat{s}}=\phi_{\hat{s}}\Delta$ where Δ is the down operator on Ω or its restriction to $k^{[N+1]}$.

(xi) Extend D to a derivation D^* on $B[t]=B^{[1]}$ by $D^*t=0$. If $\hat{s}(t)=(s_n(t))$ is a D^*-cable and $\alpha\in\ker D$ is such that $s_0(\alpha)\neq 0$, then $\hat{s}(\alpha)=(s_n(\alpha))$ is a D-cable rooted at $s_0(\alpha)$.

EXAMPLE 3.2

Let $\Omega=k[x_0,x_1,x_2,\dots]$ be the infinite polynomial ring, and let $\Delta\in\mathrm{LND}(\Omega)$ be the down operator. Then $\hat{x}=(x_j)_{j\geq 0}$ is an infinite Δ-cable, $x_0\in I_\infty$, and $\Omega=k[\hat{x}]$. Relabel the variables x_i by $y_n^{(j)}$ so that $\Omega=k[x_0,y_n^{(j)}\mid n\geq 1,1\leq j\leq n]$. Define $\tilde{\Delta}\in\mathrm{LND}(\Omega)$ so that, for $n\geq 1$,

$$\tilde{\Delta}:y_n^{(n)}\to y_n^{(n-1)}\to\cdots\to y_n^{(1)}\to y_n^{(0)}:=x_0\to 0.$$

Then $\hat{y}_n:=(y_n^{(j)})_{0\leq j\leq n}$ is a terminal $\tilde{\Delta}$-cable rooted at x_0 of length n for each $n\geq 1$. If $\tilde{I}_\infty$ is the core ideal for $\tilde{\Delta}$, then $x_0\in\tilde{I}_\infty$ but there is no infinite $\tilde{\Delta}$-cable rooted at x_0, since otherwise there would exist a homogeneous infinite $\tilde{\Delta}$-cable rooted at x_0. It is easy to check that this is not the case.

Note that an infinite D-cable $\hat{s}$ has $\hat{s}\subset DB$ and DB is an A-module, where $A=\ker D$. Therefore, addition and A-multiplication of infinite D-cables can be defined in certain situations, as described in the next result, which follows immediately from the definitions.

LEMMA 3.3

Let B be a commutative k-domain, let $D\in\mathrm{LND}(B)$, and let $A=\ker D$.

(a) *If $\hat{s}=(s_n)$ is an infinite D-cable and $a\in A$ is nonzero, then $a\hat{s}:=(as_n)$ is an infinite D-cable.*

(b) *If* $\hat{s} = (s_n)$ *and* $\hat{t} = (t_n)$ *are infinite* D*-cables and* $s_0 + t_0 \neq 0$*, then* $\hat{s} + \hat{t} := (s_n + t_n)$ *is an infinite* D*-cable.*

(c) *If* $\hat{s} = (s_n)$ *and* $\hat{t} = (t_n)$ *are infinite* D*-cables and* $m \in \mathbb{Z}$ *has* $m \geq 1$*, define the sequence* $u_n \in B$ *by* $u_n = s_n$ *if* $n < m$ *and* $u_n = s_n + t_{n-m}$ *if* $n \geq m$*. Then* $\hat{u} := (u_n)$ *is an infinite* D*-cable.*

DEFINITION 3.4

The D-cable $\hat{u}$ in Lemma 3.3(c) is called the *m-shifted sum* of $\hat{s}$ and $\hat{t}$, and is denoted by $\hat{u} = \hat{s} +_m \hat{t}$.

DEFINITION 3.5

Let $I \subset \mathbb{N}$ be either $\mathbb{N} \setminus \{0\}$ or $\{1, 2, \ldots, t\}$ for some integer $t \geq 1$. Suppose that a sequence $\vec{s} = \{\hat{s}_i\}_{i \in I}$ of infinite D-cables is given, together with a strictly increasing sequence $\vec{m} = \{m_i\}_{i \in I}$ of positive integers and a sequence $\vec{c} = \{c_i\}_{i \in I}$ with $c_i \in \ker D \setminus \{0\}$ for all $i \in I$. Define a sequence of D-cables $\hat{u}_i$ rooted at $s_1^{(0)}$ inductively by

$$\hat{u}_1 = \hat{s}_1 \qquad \text{and} \qquad \hat{u}_{i+1} = \hat{u}_i +_{m_i} c_i \hat{s}_{i+1} \quad \text{for } i \in I.$$

Note that if $\hat{u}_i = (u_i^{(j)})$, then given $j \geq 0$, there exist $u^{(j)} \in B$ and an integer N_j such that $u_i^{(j)} = u^{(j)}$ for all $i \in I$ with $i \geq N_j$. The D-cable $\hat{u} := (u^{(j)})$ so obtained is rooted at $s_1^{(0)}$ and is denoted by

$$\hat{u} = \lim(\vec{s}, \vec{m}, \vec{c}).$$

Note that, in this definition, we have $\hat{u} = \hat{u}_t$ when $I = \{1, 2, \ldots, t\}$.

EXAMPLE 3.6

Let B be a commutative k-domain, and let $D \in \mathrm{LND}(B)$. Given nonzero $f \in \ker D$, let $\exp(fD) : B \to B$ be the corresponding exponential automorphism of B. If $\hat{s} = (s_n)$ is a D-cable, then

$$D \exp(fD)(s_n) = \exp(fD)(s_{n-1}) \quad \text{for } n \geq 1.$$

Note that $\exp(fD)(s_0) = s_0$, and note that $s_i \in DB$ if and only if $\exp(fD)(s_i) \in DB$. Therefore, $\exp(fD)(\hat{s}) := (\exp(fD)(s_n))$ defines a D-cable rooted at s_0. If $\hat{s}$ is infinite, then it is given by

$$\exp(fD)(\hat{s}) = \lim(\vec{s}, \vec{m}, \vec{c}),$$

$$\text{where } \vec{s} = (\hat{s}, \hat{s}, \hat{s}, \ldots), \vec{m} = (1, 2, 3, \ldots) \text{ and } \vec{c} = \left(f, \frac{1}{2!} f^2, \frac{1}{3!} f^3, \ldots\right).$$

3.2. Quadratic Δ-cables

Note that we can view the vector space Ω_1 as being generated by the vertices of the Δ-cable $\hat{x} = (x_n)$. Similarly, Ω_2 admits a basis of homogeneous Δ-cables.

3.2.1. Monomial basis

Given $n \geq 0$, the *monomial basis* for $\Omega_{(2,n)}$ is

$$\{x_0x_n, x_1x_{n-1}, \ldots, x_{\frac{n}{2}}^2\} \quad (n \text{ even})$$

or

$$\{x_0x_n, x_1x_{n-1}, \ldots, x_{\frac{n-1}{2}}x_{\frac{n+1}{2}}\} \quad (n \text{ odd}).$$

Therefore, $\dim \Omega_{(2,n)}$ equals $(n+2)/2$ if n is even or $(n+1)/2$ if n is odd.

3.2.2. Δ-basis

Given $n \in 2\mathbb{N}$, define $\theta_n^{(0)} \in \Omega_{(2,n)} \cap \ker \Delta$ by

$$\theta_n^{(0)} = \sum_{0 \leq i \leq n} (-1)^i x_i x_{n-i}.$$

Note that, since n is even, $\theta_n^{(0)} \neq 0$. Since $\Delta : \Omega_{(2,s+1)} \to \Omega_{(2,s)}$ is surjective for all $s \geq 0$, there exists a homogeneous Δ-cable $\hat{\theta}_n = (\theta_n^{(j)})$ rooted at $\theta_n^{(0)}$. Note that $\hat{\theta}_n$ is necessarily infinite. By definition, we have $\theta_n^{(j)} \in \Omega_{(2,n+j)}$ for each $j \geq 0$. By Section 3.2.1, $\ker \Delta \cap \Omega_{(2,s)}$ equals $\{0\}$ if s is odd, and it equals $k \cdot \theta_s^{(0)}$ if s is even (cf. [6, Corollary 3.3]). Therefore, $\Delta : \Omega_{(2,n+1)} \to \Omega_{(2,n)}$ is an isomorphism. It follows that if $\hat{\theta}_n = (\theta_n^{(j)})$ is any homogeneous Δ-cable rooted at $\theta_n^{(0)}$, then $\theta_n^{(1)}$ is uniquely determined. It is given by

$$\theta_n^{(1)} = \sum_{i=1}^{n+1} (-1)^{i+1} i x_i x_{n+1-i}.$$

DEFINITION 3.7

A Δ-*basis* for Ω_2 is any set $\{\hat{\theta}_n \mid n \in 2\mathbb{N}\}$ of homogeneous Δ-cables such that, given $n \in 2\mathbb{N}$, $\hat{\theta}_n$ is rooted at $\hat{\theta}_n^{(0)}$.

LEMMA 3.8

Let $\{\hat{\theta}_n \mid n \in 2\mathbb{N}\}$ be a Δ-basis for Ω_2.

(a) *Given $j \geq 0$, the set $\{\theta_{2i}^{(j-2i)} \mid 0 \leq i \leq j/2\}$ is a basis for $\Omega_{(2,j)}$.*
(b) *The vertices of $\hat{\theta}_n$ $(n \in 2\mathbb{N})$ form a basis for Ω_2.*

Proof

To prove part (a), we proceed by induction on $j \geq 0$. We have that

$$\Omega_{(2,0)} = \langle x_0^2 \rangle = \langle \theta_0^{(0)} \rangle.$$

So the statement of part (a) holds if $j = 0$.

Assume that, for $j \geq 1$, the set $\{\theta_{2i}^{(j-1-2i)} \mid 0 \leq i \leq (j-1)/2\}$ forms a basis for $\Omega_{(2,j-1)}$. If j is odd, then $\Delta : \Omega_{(2,j)} \to \Omega_{(2,j-1)}$ is an isomorphism, and the set $\{\theta_{2i}^{(j-2i)} \mid 0 \leq i \leq j/2\}$ is a basis for $\Omega_{(2,j)}$. If j is even, then the kernel of

$\Delta : \Omega_{(2,j)} \to \Omega_{(2,j-1)}$ is $k \cdot \theta_j^{(0)}$, and again we conclude that $\{\theta_{2i}^{(j-2i)} \mid 0 \le i \le j/2\}$ is a basis for $\Omega_{(2,j)}$. This proves part (a).

Part (b) is an immediate consequence of part (a). □

3.2.3. Balanced Δ-basis

We define a particular Δ-basis for Ω_2 by using binomial coefficients $\binom{i}{j}$. Given $n \in 2\mathbb{N}$ and $j \in \mathbb{N}$, define $\beta_n^{(j)} \in \Omega_{(2,n+j)}$ by

$$\beta_n^{(j)} = \sum_{i=j}^{n+j} (-1)^{j+i} \binom{i}{j} x_i x_{n+j-i}.$$

Note that $\beta_n^{(0)} = \theta_n^{(0)}$.

LEMMA 3.9

If $n \in 2\mathbb{N}$ and $j \ge 1$, then $\Delta \beta_n^{(j)} = \beta_n^{(j-1)}$.

Proof

If $n \ge 1$ and $c_0, \ldots, c_n \in k$, then

$$\Delta \sum_{i=0}^{n} c_i x_i x_{n-i} = \sum_{i=0}^{n-1} (c_{i+1} + c_i) x_i x_{n-1-i}. \tag{2}$$

Given $i \in \mathbb{N}$ with $0 \le i < j$, we extend the definition of binomial coefficient by setting $\binom{i}{j} = 0$. Then for all $i, j \in \mathbb{N}$ we have

$$\binom{i}{j} + \binom{i}{j-1} = \binom{i+1}{j}.$$

In addition, we can write

$$\beta_n^{(j)} = \sum_{i=0}^{n+j} (-1)^{j+i} \binom{i}{j} x_i x_{n+j-i}.$$

By (2) we have

$$\begin{aligned} \Delta \beta_n^{(j)} &= \sum_{i=0}^{n+j-1} \left((-1)^{j+i+1} \binom{i+1}{j} + (-1)^{j+i} \binom{i}{j} \right) x_i x_{n+j-1-i} \\ &= \sum_{i=0}^{n+j-1} (-1)^{j+i+1} \left(\binom{i+1}{j} - \binom{i}{j} \right) x_i x_{n+j-1-i} \\ &= \sum_{i=0}^{n+j-1} (-1)^{j-1+i} \binom{i}{j-1} x_i x_{n+j-1-i} \\ &= \beta_n^{(j-1)}. \end{aligned}$$

□

We thus see that $\hat{\beta}_n = (\beta_n^{(j)})$ defines a homogeneous Δ-cable rooted at $\theta_n^{(0)}$ and that $\{\hat{\beta}_n\}$ is a Δ-basis for Ω_2, which we call the *balanced Δ-basis*.

Note that each $\beta_n^{(j)}$ involves at most $n+1$ monomials. Moreover, the monomials $x_i x_{n+j-i}$ $(j \le i \le n+j)$ are distinct if $j \ge n$, meaning that $\beta_n^{(j)}$ involves exactly $n+1$ monomials when $j \ge n$.

3.2.4. Small Δ-basis

Given $n \in 2\mathbb{N}$ and $j \in \mathbb{N}$, let

$$W_n^{(j)} = \langle x_0 x_{n+j}, x_1 x_{n+j-1}, \dots, x_{\frac{n}{2}} x_{\frac{n}{2}+j} \rangle,$$

noting that $W_n^{(j)} \subset \Omega_{(2,n+j)}$ and $\dim W_n^{(j)} = n/2+1$ for all $j \ge 0$. Then $\Delta : W_n^{(j+1)} \to W_n^{(j)}$ is an isomorphism, since $\theta_{n+j+1}^{(0)} \notin W_n^{(j+1)}$ if j is odd. Since $\theta_n^{(0)} \in W_n^{(0)}$, we conclude that there exists a unique Δ-cable $\hat{\eta}_n = (\eta_n^{(j)})$ rooted at $\theta_n^{(0)}$ such that $\eta_n^{(j)} \in W_n^{(j)}$ for each $j \ge 0$. We call $\{\hat{\eta}_n\}$ the *small Δ-basis* for Ω_2. Note that each $\eta_n^{(j)}$ involves at most $\frac{n}{2}+1$ monomials.

It is easy to check that the first three cables in this basis are given by

$$\eta_0^{(j)} = x_0 x_j, \qquad \eta_2^{(j)} = (j+2)x_0 x_{2+j} - x_1 x_{1+j},$$
$$\eta_4^{(j)} = \frac{(j+1)(j+4)}{2} x_0 x_{4+j} - (j+2) x_1 x_{3+j} + x_2 x_{2+j}.$$

In particular, $\hat{\eta}_4$ will be used to give certain 3-term recursion relations (see Remark 6.6).

3.2.5. Q-ideals

DEFINITION 3.10

Let $\{\hat{\theta}_n\}$ be a Δ-basis for Ω_2.

(1) A *Q-ideal* is an ideal of Ω generated by $\{\hat{\theta}_n \mid n \in S\}$, where $S \subset 2\mathbb{N}$ is any nonempty subset.

(2) Given $n \in 2\mathbb{N}$, the corresponding *fundamental Q-ideal* is

$$\mathcal{Q}_n = (\hat{\theta}_n, \hat{\theta}_{n+2}, \hat{\theta}_{n+4}, \dots).$$

LEMMA 3.11

For any Q-ideal I, $\Delta I = I$.

Proof

Since $\Delta\hat{\theta}_n = \hat{\theta}_n$ for each $n \in 2\mathbb{N}$, we see that $\Delta I \subset I$. To verify $\Delta I \supset I$, it suffices to show that, for each n with $\hat{\theta}_n \subset I$, each $i \ge 0$, and $f \in \Omega$, we have $f\theta_n^{(i)} \in \Delta I$. Choose m such that $\Delta^m f = 0$, and define $g \in I$ by

$$g = \sum_{j=0}^{m-1} (-1)^j \Delta^j(f) \theta_n^{(i+j+1)}.$$

Then, ΔI contains

$$\Delta(g) = \sum_{j=0}^{m-1} (-1)^j \left(\Delta^j(f)\theta_n^{(i+j)} + \Delta^{j+1}(f)\theta_n^{(i+j+1)} \right) = f\theta_n^{(i)}.$$

□

LEMMA 3.12

The following properties hold.

(a) $\mathcal{Q}_0 \supset \mathcal{Q}_2 \supset \mathcal{Q}_4 \supset \cdots$.
(b) *Given* $n \in 2\mathbb{N}$, $\mathcal{Q}_n$ *is independent of the choice of* Δ*-basis.*
(c) $\Omega_r \subset (x_0, \dots, x_{\frac{n}{2}-1})^{r-1} + \mathcal{Q}_n$ *for each integer* $r \geq 1$ *and* $n \in 2\mathbb{N}$.

Proof

Part (a) is clear from the definition.

For part (b), let $\{\hat{\theta}_m\}$ be the given Δ-basis, and let $\{\hat{\mu}_m\}$ be any other Δ-basis for Ω_2. For each $n \in 2\mathbb{N}$, define Q-ideals

$$\mathcal{Q}_n = (\hat{\theta}_n, \hat{\theta}_{n+2}, \hat{\theta}_{n+4}, \dots) \qquad \text{and} \qquad \tilde{\mathcal{Q}}_n = (\hat{\mu}_n, \hat{\mu}_{n+2}, \hat{\mu}_{n+4}, \dots).$$

By part (a), it suffices to check $\mu_n^{(j)} \in \mathcal{Q}_n$ for each integer $j \geq 0$. By Lemma 3.8(a), there exist $c_i \in k$, $0 \leq i \leq (n+j)/2$, such that

$$\mu_n^{(j)} = \sum_{0 \leq i \leq (n+j)/2} c_i \theta_{2i}^{(n+j-2i)}.$$

Since $\deg_\Delta \mu_n^{(j)} = j$, $\deg_\Delta \theta_{2i}^{(n+j-2i)} = n+j-2i$, and the integers $n+j-2i$ are distinct for distinct i, it follows that $c_i = 0$ when $n+j-2i > j$, that is, when $n > 2i$. Thus, we obtain

$$\mu_n^{(j)} = \sum_{n/2 \leq i \leq (n+j)/2} c_i \theta_{2i}^{(n+j-2i)} \in \mathcal{Q}_n.$$

This proves part (b).

We prove part (c) by induction on r, where the case $r = 1$ is clear. Fix $n \in 2\mathbb{N}$ and the integer $r \geq 2$, and let $\xi \in \Omega_r$ be given. Observe that ξ may be written as a sum of elements of $\Omega_{(r-2)} \cdot \Omega_2$. Since the vertices of the small Δ-basis $\{\hat{\eta}_m\}$ form a k-basis for Ω_2 by Lemma 3.8(b), we may write

$$\xi = \sum_{i \geq 0} \sum_{j \geq 0} L_{(2i,j)} \eta_{2i}^{(j)} = \sum_{i=0}^{n/2-1} \sum_{j \geq 0} L_{(2i,j)} \eta_{2i}^{(j)} + \sum_{i \geq n/2} \sum_{j \geq 0} L_{(2i,j)} \eta_{2i}^{(j)},$$

where $L_{(2i,j)} \in \Omega_{r-2}$. If $0 \leq i < n/2$, then $\eta_{2i}^{(j)} \in W_{2i}^{(j)} \subset \Omega_1 x_0 + \cdots + \Omega_1 x_{\frac{n}{2}-1}$. Also, by part (b) we have

$$\mathcal{Q}_n = (\hat{\eta}_n, \hat{\eta}_{n+2}, \hat{\eta}_{n+4}, \dots).$$

Together, these imply $\xi = \xi_0 x_0 + \cdots + \xi_{\frac{n}{2}-1} x_{\frac{n}{2}-1} + \xi'$ for some $\xi_0, \dots, \xi_{\frac{n}{2}-1} \in \Omega_{r-1}$ and $\xi' \in \mathcal{Q}_n$. By the induction hypothesis, we have $\xi_0, \dots, \xi_{\frac{n}{2}-1} \in (x_0, \dots, x_{\frac{n}{2}-1})^{r-2} + \mathcal{Q}_n$. Therefore, ξ belongs to $(x_0, \dots, x_{\frac{n}{2}-1})^{r-1} + \mathcal{Q}_n$. □

3.3. Cable algebras

DEFINITION 3.13

Let B be a commutative k-domain.

(a) B is a *cable algebra* if there exist nonzero $D \in \mathrm{LND}(B)$ and a finite number of D-cables $\hat{s}_1, \ldots, \hat{s}_n$ such that $B = A[\hat{s}_1, \ldots, \hat{s}_n]$, where $A = \ker D$. In this case, we say that the pair (B, D) is a *cable pair*.

(b) B is a *monogenetic* cable algebra if $B = A[\hat{s}]$ for some cable pair (B, D) with $A = \ker D$ and some D-cable $\hat{s}$.

(c) B is a *simple* cable algebra over k if $B = k[\hat{s}]$ for some D-cable $\hat{s}$, where $D \in \mathrm{LND}(B)$ is nonzero. A simple cable algebra B is of *terminal type* if $\hat{s}$ can be chosen to be a terminal D-cable.

We remark that if there exists nonzero $D \in \mathrm{LND}(B)$ for which B is finitely generated as an algebra over $\ker D$, then B is a cable algebra.

EXAMPLE 3.14

Let B be a commutative k-domain, let $D \in \mathrm{LND}(B)$, and let $A = \ker D$. If

$$S \subset B \setminus (A \cup DB) \qquad \text{and} \qquad |S| = n \geq 1,$$

then there exist terminal D-cables $\hat{s}_1, \ldots, \hat{s}_n$ such that $A[S, D] = A[\hat{s}_1, \ldots, \hat{s}_n]$. Let D' be the restriction of D to $A[S, D]$. Then $D' \neq 0$, $A[S, D]$ is a cable algebra, and $(A[S, D], D')$ is a cable pair.

EXAMPLE 3.15

Given $n \geq 1$, let $B_n = k[x_0, \ldots, x_n] = k^{[n+1]}$, and let D_n be the restriction of the down operator to B_n. The classical covariant rings $A_n = \ker D_n$ are known to be finitely generated over k, but have been calculated only for $n \leq 8$ (see [6]). Since $\partial/\partial x_n$ commutes with D_n, $\partial/\partial x_n$ restricts to A_n. If we denote this restriction by δ_n, then $\ker \delta_n = A_{n-1}$. Therefore, each A_n is a cable algebra. In particular, $A_1 = k[x_0] = k^{[1]}$ (see Lemma 3.16(a)); $A_2 = A_1[\hat{s}]$, where $\hat{s}$ is the δ_2-cable of length 1 with terminal vertex $s_1 = 2x_0x_2 - x_1^2$; $A_3 = A_2[\hat{t}]$, where $\hat{t}$ is the δ_3-cable of length 2 with terminal vertex

$$t_2 = 9x_0^2x_3^2 - 18x_0x_1x_2x_3 + 6x_1^3x_3 + 8x_0x_2^3 - 3x_1^2x_2^2;$$

and $A_4 = A_3[\hat{u}, \hat{v}]$, where $\hat{u}, \hat{v}$ are the δ_4-cables of length 1 with terminal vertices

$$u_1 = 2x_0x_4 - 2x_1x_3 + x_2^2$$

and

$$v_1 = 12x_0x_2x_4 - 6x_1^2x_4 - 9x_0x_3^2 + 6x_1x_2x_3 - 2x_2^3.$$

The rings A_2, A_3, A_4 are calculated in [5, Section 8.6]. The rings $A_5, \ldots, A_8$ are considerably more complicated, and it would be of interest to analyze their cable structures.

3.4. Simple cable algebras

A natural goal is to classify the simple cable algebras of finite transcendence degree over k according to transcendence degree. We start with the following observation.

LEMMA 3.16 (a) *$k^{[1]}$ is a simple cable algebra over k of nonterminal type.*
(b) *For each $n \geq 2$, $k^{[n]}$ is a simple cable algebra over k of terminal type.*

Proof
Let $B = k[t] = k^{[1]}$, and let d/dt denote the usual derivative. Define the sequence $t_n = \frac{1}{n!}t^n$. Then $\hat{t} = (t_n)$ is an infinite d/dt-cable and $B = k[\hat{t}]$. Therefore, $B = k^{[1]}$ is a simple cable algebra. In addition, any nonzero $D \in \mathrm{LND}(B)$ has a slice, so $\mathrm{Tr}(B, D)$ has no terminal vertices. Therefore, B is of nonterminal type. This proves part (a).

For part (b), let $B = k[x_1, \ldots, x_n] = k^{[n]}$, and define D by $Dx_{i+1} = x_i$ for $i \geq 2$ and $Dx_1 = 0$. Note that $x_n \notin (DB) = (x_1, \ldots, x_{n-1})$. Therefore, $\hat{x} = (x_i)$ is a terminal D-cable and $B = k[\hat{x}]$. □

Suppose that B is a cable algebra with $\mathrm{tr.deg}_k B = 1$. Then $B = L^{[1]}$, where L is an algebraic extension field of k (see [5, Corollary 1.24]). Therefore, when k is algebraically closed, B is simple (over k) if and only if $B = k^{[1]}$. When k is not algebraically closed, there are simple cable algebras over k other than $k^{[1]}$. For example, consider the usual derivative $D = d/dx$ on the ring $B = \mathbb{Q}[\sqrt{2}, x] = \mathbb{Q}[\sqrt{2}]^{[1]}$. We have that $\hat{s} = (\sqrt{2}x^n/n!)$ is a D-cable and $B = \mathbb{Q}[\hat{s}]$, but $B \neq \mathbb{Q}^{[1]}$.

For simple cable algebras of transcendence degree 2, we give several illustrative examples.

EXAMPLE 3.17
Let $B = k[x, v] = k^{[2]}$, and let $D = \partial/\partial v$. If $s_n = \frac{1}{n!}v^n$ for $n \geq 0$, then $\hat{s} = (s_n)$ is a D-cable rooted at 1. Let $\hat{t} = \hat{s} +_2 x\hat{s}$ be given by $\hat{t} = (t_n)$. Then $B = k[\hat{t}]$, since $k[\hat{t}]$ contains $t_1 = v$ and $t_2 = x + \frac{1}{2}v^2$. This shows that a simple cable algebra of terminal type can also be generated by an infinite D-cable for some D.

EXAMPLE 3.18
Continuing the notation of the preceding example, we see that the subring $k[x\hat{s}]$ of $k[x, v]$ is a simple cable algebra which is not finitely generated as a k-algebra and therefore not of terminal type. More generally, let $D = \partial/\partial v$, and let $p_n(v)$ be any infinite sequence of polynomials in $k[x, v]$ with $Dp_n(v) = p_{n-1}(v)$ for $n \geq 1$ and $p_0(v) \in k[x] \setminus k$. Then $\hat{p} := (p_n(v))$ is a D-cable and $k[\hat{p}]$ is a simple cable algebra of transcendence degree 2 over k.

EXAMPLE 3.19
Let $B = k[y_0, y_1, y_2]$ where $2y_0y_2 = y_1^2$. Define $D \in \mathrm{LND}(B)$ by $y_2 \to y_1 \to y_0 \to 0$.

It is easy to see that $y_2 \notin DB$. Therefore, $\hat{y} := (y_n)$ is a terminal D-cable and $B = k[\hat{y}]$.

EXAMPLE 3.20

The ring $B = k[z_0, z_1, z_2]$ where $2z_0^2 z_2 = z_1^2$ is not a simple cable algebra. To see this, let $D \in \mathrm{LND}(B)$, and let a D-cable $\hat{s} = (s_n)$ be given. Define $E \in \mathrm{LND}(B)$ by $z_2 \to z_1 \to z_0^2 \to 0$. It is known that $\mathrm{LND}(B) = k[z_0] \cdot E$ (see [10]). Therefore, $DB \subset J = (z_0^2, z_1)$. Assume that $k[\hat{s}] = B$. If $s_n \in DB$ for every $n \geq 0$, then $B/J = k$. However, if $\pi : B \to B/J$ is the canonical surjection, then $\pi(z_2)$ is transcendental over k, so this case cannot occur. Therefore, $s_n \notin DB$ for some $n \geq 0$, meaning that s_n is a terminal vertex and $s_0, \ldots, s_{n-1} \in J$. It follows that $B/J = k[\pi(s_n)] \cong k^{[1]}/(p)$ for some $p \in k^{[1]} \setminus k^*$. If $p = 0$, then B/J is an integral domain, a contradiction. If $p \neq 0$, then every element of B/J is algebraic over k, a contradiction. Therefore, $k[\hat{s}] \neq B$.

3.5. Cable relations for S

Define the simple cable algebra $S \subset k[x, v] = k^{[2]}$ by $S = k[x\hat{s}]$, where $\hat{s} = (\frac{1}{n!}v^n)$.

THEOREM 3.21

We have $S \cong_k \Omega/\mathcal{Q}_2$. Consequently, $\mathcal{Q}_2$ is a prime ideal of Ω.

Proof

The surjections $\phi_{\hat{s}} : \Omega \to k[v]$ and $\phi_{x\hat{s}} : \Omega \to S$ are given by

$$\phi_{\hat{s}}(x_i) = s_i \qquad \text{and} \qquad \phi_{x\hat{s}}(x_i) = xs_i \quad (i \geq 0).$$

Let $g \in \ker \Delta$ be given, and let $\{\hat{\theta}_n\}$ be a Δ-basis for Ω_2. If d/dv denotes the standard derivative on $k[v]$, then we have

$$(3) \qquad 0 = \phi_{\hat{s}}\Delta g = \frac{d}{dv}\phi_{\hat{s}} g \quad \Rightarrow \quad \phi_{\hat{s}} g \in \ker \frac{d}{dv} = k \quad \Rightarrow \quad g \in k + \ker \phi_{\hat{s}}.$$

If $n \geq 2$ is even, then $\phi_{\hat{s}}\theta_n^{(0)} = \lambda v^n$ for some $\lambda \in k$. Therefore, $\theta_n^{(0)} \in \ker \phi_{\hat{s}}$ for each even $n \geq 2$.

Given an even integer $n \geq 2$, assume that $\theta_n^{(j)} \in \ker \phi_{\hat{s}}$ for some $j \geq 0$. We have

$$0 = \phi_{\hat{s}}\theta_n^{(j)} = \phi_{\hat{s}}\Delta\theta_n^{(j+1)} = \frac{d}{dv}\phi_{\hat{s}}\theta_n^{(j+1)} \quad \Rightarrow \quad \phi_{\hat{s}}\theta_n^{(j+1)} \in \ker \frac{d}{dv} = k.$$

As before, since $n \geq 2$, we must have $\phi_{\hat{s}}\theta_n^{(j+1)} = 0$. It follows by induction that $\theta_n^{(j)} \in \ker \phi_{\hat{s}}$ for every even $n \geq 2$ and every $j \geq 0$. Therefore, $\mathcal{Q}_2 \subset \ker \phi_{\hat{s}}$.

Given $r \geq 2$ and $P \in \Omega_r$, note that $\phi_{x\hat{s}} P = x^r \phi_{\hat{s}} P$. Therefore, if $P \in \Omega$ is homogeneous, then $P \in \ker \phi_{x\hat{s}}$ if and only if $P \in \ker \phi_{\hat{s}}$. In particular, this implies $\mathcal{Q}_2 \subset \ker \phi_{x\hat{s}}$.

Suppose that $P \in \Omega_r \cap \ker \phi_{x\hat{s}}$. By Lemma 3.12(c), we see that $P \in (x_0)^{r-1} + \mathcal{Q}_2$. Write $P = x_0^{r-1} L + Q$ for $L \in \Omega$ and $Q \in \mathcal{Q}_2$. Since the element P and the ideals $(x_0)^{r-1}$ and $\mathcal{Q}_2$ are homogeneous, we may assume that L and Q are homogeneous. By degree considerations, $L \in \Omega_1$. We have that $x_0^{r-1}L \in \ker \phi_{x\hat{s}}$. If $L \neq 0$, then since $\ker \phi_{x\hat{s}}$ is a prime ideal, either $x_0 \in \ker \phi_{x\hat{s}}$ or $L \in \ker \phi_{x\hat{s}}$, a contradiction. Therefore, $L = 0$ and $P \in \mathcal{Q}_2$.

We have thus shown $\Omega_r \cap \ker \phi_{x\hat{s}} \subset \mathcal{Q}_2$ for all $r \geq 2$. This suffices to prove $\ker \phi_{x\hat{s}} = \mathcal{Q}_2$. □

4. The derivation D in dimension 5

4.1. Definitions

Define the polynomial ring $B = k[a,x,y,z,v] = k^{[5]}$. We define the locally nilpotent derivation D of B by its action on a set of generators

$$z \to y \to x \to a^3, \qquad v \to a^2, \qquad a \to 0.$$

Define $A = \ker D$ and $R = k[a,x,y,z]$, noting that D restricts to R. In fact, D restricts to a linear derivation of the subring $k[a^3,x,y,z]$, and this kernel is well known. Let $k[t,x,y,z] = k^{[4]}$, and define the linear derivation $\tilde{D}$ on this ring by $z \to y \to x \to t \to 0$. Then $\ker \tilde{D} = k[t,\tilde{F},\tilde{G},\tilde{h}]$, where (see [5, Example 8.9])

$$\tilde{F} = 2ty - x^2, \qquad \tilde{G} = 3t^2z - 3txy + x^3, \qquad \text{and} \qquad t^2\tilde{h} = \tilde{F}^3 + \tilde{G}^2.$$

Note that the restriction of D to R is equal to the $k[a]$-derivation $\mathrm{id}_{k[a]} \otimes \tilde{D}$ on $k[a] \otimes_{k[t]} k[t,x,y,z] = R$, and its kernel $R \cap A$ is equal to $\ker(\mathrm{id}_{k[a]} \otimes \tilde{D}) = k[a] \otimes_{k[t]} \ker \tilde{D}$. Therefore, if $F = \tilde{F}|_{t=a^3}$, $G = \tilde{G}|_{t=a^3}$, and $h = \tilde{h}|_{t=a^3}$, then

$$R \cap A = k[a,F,G,h], \quad \text{where } a^6h = F^3 + G^2.$$

Specifically,

$$F = 2a^3y - x^2, \qquad G = 3a^6z - 3a^3xy + x^3,$$
$$h = 9a^6z^2 - 18a^3xyz + 8a^3y^3 + 6x^3z - 3x^2y^2.$$

Define a $\mathbb{Z}^2$-grading of B by declaring that a,x,y,z,v are homogeneous and

$$\deg(a,x,y,z,v) = \big((1,0),(3,1),(3,2),(3,3),(2,1)\big).$$

Then D is a homogeneous derivation of degree $(0,-1)$ and A is a graded subring of B. Given integers $r,s \geq 0$, let $B_{(r,s)}$ be the vector space of homogeneous polynomials in B of degree (r,s), and define

$$A_{(r,s)} = A \cap B_{(r,s)}.$$

Then we have

$$F \in A_{(6,2)}, \qquad G \in A_{(9,3)}, \qquad h \in A_{(12,6)}.$$

Since $k[a,F,G,h] = R \cap A = \ker D|_R$ is factorially closed in R, we see that F, G, and h are irreducible by degree considerations. Note that $[D, \partial/\partial v] = 0$, that is, D commutes with the partial derivative $\partial/\partial v$ on B. Therefore, $\partial/\partial v$ restricts to A. If ∂ denotes the restriction of $\partial/\partial v$ to A, then $\partial \in \mathrm{LND}(A)$ and ∂ is homogeneous of degree $(-2,-1)$.

The following result is needed below.

LEMMA 4.1

Given $n \geq 0$, write $n = 6e + \ell$ for $e \geq 0$ and $0 \leq \ell \leq 5$.

(a)

$$R \cap A_{(2n+1,n)} = \begin{cases} \langle ah^e \rangle & \ell = 0, \\ \{0\} & \ell \neq 0. \end{cases}$$

(b)

$$R \cap A_{(2n+2,n)} = \begin{cases} \langle a^2h^e \rangle & \ell = 0, \\ \langle Fh^e \rangle & \ell = 2, \\ \{0\} & \ell = 1,3,4,5. \end{cases}$$

Proof

Since $R \cap A = k[a, F, G, h]$ with a, F, G, and h homogeneous, each k-vector space $R \cap A_{(r,s)}$ is spanned by monomials in a, F, G, and h. If the monomial $a^{e_1}F^{e_2}G^{e_3}h^{e_4} \in R$ ($e_i \in \mathbb{N}$) has degree $(2n+1, n)$, then

$$\begin{cases} e_1 + 6e_2 + 9e_3 + 12e_4 = 2n + 1, \\ 2e_2 + 3e_3 + 6e_4 = n. \end{cases}$$

The solutions to this system are $e_1 = 1$, $e_2 = e_3 = 0$, and $6e_4 = n$. This proves part (a).

Similarly, if $\deg(a^{e_1}F^{e_2}G^{e_3}h^{e_4}) = (2n+2, n)$, then

$$\begin{cases} e_1 + 6e_2 + 9e_3 + 12e_4 = 2n + 2, \\ 2e_2 + 3e_3 + 6e_4 = n. \end{cases}$$

The solutions to this system are

$$\{e_1 = 2, e_2 = e_3 = 0, n = 6e_4\} \qquad \text{and} \qquad \{e_1 = e_3 = 0, e_2 = 1, n = 6e_4 + 2\}.$$

This proves part (b). □

4.2. Homogeneous ∂-cables

Let $\mathcal{S}_a$ denote the set of infinite homogeneous ∂-cables rooted at a.

THEOREM 4.2

We have $\mathcal{S}_a \neq \emptyset$.

Proof

We show that there exists a sequence $s_n \in A$, $n \geq 0$, such that

(a) $s_0 = a$,
(b) $s_n \in A_{(2n+1,n)}$ for each $n \geq 0$,
(c) $\partial s_n = s_{n-1}$ for each $n \geq 1$.

Let d denote the restriction of D to the subring $Q \subset B$ defined by $Q = k[t, x, y, z] \cong k^{[4]}$, where $t = a^3$. Then d is a linear derivation defined by

$$z \to y \to x \to t \to 0.$$

In addition, d is homogeneous of degree $(0, -1)$ for the $\mathbb{Z}^2$-grading of Q for which

$$\deg(t, x, y, z) = \big((1,0), (1,1), (1,2), (1,3)\big).$$

Let $Q_{(r,s)}$ denote the vector space of homogeneous polynomials in Q of degree (r, s). Then according to [6, Proposition 4.1], the mapping

$$d : Q_{(r,s+1)} \to Q_{(r,s)}$$

is surjective if $2s < 3r$. Thus, given $m \geq 1$, each mapping in the following sequences of maps is surjective:

$$t \cdot Q_{(2m,3m)} \subset Q_{(2m+1,3m)} \overset{d}{\leftarrow} Q_{(2m+1,3m+1)} \overset{d}{\leftarrow} Q_{(2m+1,3m+2)}$$

and

$$t \cdot Q_{(2m-1,3m-1)} \subset Q_{(2m,3m-1)} \overset{d}{\leftarrow} Q_{(2m,3m)}.$$

Consequently, there exists a sequence $w_n \in Q$, $n \geq 0$, such that $w_0 = 1$, and for all $m \geq 0$,

$$w_{3m} \in Q_{(2m,3m)}, \qquad w_{3m+1} \in Q_{(2m+1,3m+1)}, \qquad w_{3m+2} \in Q_{(2m+1,3m+2)},$$

where

$$dw_{3m+3} = t \cdot w_{3m+2}, \qquad dw_{3m+2} = w_{3m+1}, \qquad dw_{3m+1} = t \cdot w_{3m}.$$

With the sequence w_n so constructed, it follows that, for $m \geq 1$,

$$D^{3i} w_{3m} = d^{3i} w_{3m} = t^{2i} w_{3(m-i)} = a^{6i} w_{3(m-i)} = (Dv)^{3i} w_{3(m-i)} \quad (0 \leq i \leq m).$$

Therefore, for $0 \leq i \leq m$, we have

(i) $D^{3i}(aw_{3m}) = a(Dv)^{3i} w_{3(m-i)}$,

(ii) $D^{3i+1}(aw_{3m}) = d(a(Dv)^{3i} w_{3(m-i)}) = a(Dv)^{3i} t w_{3(m-i)-1} = a^2(Dv)^{3i+1} w_{3(m-i)-1}$,

(iii) $D^{3i+2}(aw_{3m}) = d(a^2(Dv)^{3i+1} w_{3(m-i)-1}) = a^2(Dv)^{3i+1} w_{3(m-i)-2} = (Dv)^{3i+2} w_{3(m-i)-2}$.

We see that

$$(Dv)^j \text{ divides } D^j(aw_{3m}) \text{ for each } j \ (0 \leq j \leq 3m). \tag{4}$$

Therefore, if we define $s_{3m} = (-1)^{3m} \pi_v(aw_{3m})$ for $m \geq 0$, then $s_{3m} \in A$ for each $m \geq 0$. Using (1) in Section 2.1, it follows that for $m \geq 1$

$$\begin{aligned} \frac{\partial^3}{\partial v^3} s_{3m} &= \frac{\partial^2}{\partial v^2} (-1)^{3m-1} \pi_v(aDw_{3m}) \frac{\partial}{\partial v} \frac{v}{a^2} \\ &= \frac{\partial}{\partial v} (-1)^{3m-2} \pi_v(aD^2 w_{3m}) \frac{1}{a^2} \frac{\partial}{\partial v} \frac{v}{a^2} \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{3m-3}\pi_v(aD^3 w_{3m})\frac{1}{a^4}\frac{\partial}{\partial v}\frac{v}{a^2} \\
&= (-1)^{3m-3}\pi_v\big(a(a^2)^3 w_{3(m-1)}\big)\frac{1}{a^6} \\
&= (-1)^{3(m-1)}\pi_v(a w_{3(m-1)}) \\
&= s_{3(m-1)}.
\end{aligned}
$$

Define

$$
s_{3m-1} = \frac{\partial}{\partial v}s_{3m} \qquad \text{and} \qquad s_{3m-2} = \frac{\partial}{\partial v}s_{3m-1} \quad (m \geq 1).
$$

Then $\hat{s} := (s_n)$ is a ∂-cable rooted at a with $s_n \in A_{(2n+1,n)}$ for each $n \geq 0$. □

REMARK 4.3

Let $\hat{s} = (s_n) \in \mathcal{S}_a$ be given. Since $\dim A_{(2n+1,n)} = 1$ for $n = 0, \ldots, 5$, the elements $s_0, \ldots, s_5$ are uniquely determined (see Corollary 5.5(a)). They are given by

$$
\begin{aligned}
0!s_0 &= a, \\
1!s_1 &= av - x, \\
2!s_2 &= av^2 - 2xv + 2a^2y, \\
3!s_3 &= av^3 - 3xv^2 + 6a^2yv - 6a^4z, \\
4!s_4 &= av^4 - 4xv^3 + 12a^2yv^2 - 24a^4zv + 24a^3xz - 12a^3y^2, \\
5!s_5 &= av^5 - 5xv^4 + 20a^2yv^3 - 60a^4zv^2 + 120a^3xzv - 60a^3y^2v - 72x^2a^2z \\
&\quad + 36xa^2y^2 + 24a^5yz.
\end{aligned}
$$

Note the identities

$$
\begin{aligned}
F &= 2s_0s_2 - s_1^2, & -G &= 3s_0^2s_3 - 3s_0s_1s_2 + s_1^3, \\
2s_0s_4 &= 2s_1s_3 - s_2^2, & 5s_0s_5 &= 3s_1s_4 - s_2s_3.
\end{aligned}
\tag{5}
$$

5. Generators of $\bar{A}$ and A

The main result of this section is the following.

THEOREM 5.1

Let $\hat{s} = (s_n) \in \mathcal{S}_a$ be given.

(a) *$A = k[h, \hat{s}]$.*
(b) *A is not finitely generated as a k-algebra.*
(c) *The generating set $\{h, s_n\}_{n\geq 0}$ is minimal in the sense that no proper subset generates A.*

5.1. Generators of $\bar{A}$

Let $\pi : B \to B/hB$ be the canonical surjection. Given $b \in B$, let $\bar{b}$ denote $\pi(b)$, and for a subalgebra $M \subset B$, let $\bar{M} = \pi(M)$. Since h is homogeneous, π induces a $\mathbb{Z}^2$-grading on $\bar{B}$, and $\bar{A}$ is a graded subring with

$$\bar{A}_{(r,s)} = \pi(A_{(r,s)}).$$

Note that, since h is irreducible, hB is a prime ideal of B. Hence, B/hB and its subring $\bar{A}$ are integral domains. Since $D(h) = 0$, we have $hB \cap A = hA$. Indeed, if $P \in B$ is such that $hP \in A$, then $hDP = D(hP) = 0$, and hence $DP = 0$. Thus, $\bar{A} \cong A/hA$ and so hA is a prime ideal of A. Since $h \in \ker\partial$, we can define $\delta \in \mathrm{LND}(\bar{A})$ by $\delta\pi(g) = \pi\partial(g)$. Then δ is a homogeneous locally nilpotent derivation of $\bar{A}$ of degree $(-2,-1)$. Recall that $\ker\partial = R \cap A = k[a,F,G,h]$.

LEMMA 5.2

We have $\ker\delta = \pi(\ker\partial) = k[\bar{a},\bar{F},\bar{G}]$.

Proof

It must be shown that $\partial^{-1}(hA) = R \cap A + hA$. The inclusion $R \cap A + hA \subset \partial^{-1}(hA)$ is clear. For the converse, we first show that if $H = R \cap A + hB$, then $H \cap aB = aH$.

Since $R \cap A = k[a,F,G,h]$ and $F^3 + G^2 \in hR$, we have

$$H = k[a,F] + k[a,F]G + hB.$$

In addition, H is a graded subring of B, and if $g \in H_{(r,s)}$, then $g \in k[a,F] + hB$ for s even and $g \in k[a,F]G + hB$ for s odd. Write $g = p(a,F)G^{\epsilon} + h\rho$, where $p \in k^{[2]}$, $\rho \in B$, and $\epsilon \in \{0,1\}$. If $g \in aB$, then setting $a = 0$ yields the following equation in $k[x,y,z,v]$:

$$(h\rho)|_{a=0} = 3x^2(2xz - y^2)\rho|_{a=0} = -p(0,-x^2)x^{3\epsilon} \in k[x].$$

This means $\rho \in aB$, since $2xz - y^2$ is transcendental over $k[x]$. Therefore, $p(a,F) \in aB$, and since $R \cap A$ is factorially closed in B it follows that $p(a,F) \in a(R \cap A)$. So $g \in aH$. This shows that $H \cap aB = aH$.

Suppose that $f \in A$ and $\partial f \in hA$. Let $L \in R^{[1]}$ be such that $f = L(v) = \sum_i \frac{1}{i!} L^{(i)}(0)v^i$. We have

$$\partial^i f = L^{(i)}(v) \in hA \quad \forall i \geq 1 \quad \Rightarrow \quad L^{(i)}(0) \in hR \quad \forall i \geq 1.$$

Therefore, $f = hq + r$ for $q \in B$ and $r = L(0) \in R$. It follows that $0 = Df = hDq + Dr$, which implies $Dr \in R \cap hB = hR$.

The restriction of D to R has kernel $R \cap A$ and local slice x. So there exist $n \geq 0$ and $P \in (R \cap A)^{[1]}$ with $a^n r = P(x) = \sum_i \frac{1}{i!} P^{(i)}(0)x^i$. We thus have

$$\begin{aligned} a^n D^i r = P^{(i)}(x)a^{3i} \in hR \quad \forall i \geq 1 \quad &\Rightarrow \quad P^{(i)}(x) \in hR \quad \forall i \geq 1 \\ &\Rightarrow \quad P^{(i)}(0) \in h(R \cap A) \quad \forall i \geq 1. \end{aligned}$$

Therefore, $a^n r \in (R \cap A) + h(R \cap A)[x] \subset H$.

By repeated application of the identity $H\cap aB=aH$, we have that $H\cap a^nB=a^nH$. It follows that $a^nr\in H\cap a^nB=a^nH$. Therefore, $r\in H$ and $f=hq+r\in hB+H=H$. Since A is factorially closed in B, we conclude that $f\in R\cap A+hA$. □

Given $\hat{s}=(s_n)\in\mathcal{S}_a$, we have $s_0=a\notin hB$, and so $\bar{s}_0\neq 0$. Since $\delta\pi=\pi\partial$, we see that $\pi\hat{s}:=(\bar{s}_n)$ is a δ-cable. If $\phi_{\pi\hat{s}}:\Omega\to\bar{A}$ is the associated mapping, then $\phi_{\pi\hat{s}}\Delta=\delta\phi_{\pi\hat{s}}$ (cf. Section 3.1(x)). We also note that $\ker\phi_{\pi\hat{s}}$ is a homogeneous ideal of Ω, since $\phi_{\pi\hat{s}}(\Omega_{(r,s)})\subset\bar{A}_{(2s+r,s)}$ for each $r,s\geq 0$.

THEOREM 5.3
We have that $\phi_{\pi\hat{s}}$ is surjective.

Proof
Define

$$A'=\phi_{\pi\hat{s}}(\Omega)=k[\pi\hat{s}],\qquad A'_+=\phi_{\pi\hat{s}}(\Omega_+),\qquad\text{and}\qquad A'_{(r,s)}=A'\cap\bar{A}_{(r,s)}.$$

Since $\Delta:\Omega_+\to\Omega_+$ is surjective and $\phi_{\pi\hat{s}}\Delta=\delta\phi_{\pi\hat{s}}$, it follows that the mapping $\delta:A'_+\to A'_+$ is surjective. In addition, define

$$C=\ker\delta\qquad\text{and}\qquad C_{(r,s)}=C\cap\bar{A}_{(r,s)}.$$

Then from Lemma 5.2 and (5) we see that

$$C=k[\bar{a},\bar{F},\bar{G}],\qquad \bar{F}=2\bar{s}_0\bar{s}_2-\bar{s}_1^2,\qquad -\bar{G}=3\bar{s}_0^2\bar{s}_3-3\bar{s}_0\bar{s}_1\bar{s}_2+\bar{s}_1^3. \tag{6}$$

Therefore, $C\subset A'$ and $\ker\delta|_{A'}=C$.

Fix $\ell\in\mathbb{Z}$. We show by induction on n that, for each integer $n\geq 0$,

$$A'_{(2n+\ell,n)}=\bar{A}_{(2n+\ell,n)}. \tag{7}$$

For $n=0$, it is easy to see that $\bar{A}_{(\ell,0)}=\{0\}$ if $\ell<0$. If $\ell\geq 0$, then $\bar{A}_{(\ell,0)}=\langle a^\ell\rangle=\langle\bar{s}_0^\ell\rangle$, since $B_{(\ell,0)}=\langle a^\ell\rangle$. So (7) holds for $n=0$. Since $B_{(2,1)}=\langle v\rangle$, we have $A'_{(2,1)}=\bar{A}_{(2,1)}=\{0\}$. Hence, (7) also holds for $n=1$ and $\ell=0$.

Given $n\geq 1$, assume that

$$(n,\ell)\neq(1,0)\qquad\text{and}\qquad A'_{(2(n-1)+\ell,n-1)}=\bar{A}_{(2(n-1)+\ell,n-1)}.$$

Since $\delta:A'_+\to A'_+$ is surjective and $A'_+=\bigoplus_{(r,s)\neq(0,0)}A'_{(r,s)}$, it follows that

$$\delta A'_{(2n+\ell,n)}=A'_{(2(n-1)+\ell,n-1)}=\bar{A}_{(2(n-1)+\ell,n-1)}.$$

Since $A'_{(2n+\ell,n)}\subset\bar{A}_{(2n+\ell,n)}$, we have

$$\bar{A}_{(2(n-1)+\ell,n-1)}=\delta A'_{(2n+\ell,n)}\subset\delta\bar{A}_{(2n+\ell,n)}\subset\bar{A}_{(2(n-1)+\ell,n-1)},$$

which implies $\delta A'_{(2n+\ell,n)}=\delta\bar{A}_{(2n+\ell,n)}$. Therefore,

$$\begin{aligned}\dim\bar{A}_{(2n+\ell,n)}&=\dim C_{(2n+\ell,n)}+\dim\delta\bar{A}_{(2n+\ell,n)}\\&=\dim C_{(2n+\ell,n)}+\dim\delta A'_{(2n+\ell,n)}=\dim A'_{(2n+\ell,n)}.\end{aligned}$$

It follows that $A'_{(2n+\ell,n)} = \bar{A}_{(2n+\ell,n)}$. By induction, we conclude that (7) holds for all $n \geq 0$. □

COROLLARY 5.4
Let $\hat{s} = (s_n) \in \mathcal{S}_a$ *be given.*

(a) $\bar{A} = k[\pi\hat{s}]$.
(b) $\bar{A}$ *is not finitely generated as a* k*-algebra.*
(c) *The generating set* $\{\bar{s}_n\}_{n\geq 0}$ *is minimal in the sense that no proper subset generates* $\bar{A}$.

Proof
Part (a) is implied by Theorem 5.3. For part (b), let $\Sigma \subset \mathbb{N}^2$ be the degree semigroup of A. Then part (a) implies that

$$\Sigma = \langle (2n+1, n) \mid n \geq 0 \rangle.$$

It will suffice to show that Σ is not finitely generated as a semigroup. However, this is obvious, since the element $(2n+1, n)$ does not belong to the subsemigroup generated by $(2i+1, i)$ for $i < n$. This proves part (b). In fact, $(2n+1, n)$ does not even belong to the larger subsemigroup generated by $(2i+1, i)$ for $i \neq n$, and this implies part (c). □

5.2. Proof of Theorem 5.1

Set $\Gamma = k[\hat{s}]$. Then Γ is a graded subring of A, where $\Gamma_{(r,s)} = \Gamma \cap A_{(r,s)}$. By Corollary 5.4(a), each $g \in A$ has the form $g = \gamma + h \cdot \alpha$, where $\gamma \in \Gamma$ and $\alpha \in B$. Since $g, \gamma, h \in A$, it follows that $\alpha \in A$. Write

$$\gamma = \sum \gamma_{(r,s)} \qquad \text{and} \qquad \alpha = \sum \alpha_{(r,s)},$$

where $\gamma_{(r,s)} \in \Gamma_{(r,s)}$ and $\alpha_{(r,s)} \in A_{(r,s)}$ for each $r, s \in \mathbb{Z}$. Then the homogeneous decomposition of g is

$$g = \sum_{(r,s)} \big(\gamma_{(r,s)} + h \cdot \alpha_{(r-12,s-6)}\big).$$

When g is homogeneous, there exists (r,s) such that $g = \gamma_{(r,s)} + h \cdot \alpha_{(r-12,s-6)}$.

For each fixed $r \geq 0$, we show by induction on s that $A_{(r,s)} \subset \Gamma[h]$. We have $A_{(r,0)} = k \cdot a^r \subset \Gamma$, which gives the basis for induction. Given $s \geq 1$, suppose that $A_{(r,i)} \subset \Gamma[h]$ for $0 \leq i \leq s-1$. Given $g \in A_{(r,s)}$, write $g = \gamma_{(r,s)} + h \cdot \alpha_{(r-12,s-6)}$ as above. By the induction hypothesis, we have that $\alpha_{(r-12,s-6)} \in \Gamma[h]$. Therefore, $g \in \Gamma[h]$. We conclude that $A_{(r,s)} \subset \Gamma[h]$ for all (r,s) with $r, s \geq 0$, and therefore, $A \subset \Gamma[h]$. This proves part (a).

Part (b) is immediately implied by Corollary 5.4(b) and the fact that $\bar{A}$ is the image of A under a k-algebra homomorphism.

For part (c), note that Corollary 5.4(c) implies that any generating subset of $\{h, s_n\}_{n\geq 0}$ must include each s_n. We also cannot exclude h, since $(12, 6)$ does not belong to the degree semigroup generated by $\{(2n+1,n) \mid n \geq 0\}$. This proves part (c) and completes the proof of Theorem 5.1.

For the next result, the reader is reminded that $A_{(r,s)} = \{0\}$ if $r < 0$ or $s < 0$.

COROLLARY 5.5

Let $\hat{s} = (s_n) \in \mathcal{S}_a$. Given $n \geq 0$, let $e \geq 0$ be such that $0 \leq n - 6e \leq 5$.

(a) $A_{(2n+1,n)} = k \cdot s_n \oplus h \cdot A_{(2(n-6)+1,n-6)}$.
(b) $\dim A_{(2n+1,n)} = e + 1$.
(c) *A basis for $A_{(2n+1,n)}$ is given by $\{s_n, s_{n-6}h, s_{n-12}h^2, \ldots, s_{n-6e}h^e\}$.*

Proof

Part (a) is implicit in the first paragraph of the proof of Theorem 5.1 with $(r,s) = (2n+1,n)$, since $\Gamma_{(2n+1,n)} = k \cdot s_n$ and $s_n \notin hB$. It follows that $A_{(2n+1,n)} = k \cdot s_n$ for $n = 0, \ldots, 5$. Therefore, using part (a), we get parts (b) and (c) by induction on n. □

REMARK 5.6

Consider the field $k(h) = k^{(1)}$ and the $k(h)$-algebra $k(h) \otimes_{k[h]} A = k(h)[\hat{s}]$. Since $\partial h = 0$, ∂ extends to a locally nilpotent derivation $\tilde{\partial}$ of $k(h)[\hat{s}]$, $\hat{s}$ is a $\tilde{\partial}$-cable, and $k(h)[\hat{s}]$ is a simple cable algebra over $k(h)$ which is of transcendence degree 3 over $k(h)$.

5.3. The ∂-cable $\hat{\sigma}$

THEOREM 5.7

There exists a unique $\hat{\sigma} = (\sigma_n) \in \mathcal{S}_a$ such that $n!\sigma_n \equiv -nxv^{n-1} \pmod{aB}$ for each $n \geq 1$. In addition, $\hat{\sigma}$ satisfies the following.

(a) *If $n, e \geq 0$ with $n \neq 1$, then $\sigma_0\sigma_1 h^e \notin \langle \sigma_i\sigma_{n-i} \mid 0 \leq i \leq n/2, i \neq 1\rangle$.*
(b) *If $n, e \geq 0$ with $n \neq 2$, then $Fh^e \notin \langle \sigma_i\sigma_{n-i} \mid 0 \leq i \leq n/2\rangle$.*

Proof

Given $P \in B$, let $P(0)$ denote evaluation at $v = 0$. An explicit sequence $w_n \in k[t,x,y,z]$ of the type used in the proof of Theorem 4.2 is constructed in [5, Section 7.2.1], and in this example, w_n has the property that t divides w_n whenever $n \geq 4$ and $n \equiv 1 \pmod 3$. Let $\hat{\sigma} = (\sigma_n) \in \mathcal{S}_a$ be the ∂-cable constructed from this sequence. Given $m \geq 1$, it follows from the definition of the functions $s_n = \sigma_n$ given in the proof of Theorem 4.2 that

$$\sigma_{3m} = (-1)^{3m} a w_{3m} - D\big((-1)^{3m} a w_{3m}\big)\frac{v}{a^2} + \frac{1}{2}D^2\big((-1)^{3m} a w_{3m}\big)\frac{v^2}{a^4} + \cdots.$$

Since $\partial^i \sigma_{3m}/\partial v^i = \sigma_{3m-i}$ for $0 \leq i \leq 3m$, this implies that

$$\sigma_{3m}(0) = (-1)^{3m} a w_{3m}, \qquad \sigma_{3m-1}(0) = (-1)^{3m-1} a^2 w_{3m-1},$$
$$\sigma_{3m-2}(0) = (-1)^{3m-2} w_{3m-2}.$$

Since $t=a^3$ divides w_{3m-2} for $m\geq 2$ and $\sigma_0(0)=\sigma_0=a$, it follows that a divides $\sigma_n(0)$ for all $n\geq 0$ with $n\neq 1$. We now show by induction on n that

$$\text{(8)} \qquad a \text{ divides } P_n(v):=(n-1)!\sigma_n+xv^{n-1} \quad (n\geq 1).$$

First, observe that Corollary 5.5(b) implies that the functions $\sigma_0,\ldots,\sigma_5$ are uniquely determined. In particular, we have $\sigma_1=av-x$ (see Remark 4.3). Hence, property (8) holds for $n=1$.

Given $n\geq 2$, assume that a divides $P_i(v)$ for $1\leq i\leq n-1$. We have

$$P_n'(v)=(n-1)!\sigma_{n-1}+(n-1)xv^{n-2}=(n-1)P_{n-1}(v).$$

The inductive hypothesis implies that $P_n'(v)\in aB$, which means $P_n(v)-P_n(0)\in aB$. Since $P_n(0)=(n-1)!\sigma_n(0)\in aB$, we conclude that $P_n(v)\in aB$ for all $n\geq 1$. This proves the existence of $\hat\sigma=(\sigma_n)\in\mathcal{S}_a$ such that $n!\sigma_n\equiv -nxv^{n-1} \pmod{aB}$.

For uniqueness, let $\hat s=(s_n)\in\mathcal{S}_a$ be such that $n!s_n\equiv -nxv^{n-1} \pmod{aB}$ for $n\geq 1$. Choose $N\geq 1$ such that 6 does not divide N, and let $e\geq 0$ be such that $1\leq N-6e\leq 5$. By Corollary 5.5(c), a basis for $A_{(2N+1,N)}$ is given by

$$s_N', s_{N-6}'h, s_{N-12}'h^2,\ldots,s_{N-6e}'h^e, \quad \text{where } s_n':=n!s_n.$$

Therefore, there exist $c_i\in k$ with $N!\sigma_N=c_0s_N'+c_1s_{N-6}'h+\cdots+c_es_{N-6e}'h^e$. The substitution $a\mapsto 0$ yields

$$-Nxv^{N-1}=-c_0Nxv^{N-1}-c_1(N-6)xv^{N-7}h'-\cdots-c_e(N-6e)xv^{N-6e-1}(h')^e,$$

where $h'=3x^2(2xz-y^2)$. This implies that $c_0=1$ and $c_1=\cdots=c_e=0$, meaning that $\sigma_N=s_N$. Therefore, $\hat\sigma$ and $\hat s$ agree on an infinite number of vertices, which implies that $\hat\sigma=\hat s$ (see Section 3.1(vi)). This proves the uniqueness assertion.

To prove properties (a) and (b), recall that $\sigma_i(0)\in aB$ for all $i\geq 0$ with $i\neq 1$. Hence, $\sigma_i(0)\sigma_{n-i}(0)\in a^2B$ $(0\leq i\leq n/2,\ i\neq 1)$ if $n\neq 1$, and $\sigma_i(0)\sigma_{n-i}(0)\in aB$ $(0\leq i\leq n/2)$ if $n\neq 2$. To show (a), suppose that $\sigma_0\sigma_1h^e\in\langle\sigma_i\sigma_{n-i}\mid 0\leq i\leq n/2, i\neq 1\rangle$. Then, we have

$$-axh^e=(\sigma_0\sigma_1h^e)|_{v=0}\in\big\langle\sigma_i(0)\sigma_{n-i}(0)\,\big|\,0\leq i\leq n/2, i\neq 1\big\rangle\subset a^2B,$$

and so $xh^e\in aB$, a contradiction. Since $Fh^e\in R\setminus aB$, property (b) is proved similarly. □

We remark that Theorem 5.7(b), together with Lemma 4.1(b), implies $R\cap A_{(2n+2,n)}\cap\phi_{\hat\sigma}(\Omega_{(2,n)})=\{0\}$ if $n\equiv 2 \pmod 6$ and $n\neq 2$.

COROLLARY 5.8

Let $S\subset k[x,v]=k^{[2]}$ be the subalgebra $S=k[x,xv,xv^2,\ldots]$. Given $\lambda\in k$, put $J_\lambda=aA+(h-\lambda)A$. Then A/J_λ is isomorphic to S. In particular, J_λ is a prime ideal of A for each $\lambda\in k$.

Proof

Let $\hat\sigma\in\mathcal{S}_a$ be as in Theorem 5.7. By Theorem 5.1, we have $A=k[h,\hat\sigma]$.

Given $f \in B$, let $f(0)$ denote the evaluation of f at $a = 0$. Since $D(a) = 0$, we have $aB \cap A = aA$. Indeed, if $b \in B$ is such that $ab \in A$, then $aD(b) = D(ab) = 0$, and so $D(b) = 0$. Hence, the kernel of the map $A \to B$ defined by $f \to f(0)$ equals aA. Therefore,

$$\begin{aligned}\mathfrak{A} := A/aA &\cong k\big[h(0), \sigma_0(0), \sigma_1(0), \sigma_2(0), \ldots\big] = k\big[h(0), x, xv, xv^2, \ldots\big] \\ &= S\big[h(0)\big] = S^{[1]}.\end{aligned}$$

The last equality holds because $h(0) = 6x^3z - 3x^2y^2$ is transcendental over $k[x, v]$. We conclude that

$$A/J_\lambda \cong \mathfrak{A}/\big(h(0) - \lambda\big)\mathfrak{A} \cong S. \qquad \square$$

6. Relations in $\bar{A}$

We continue the notation of the preceding section. The main goal of this section is to show the following.

THEOREM 6.1

For every $\hat{s} \in \mathcal{S}_a$, we have $\ker\phi_{\pi\hat{s}} = \mathcal{Q}_4$. Consequently, $\bar{A} \cong_k \Omega/\mathcal{Q}_4$ and $\mathcal{Q}_4$ is a homogeneous prime ideal of Ω.

6.1. Quadratic relations

Let $\hat{s} \in \mathcal{S}_a$ be given, and let $\{\hat{\theta}_n\}$ be a Δ-basis for Ω_2, where $\hat{\theta}_n = (\theta_n^{(j)})$ for given n.

LEMMA 6.2 (a) *If $n \geq 4$ is even, then $\theta_n^{(j)} \in \ker\phi_{\pi\hat{s}}$ holds for any $j \geq 0$.*

(b) *$\langle\theta_0^{(j)}, \theta_2^{(j-2)}\rangle \cap \ker\phi_{\pi\hat{s}} = \{0\}$ holds for every $j \geq 0$, where $\theta_2^{(j-2)} = 0$ if $j = 0, 1$.*

Proof

(a) Fixing $n \geq 4$, we proceed by induction on j to show that $\theta_n^{(j)} \in \ker\phi_{\pi\hat{s}}$ for each $j \geq 0$. We have

$$\delta\phi_{\pi\hat{s}}(\theta_n^{(0)}) = \phi_{\pi\hat{s}}\Delta(\theta_n^{(0)}) = 0 \quad \Rightarrow \quad \phi_{\pi\hat{s}}(\theta_n^{(0)}) \in \ker\delta = k[\bar{a}, \bar{F}, \bar{G}].$$

From line (5) in Remark 4.3, we have that $\bar{F} = \phi_{\pi\hat{s}}(2x_0x_2 - x_1^2)$ and $-\bar{G} = \phi_{\pi\hat{s}}(3x_0^2x_3 - 3x_0x_1x_2 + x_1^3)$. Therefore, there exists $P \in \ker\phi_{\pi\hat{s}} \cap \Omega_{(2,n)}$ such that

$$\begin{aligned}\theta_n^{(0)} - P &\in k[x_0, 2x_0x_2 - x_1^2, 3x_0^2x_3 - 3x_0x_1x_2 + x_1^3] \cap \Omega_2 \\ &= k \cdot x_0^2 + k \cdot (2x_0x_2 - x_1^2) \subset \Omega_{(2,0)} + \Omega_{(2,2)}.\end{aligned}$$

Since $\theta_n^{(0)}, P \in \Omega_{(2,n)}$ and $n \geq 4$, we conclude that $\theta_n^{(0)} = P \in \ker\phi_{\pi\hat{s}}$. This gives the basis for induction.

Assume that $\theta_n^{(j-1)} \in \ker\phi_{\pi\hat{s}}$ for $j \geq 1$. Then

$$0 = \phi_{\pi\hat{s}}(\theta_n^{(j-1)}) = \phi_{\pi\hat{s}}\Delta(\theta_n^{(j)}) = \delta\phi_{\pi\hat{s}}(\theta_n^{(j)}) \quad \Rightarrow \quad \phi_{\pi\hat{s}}(\theta_n^{(j)}) \in \ker\delta.$$

Since $\theta_n^{(j)} \in \Omega_{(2,n+j)}$, we conclude as above that $\theta_n^{(j)} \in \ker \phi_{\pi\hat{s}}$. This proves part (a).

(b) Since $\theta_0^{(j)} = x_0 x_j \notin \ker \phi_{\pi\hat{s}}$ for $j = 0,1$, the assertion holds for $j = 0,1$. By Lemma 3.8(a), we have

$$\langle \phi_{\hat{s}}(\theta_0^{(2)}), \phi_{\hat{s}}(\theta_2^{(0)}) \rangle = \phi_{\hat{s}}(\Omega_{(2,2)}) = \phi_{\hat{s}}(\langle \beta_0^{(2)}, \beta_2^{(0)} \rangle) = \langle as_2, s_1^2 \rangle.$$

Since $\dim\langle as_2, s_1^2\rangle = 2$ and $\langle as_2, s_1^2\rangle \cap hB \subset B_{(6,2)} \cap hB = \{0\}$, the assertion also holds for $j = 2$. We prove the case $j \geq 3$ by contradiction. Let $j \geq 3$ be the smallest integer for which there exists $(0,0) \neq (\alpha, \beta) \in k^2$ such that $f := \alpha\theta_0^{(j)} + \beta\theta_2^{(j-2)} \in \ker \phi_{\pi\hat{s}}$. Then

$$0 = \phi_{\pi\hat{s}}(f) \quad \Rightarrow \quad 0 = \delta\phi_{\pi\hat{s}}(f) = \phi_{\pi\hat{s}}\Delta(f) = \phi_{\pi\hat{s}}(\alpha\theta_0^{(j-1)} + \beta\theta_2^{(j-3)})$$

and so $\alpha\theta_0^{(j-1)} + \beta\theta_2^{(j-3)} \in \ker \phi_{\pi\hat{s}}$. This contradicts the minimality of j, proving part (b). □

Combining Lemmas 3.8 and 6.2, we obtain the following result.

LEMMA 6.3 (a) *Given $j \geq 4$, the set $\{\theta_{2i}^{(j-2i)} \mid 2 \leq i \leq j/2\}$ is a basis for $\Omega_{(2,j)} \cap \ker \phi_{\pi\hat{s}}$.*

(b) *The vertices of $\hat{\theta}_n$ ($n \in 2\mathbb{N}$, $n \geq 4$) form a basis for $\Omega_2 \cap \ker \phi_{\pi\hat{s}}$.*

6.2. Proof of Theorem 6.1

Note that, by Corollary 5.5(a), if $\hat{t} \in \mathcal{S}_a$, then $\pi\hat{t} = \pi\hat{s}$. So there is no loss in generality in assuming that $\hat{s} = \hat{\sigma}$, where $\hat{\sigma}$ is the ∂-cable specified in Theorem 5.7.

By Lemma 6.3(b), the ideal generated by $\Omega_2 \cap \ker \phi_{\pi\hat{\sigma}}$ equals $\mathcal{Q}_4$. Since $\phi_{\pi\hat{\sigma}}$ is a homogeneous ideal of Ω, it suffices to show that

$$\Omega_{(r,s)} \cap \ker \phi_{\pi\hat{\sigma}} \subset \mathcal{Q}_4 \quad (r, s \geq 0).$$

Let nonzero $\zeta \in \Omega_{(r,s)} \cap \ker \phi_{\pi\hat{\sigma}}$ be given $(r, s \geq 0)$. Then $r \geq 2$. We prove $\zeta \in \mathcal{Q}_4$ by induction on r, where the case $r = 2$ holds as mentioned. Assume that $r \geq 3$. By Theorem 3.12(c) we have

$$\Omega_r \subset (x_0, x_1)^{r-1} + \mathcal{Q}_4.$$

So it suffices to assume that $\zeta \in (x_0, x_1)^{r-1}$. By degree considerations, we see that ζ is a linear combination of the monomials

$$x_0^{r-i-1} x_1^i x_{s-i} \qquad \text{such that } r - i - 1, i, s - i \geq 0.$$

Suppose that x_0 does not divide ζ. Then $s - r + 1 \geq 1$, and there exist $\zeta_0 \in \Omega_{r-1}$ and nonzero $c \in k$ with $\zeta = x_0\zeta_0 + c x_1^{r-1} x_{s-r+1}$. Since $\zeta \in \ker \phi_{\pi\hat{\sigma}}$, we see that $\phi_{\hat{\sigma}}(\zeta) \in hA$, which implies that, for some $q \in A$,

$$c\sigma_1^{r-1}\sigma_{s-r+1} = hq - a\phi_{\hat{\sigma}}(\zeta_0). \tag{9}$$

By Theorem 5.7, we have that $n!\sigma_n \equiv -nxv^{n-1} (\mathrm{mod}\ aB)$ for each $n \geq 1$. From (9), it follows that

$$\frac{c}{(s-r)!}(-x)^r v^{s-r} = 3x^2(2xz - y^2) \cdot q|_{a=0}.$$

Since $c \neq 0$, this is a contradiction. Therefore, x_0 divides ζ. If $\zeta = x_0\zeta_0$ for $\zeta_0 \in \Omega$, then $\zeta_0 \in \Omega_{(r-1,s)} \cap \ker \phi_{\pi\hat{\sigma}}$. We conclude by induction on r that $\zeta_0 \in \mathcal{Q}_4$. Therefore, $\zeta \in \mathcal{Q}_4$. This completes the proof of Theorem 6.1. □

EXAMPLE 6.4

Consider the well-known cubic Δ-invariant given by

$$\xi = 2x_2^3 + 9x_0x_3^2 - 6x_1x_2x_3 - 12x_0x_2x_4 + 6x_1^2x_4.$$

Let $\hat{\theta}_4$ be a Δ-cable rooted at $\theta_4^{(0)}$ such that

$$\theta_4^{(2)} = 5x_1x_5 - 8x_2x_4 + \frac{9}{2}x_3^2, \qquad \theta_4^{(1)} = 5x_0x_5 - 3x_1x_4 + x_2x_3,$$
$$\theta_4^{(0)} = 2x_0x_4 - 2x_1x_3 + x_2^2.$$

We have

$$\frac{1}{2}\xi = x_0\theta_4^{(2)} - x_1\theta_4^{(1)} + x_2\theta_4^{(0)} \in \mathcal{Q}_4.$$

Notice that, to express $\xi \in k[x_0, x_1, x_2, x_3, x_4]$ by using quadratics in $\mathcal{Q}_4$, it was necessary to use x_5.

EXAMPLE 6.5

Since the transcendence degree of $\bar{A}$ over k is 3, $\bar{s}_0, \bar{s}_1, \bar{s}_2, \bar{s}_3$ are algebraically dependent in $\bar{A}$. Their minimal algebraic relation is quartic and can be obtained as follows.

Let ξ be as in the preceding example. The x_4-coefficient of ξ is $-6\theta_2^{(0)}$, and the x_4-coefficient of $\theta_4^{(0)}$ is $2x_0$. Thus, to eliminate x_4, we take

$$\chi := 3\theta_2^{(0)}\theta_4^{(0)} + x_0\xi = 9x_0^2x_3^2 - 3x_1^2x_2^2 + 8x_0x_2^3 - 18x_0x_1x_2x_3 + 6x_1^3x_3.$$

We see that $\chi \in k[x_0, x_1, x_2, x_3] \cap \ker \Delta \cap \mathcal{Q}_4$. Since χ is irreducible, χ is a minimal algebraic relation among $\bar{s}_0$, $\bar{s}_1$, $\bar{s}_2$, and $\bar{s}_3$.

REMARK 6.6

Let $\hat{\eta}_4$ be the Δ-cable belonging to the small Δ-basis for Ω_2. According to Lemma 6.2, $\hat{\eta}_4 \subset \ker \phi_{\pi\hat{s}}$ for every $\hat{s} \in \mathcal{S}_a$. Recall that

$$\eta_4^{(j)} = \frac{(j+1)(j+4)}{2}x_0x_{4+j} - (j+2)x_1x_{3+j} + x_2x_{2+j}.$$

Since we know $\bar{s}_0, \bar{s}_1, \bar{s}_2, \bar{s}_3$ (see Remark 4.3), we can easily determine the δ-cable $\pi\hat{s}$ by using these 3-term recursion relations in $\bar{A}$.

7. Relations in A

Let $\Omega[t] = \Omega^{[1]}$, and extend the $\mathbb{Z}^2$-grading on Ω to $\Omega[t]$ by setting $\deg t = (0,6)$. Note that $\Omega[t]_r = \Omega_r[t]$ for each $r \geq 0$. In addition,

$$\Omega[t]_{(r,n)} = \Omega_{(r,n)} \oplus t \cdot \Omega_{(r,n-6)} \oplus \cdots \oplus t^e \cdot \Omega_{(r,n-6e)} \quad \text{where } 0 \leq n - 6e \leq 5.$$

Extend Δ to $\tilde{\Delta}$ on $\Omega[t]$ by setting $\tilde{\Delta}(t) = 0$. Then $\tilde{\Delta}$ is homogeneous and $\deg \tilde{\Delta} = (0,-1)$. Since $\Delta : \Omega_{(r,s)} \to \Omega_{(r,s-1)}$ is surjective for each $r,s \geq 1$, we see that $\tilde{\Delta} : \Omega[t]_{(r,n)} \to \Omega[t]_{(r,n-1)}$ is surjective for each $r,n \geq 1$. Given $n \geq 0$, define the vector space

$$V_n = \Omega[t]_{(2,n)} \cap \ker \tilde{\Delta} = \Omega[t]_{(2,n)} \cap (\ker \Delta)[t].$$

Since $\ker \Delta \cap \Omega_{(2,s)}$ equals $\{0\}$ if s is odd and equals $k \cdot \theta_s^{(0)}$ if s is even as mentioned in Section 3.2.2, the reader can easily check that $V_n = \{0\}$ if n is odd and that for n even

(10) $$V_n = \langle \theta_n^{(0)}, t\theta_{n-6}^{(0)}, \ldots, t^e \theta_{n-6e}^{(0)} \rangle \quad \text{where } n - 6e \in \{0,2,4\}.$$

7.1. The mapping $\Phi_{\hat{s}}$

Let $\hat{s} \in \mathcal{S}_a$. By Theorem 5.1(a), $\phi_{\hat{s}} : \Omega \to A$ extends to the surjection

$$\Phi_{\hat{s}} : \Omega[t] \to A, \qquad \Phi_{\hat{s}}(t) = h.$$

Note that $\Phi_{\hat{s}} \hat{\Delta} = \partial \Phi_{\hat{s}}$, since $\phi_{\hat{s}} \Delta = \partial \phi_{\hat{s}}$, $\Phi_{\hat{s}} \hat{\Delta} t = 0$, and $\partial \Phi_{\hat{s}} t = 0$.

THEOREM 7.1

There exists a set $\{\hat{\Theta}_4, \hat{\Theta}_6, \hat{\Theta}_8, \ldots\}$ of homogeneous $\tilde{\Delta}$-cables such that $\hat{\Theta}_n$ is rooted in V_n for each n and

$$\ker \Phi_{\hat{s}} = (\hat{\Theta}_4, \hat{\Theta}_6, \hat{\Theta}_8, \ldots).$$

Proof

The proof proceeds in three steps.

Step 1. This step constructs a set $\{\hat{\Theta}_4, \hat{\Theta}_6, \hat{\Theta}_8, \ldots\}$ of homogeneous $\tilde{\Delta}$-cables such that $\hat{\Theta}_n$ is rooted in V_n for each n and $(\hat{\Theta}_4, \hat{\Theta}_6, \hat{\Theta}_8, \ldots) \subset \ker \Phi_{\hat{s}}$. For the integer $n \geq 4$, write $n = 6e + \ell$ $(e \geq 0, 0 \leq \ell \leq 5)$. Given $P \in V_n$, we have

$$0 = \Phi_{\hat{s}} \tilde{\Delta}(P) = \partial \Phi_{\hat{s}}(P) \quad \Rightarrow \quad \Phi_{\hat{s}}(V_n) \subset \ker \partial = R \cap A.$$

Since $\Phi_{\hat{s}}(V_n) \subset \Phi_{\hat{s}}(\Omega[t]_{(2,n)}) \subset A_{(2n+2,n)}$, it follows that

(11) $$\Phi_{\hat{s}}(V_n) \subset R \cap A_{(2n+2,n)} = \begin{cases} \langle a^2 h^e \rangle & \ell = 0, \\ \langle F h^e \rangle & \ell = 2, \\ \{0\} & \text{otherwise,} \end{cases}$$

by Lemma 4.1(b). Now assume that n is even. In view of (10), there exists $c_n \in k$ such that $\Phi_{\hat{s}}(\theta_n^{(0)}) = c_n \Phi_{\hat{s}}(t^e \theta_\ell^{(0)})$. Note that we may take $c_n = 0$ when $\ell = 4$. Then, we have

(12) $$\Theta_n^{(0)} := \theta_n^{(0)} - c_n t^e \theta_\ell^{(0)} \in \ker \Phi_{\hat{s}} - \{0\},$$

since $e \geq 1$ except when $n = 4$. Suppose that, for some $j \geq 0$, we have constructed $\Theta_n^{(0)}, \ldots, \Theta_n^{(j)} \in \ker \Phi_{\hat{s}}$, which satisfy $\Theta_n^{(i)} \in \Omega[t]_{(2,n+i)}$ and $\tilde{\Delta}\Theta_n^{(i)} = \Theta_n^{(i-1)}$, $1 \leq i \leq j$. Since the mapping

$$\tilde{\Delta} : \Omega[t]_{(2,n+j+1)} \to \Omega[t]_{(2,n+j)}$$

is surjective, we may choose $P \in \Omega[t]_{(2,n+j+1)}$ with $\tilde{\Delta}P = \Theta_n^{(j)}$. We have

$$0 = \Phi_{\hat{s}}\Theta_n^{(j)} = \Phi_{\hat{s}}\tilde{\Delta}(P) = \partial\Phi_{\hat{s}}(P) \quad \Rightarrow \quad \Phi_{\hat{s}}(P) \in R \cap A_{(2(n+j+1)+2,n+j+1)}.$$

We again apply the equality in (11). In fact, if $\ell \in \{0,2\}$, then $\theta_{n-6e}^{(0)} = \theta_\ell^{(0)} \notin \ker \phi_{\pi\hat{s}}$ by Lemma 6.2(b), and so $\Phi_{\hat{s}}(t^e\theta_{n-6e}^{(0)}) = h^e\phi_{\hat{s}}(\theta_{n-6e}^{(0)}) \neq 0$. Thus, as above, there exist $\kappa \in k$ and $\epsilon, l \in \mathbb{N}$ with

$$\Theta_n^{(j+1)} := P - \kappa t^\epsilon \theta_l^{(0)} \in \ker \Phi_{\hat{s}} \cap \Omega[t]_{(2,n+j+1)},$$

where $\kappa = 0$ if $n+j+1$ is odd, since $V_{n+j+1} = \{0\}$. Then, we have $\tilde{\Delta}\Theta_n^{(j+1)} = \Theta_n^{(j)}$, since $\tilde{\Delta}(t^\epsilon\theta_l^{(0)}) = 0$. Therefore, for each even $n \geq 4$, there exists a homogeneous $\tilde{\Delta}$-cable $\hat{\Theta}_n$ rooted in V_n and contained in $\ker \Phi_{\hat{s}} \cap \Omega[t]_2$. Note that $\Theta_4^{(j)} = \theta_4^{(j)}$ for $j = 0, 1$ by construction.

Step 2. By construction, the ideal $J := (\hat{\Theta}_4, \hat{\Theta}_6, \hat{\Theta}_8, \ldots)$ of $\Omega[t]$ is contained in $\ker \Phi_{\hat{s}}$. This step shows that $\ker \Phi_{\hat{s}} \subset J + (t)$. Define polynomials $H_n^{(j)} \in \Omega_{(2,n+j)}$ ($n \in 2\mathbb{N}$, $n \geq 4$, $j \geq 0$) by $H_n^{(j)} = \Theta_n^{(j)}|_{t=0}$. Note that, by (12), we have $H_n^{(0)} = \theta_n^{(0)} \neq 0$. Therefore, by Section 3.1(xi), for each even $n \geq 4$, $\hat{H}_n := (H_n^{(j)})$ is a homogeneous Δ-cable rooted at $\theta_n^{(0)}$. By Definition 3.10(2) and Lemma 3.12(b), we get

$$\mathcal{Q}_4 + (t) = (\hat{H}_4, \hat{H}_6, \hat{H}_8, \ldots) + (t) = (\hat{\Theta}_4, \hat{\Theta}_6, \hat{\Theta}_8, \ldots) + (t) = J + (t).$$

Consider the map $\pi\Phi_{\hat{s}} : \Omega[t] \xrightarrow{\Phi_{\hat{s}}} A \xrightarrow{\pi} A/hA$. Since $\pi\Phi_{\hat{s}}|_\Omega = \phi_{\pi\hat{s}}$, we see from Theorem 6.1 that

$$\ker \Phi_{\hat{s}} \subset \ker \pi\Phi_{\hat{s}} = \mathcal{Q}_4 + (t) = J + (t).$$

Step 3. This step shows that $J = \ker \Phi_{\hat{s}}$. Since $\Phi_{\hat{s}}(\Omega[t]_{(r,s)}) \subset A_{(2s+r,s)}$ for each $r, s \geq 0$, we see that $\ker \Phi_{\hat{s}}$ is a homogeneous ideal of $\Omega[t]$. So, given integers $r, N \geq 0$, we show by induction on N that

$$\ker \Phi_{\hat{s}} \cap \Omega[t]_{(r,N)} \subset J. \tag{13}$$

If $r \leq 1$, then $\ker \Phi_{\hat{s}} \cap \Omega[t]_{(r,N)} = \{0\}$, so assume that $r \geq 2$.

Consider first the case in which $0 \leq N \leq 5$. In this case, $\Omega[t]_{(r,N)} = \Omega_{(r,N)} = k[x_0, \ldots, x_N]_{(r,N)}$, since $\deg t = (0,6)$. Let

$$P \in \ker \Phi_{\hat{s}} \cap \Omega[t]_{(r,N)} = \ker \phi_{\hat{s}} \cap k[x_0, \ldots, x_N]_{(r,N)}$$

be given. If $N \leq 3$, then $P = 0$, since s_0, s_1, s_2, s_3 are algebraically independent over k (see Remark 4.3).

Suppose that $N = 4$. The only monomial in $k[x_0, \ldots, x_4]_{(r,4)}$ in which x_4 appears is $x_0^{r-1}x_4$. Therefore, noting that $\theta_4^{(0)} = 2(x_0x_4 - x_1x_3) + x_2^2$, we have

$$k[x_0, \ldots, x_4]_{(r,4)} = k \cdot x_0^{r-1}x_4 \oplus k[x_0, \ldots, x_3]_{(r,4)} = k \cdot x_0^{r-2}\theta_4^{(0)} \oplus k[x_0, \ldots, x_3]_{(r,4)}.$$

So there exists $\lambda \in k$ such that $P - \lambda x_0^{r-2}\theta_4^{(0)} \in k[x_0, \ldots, x_3]$. Since $\theta_4^{(0)} \in \ker \phi_{\hat{s}}$ by Lemma 7.2(a) below, we get $P - \lambda x_0^{r-2}\theta_4^{(0)} \in \ker \phi_{\hat{s}} \cap k[x_0, \ldots, x_3] = \{0\}$. Since $\theta_4^{(0)} = \Theta_4^{(0)} \in J$, $P \in J$ in this case.

Suppose that $N = 5$. The only monomial in $k[x_0, \ldots, x_5]_{(r,5)}$ in which x_5 appears is $x_0^{r-1}x_5$. Therefore, noting that $\theta_4^{(1)} = 5x_0x_5 - 3x_1x_4 + x_2x_3$, we have

$$k[x_0, \ldots, x_5]_{(r,5)} = k \cdot x_0^{r-1}x_5 \oplus k[x_0, \ldots, x_4]_{(r,5)} = k \cdot x_0^{r-2}\theta_4^{(1)} \oplus k[x_0, \ldots, x_4]_{(r,5)}.$$

Since $\theta_4^{(1)} \in \ker \phi_{\hat{s}}$ by Lemma 7.2(a) below, there exists $\lambda \in k$ such that

$$P - \lambda x_0^{r-2}\theta_4^{(1)} \in \ker \phi_{\hat{s}} \cap k[x_0, \ldots, x_4]_{(r,5)}$$

as above. Similarly, the only monomial in $k[x_0, \ldots, x_4]_{(r+1,5)}$ in which x_4 appears is $x_0^{r-1}x_1x_4$. Therefore,

$$\begin{aligned} k[x_0, \ldots, x_4]_{(r+1,5)} &= k \cdot x_0^{r-1}x_1x_4 \oplus k[x_0, \ldots, x_3]_{(r+1,5)} \\ &= k \cdot x_0^{r-2}x_1\theta_4^{(0)} \oplus k[x_0, \ldots, x_3]_{(r+1,5)}. \end{aligned}$$

So there exists $\mu \in k$ such that

$$x_0P - \lambda x_0^{r-1}\theta_4^{(1)} - \mu x_0^{r-2}x_1\theta_4^{(0)} \in \ker \phi_{\hat{s}} \cap k[x_0, \ldots, x_3] = \{0\}.$$

If $r = 2$, then $\mu x_1\theta_4^{(0)} \in x_0\Omega$ implies $\mu = 0$ and $P = \lambda x_0^{r-2}\theta_4^{(1)}$. If $r \geq 3$, then

$$P = \lambda x_0^{r-2}\theta_4^{(1)} + \mu x_0^{r-3}x_1\theta_4^{(0)}.$$

In either case, $P \in J$, since $\theta_4^{(1)} = \Theta_4^{(1)} \in J$. Therefore, the inclusion (13) holds when $0 \leq N \leq 5$, which gives the basis for induction.

Suppose that N_0 is an integer such that $N_0 \geq 5$ and (13) holds for all integers $0 \leq N \leq N_0$. Let $P \in \ker \Phi_{\hat{s}} \cap \Omega[t]_{(r,M)}$ be given, where $N_0 < M \leq N_0 + 6$. We show that P is of the form

$$\text{(14)} \qquad P = P_J + tQ \quad \text{where } P_J \in J \cap \Omega[t]_{(r,M)} \text{ and } Q \in \Omega[t]_{(r,M-6)}.$$

Since $\ker \Phi_{\hat{s}} \subset J + (t)$ by Step 2, we may write $P = E + C$ for $E \in J$ and $C \in t \cdot \Omega[t]$. Since J and $t \cdot \Omega[t]$ are homogeneous ideals, each homogeneous summand of E belongs to J, and each homogeneous summand of C belongs to $t \cdot \Omega[t]$. Since P is homogeneous, statement (14) holds.

In addition, since $P_J \in J \subset \ker \Phi_{\hat{s}}$, we have

$$tQ = P - P_J \in \ker \Phi_{\hat{s}} \quad \Rightarrow \quad Q \in \ker \Phi_{\hat{s}} \cap \Omega[t]_{(r,M-6)}.$$

By the inductive hypothesis, $Q \in J$, which implies $P \in J$. Therefore, statement (13) holds for all $N \geq 0$. This proves $J = \ker \Phi_{\hat{s}}$. □

7.2. The cable $\hat{\sigma}$

Let $\hat{\sigma} \in \mathcal{S}_a$ be the ∂-cable defined in Theorem 5.7. The goal of this section is to give an explicit recursive definition of $\hat{\sigma}$ (see Theorem 7.6).

LEMMA 7.2

Let $n \in 2\mathbb{N}$, $n \geq 4$, and $\hat{s} \in \mathcal{S}_a$ be given.

(a) *If* $n \equiv 4 \pmod 6$*, then* $\theta_n^{(0)}, \theta_n^{(1)} \in \ker \phi_{\hat{s}}$ *for every* $\hat{s} \in \mathcal{S}_a$*.*
(b) *If* $n \equiv 2 \pmod 6$*, then* $\theta_n^{(0)}, \theta_n^{(1)} \in \ker \phi_{\hat{\sigma}}$*.*

Proof
For both (a) and (b), it suffices to show that $\theta_n^{(0)} \in \ker \phi_{\hat{s}}$, since

$$0 = \phi_{\hat{s}}(\theta_n^{(0)}) = \phi_{\hat{s}}\Delta(\theta_n^{(1)}) = \partial\phi_{\hat{s}}(\theta_n^{(1)}) \quad \Rightarrow$$
$$\phi_{\hat{s}}(\theta_n^{(1)}) \in \ker \partial|_{A_{(2n+4,n+1)}} = R \cap A_{(2n+4,n+1)} = \{0\}$$

by Lemma 4.1(b) with $\ell = 5, 3$. If $n \equiv 4 \pmod 6$, then inclusion (11) shows that $\theta_n^{(0)} \in \ker \phi_{\hat{s}}$. This proves part (a). For part (b), write $n = 6e+2$ for some $e \geq 1$. Inclusion (11) shows that

$$\phi_{\hat{\sigma}}(\theta_n^{(0)}) = cFh^e \quad (c \in k).$$

By Theorem 5.7(b), it follows that $\phi_{\hat{\sigma}}(\theta_n^{(0)}) = 0$. This proves part (b). □

For $n \in 2\mathbb{N}$, let J_n be the set of integers $j \geq 3$ such that $n + j \equiv 1 \pmod 6$. In particular, each $j \in J_n$ is odd.

Let $\{\hat{\theta}_n\}$ be a Δ-basis for Ω_2. Given $n \in 2\mathbb{N}$ and $j \in \mathbb{N}$ (and $j \geq 1$ if $n = 0$), let $\xi(\theta_n^{(j)}) \in k$ be the coefficient of $x_1 x_{n+j-1}$ in $\theta_n^{(j)}$. Note that $\xi(\theta_n^{(j)}) = 0$ if and only if $\theta_n^{(j)} \in k[x_0, x_2, x_3, \ldots, x_{n+j}]$, since $\theta_n^{(j)} \in \Omega_{(2,n+j)}$. Define

$$\mu(\hat{\theta}_n) = \min\{j \in J_n \mid \xi(\theta_n^{(j)}) \neq 0\},$$

where it is understood that $\mu(\hat{\theta}_n) = \infty$ if $\xi(\theta_n^{(j)}) = 0$ for all $j \in J_n$.

LEMMA 7.3
If $\mu(\hat{\theta}_n) = \infty$*, then the following are equivalent.*

(i) $\theta_n^{(j)} \in \ker \phi_{\hat{\sigma}}$ *for some* $j \geq 0$*.*
(ii) $\theta_n^{(0)} \in \ker \phi_{\hat{\sigma}}$*.*
(iii) $\theta_n^{(j)} \in \ker \phi_{\hat{\sigma}}$ *for all* $j \geq 0$*.*

Proof
It is clear that (i) $\Leftarrow$ (ii) $\Leftarrow$ (iii). We also have (i) $\Rightarrow$ (ii), since

$$\phi_{\hat{\sigma}}(\theta_n^{(0)}) = \phi_{\hat{\sigma}}(\Delta^j \theta_n^{(j)}) = \partial^j \phi_{\hat{\sigma}}(\theta_n^{(j)}) = 0.$$

We show (ii) $\Rightarrow$ (iii). Suppose that $\theta_n^{(0)} \in \ker \phi_{\hat{\sigma}}$, noting that $n \geq 4$, since $\phi_{\hat{\sigma}}(\theta_0^{(0)})$ and $\phi_{\hat{\sigma}}(\theta_2^{(0)})$ cannot be zero by Lemma 6.2(b). We prove by induction on j that $\theta_n^{(j)} \in \ker \phi_{\hat{\sigma}}$ for all $j \geq 0$.

Assume that $\theta_n^{(j)} \in \ker \phi_{\hat{\sigma}}$ for some $j \geq 0$. Then, $\partial\phi_{\hat{\sigma}}(\theta_n^{(j+1)}) = \phi_{\hat{\sigma}}(\Delta\theta_n^{(j+1)}) = \phi_{\hat{\sigma}}(\theta_n^{(j)}) = 0$. Hence, we get

$$\phi_{\hat{\sigma}}(\theta_n^{(j+1)}) \in \ker \partial|_{A_{(2(n+j+1)+2,n+j+1)}} = R \cap A_{(2(n+j+1)+2,n+j+1)}.$$

Now, suppose that $\theta_n^{(j+1)} \notin \ker \phi_{\hat{\sigma}}$. Then, by Lemma 4.1(b) and the remark after Theorem 5.7, we have $n+j+1 \equiv 0 \pmod 6$ and $\phi_{\hat{\sigma}}(\theta_n^{(j+1)}) = \lambda a^2 h^e$ for some $\lambda \in k^*$ and $e \geq 0$. Note that

$$\partial : A_{(2(n+j+2)+2, n+j+2)} \to A_{(2(n+j+1)+2, n+j+1)}$$

is an injection by Lemma 4.1(b), since $n+j+2 \equiv 1 \pmod 6$. Because $\partial \phi_{\hat{\sigma}}(\theta_n^{(j+2)}) = \phi_{\hat{\sigma}}(\Delta \theta_n^{(j+2)}) = \phi_{\hat{\sigma}}(\theta_n^{(j+1)})$ and $\partial \sigma_0 \sigma_1 h^e = a^2 h^e$, it follows that $\phi_{\hat{\sigma}}(\theta_n^{(j+2)}) = \lambda \sigma_0 \sigma_1 h^e$. By assumption, the monomial $x_1 x_{n+j+1}$ does not appear in $\theta_n^{(j+2)}$. Hence, $\theta_n^{(j+2)}$ is a k-linear combination of $x_i x_{n+j+2-i}$ for $0 \leq i \leq (n+j+2)/2$ with $i \neq 1$. This contradicts Theorem 5.7(a). Therefore, we must have $\theta_n^{(j+1)} \in \ker \phi_{\hat{\sigma}}$. It follows by induction that $\theta_n^{(j)} \in \ker \phi_{\hat{\sigma}}$ for all $j \geq 0$. This completes the proof. □

Combining Lemmas 7.2 and 7.3 gives the following result.

LEMMA 7.4

Suppose that $\{\hat{\theta}_n\}$ is a Δ-basis such that $\mu(\hat{\theta}_n) = \infty$ for each $n = 6e \pm 2$, $e \geq 1$. Define the Q-ideal $\mathcal{J}$ by $\mathcal{J} = (\hat{\theta}_n \mid n = 6e \pm 2, e \geq 1)$. Then $\mathcal{J} \subset \ker \phi_{\hat{\sigma}}$.

We next describe a procedure to modify a given Δ-basis $\{\hat{\theta}_n\}$ to obtain a Δ-basis $\{\hat{\psi}_n\}$ for which $\mu(\hat{\psi}_n) = \infty$ for each n.

Given $n \in 2\mathbb{N}$, if $\mu(\hat{\theta}_n) = \infty$, set $\hat{\psi}_n = \hat{\theta}_n$. If $\mu(\hat{\theta}_n) < \infty$, then define constants

$$j = \mu(\hat{\theta}_n), \qquad m = j - 1, \qquad \text{and} \qquad c = \frac{\xi(\theta_n^{(j)})}{n+j-2},$$

noting that $j \geq 3$ is odd and $\xi(\theta_{n+j-1}^{(1)}) = -(n+j-2) \neq 0$. It follows that

$$\mu(\hat{\theta}_n) < \mu(\hat{\theta}_n +_m c\hat{\theta}_{n+m}).$$

If $\mu(\hat{\theta}_n +_m c\hat{\theta}_{n+m}) = \infty$, set $\hat{\psi}_n = \hat{\theta}_n +_m c\hat{\theta}_{n+m}$. If $\mu(\hat{\theta}_n +_m c\hat{\theta}_{n+m}) < \infty$, the process can be repeated. Continuing in this way, we construct a strictly increasing sequence $\vec{m} = \{m_i\}_{i \in I}$ of positive integers, together with sequences $\vec{c} = \{c_i\}_{i \in I}$ for $c_i \in k^*$ and $\vec{s} = \{\hat{\theta}_{n+m_i}\}_{i \in I}$ such that if $\hat{\psi}_n = \lim(\vec{s}, \vec{m}, \vec{c})$, then $\mu(\hat{\psi}_n) = \infty$.

Note that, with this algorithm, $\{\hat{\psi}_n\}$ is uniquely determined by $\{\hat{\theta}_n\}$. The resulting Δ-basis $\{\hat{\psi}_n\}$ is the *reduction* of $\{\hat{\theta}_n\}$.

To illustrate, let $\{\hat{\psi}_n\}$ be the reduction of the balanced Δ-basis $\{\hat{\beta}_n\}$. Assume that $n \equiv 4 \pmod 6$. Then $\xi(\beta_n^{(3)}) = -\binom{n+2}{3}$, and if

$$c = -\frac{\binom{n+2}{3}}{n+3-2} = -\frac{n(n+2)}{6},$$

then the first eight terms of $\hat{\psi}_n$ equal those of $\hat{\beta}_n +_2 c\hat{\beta}_{n+2}$. In particular, we have

$$(15) \quad \psi_n^{(2)} = \beta_n^{(2)} - \frac{n(n+2)}{6}\beta_{n+2}^{(0)} = \frac{1}{6}\sum_{i=0}^{n+2}(-1)^i\big(3i(i-1) - n(n+2)\big)x_i x_{n+2-i}$$

and

$$\psi_n^{(3)} = \beta_n^{(3)} - \frac{n(n+2)}{6}\beta_{n+2}^{(1)} \tag{16}$$
$$= \frac{1}{6}\sum_{i=1}^{n+3}(-1)^{i+1}\big((i-1)(i-2)-n(n+2)\big)ix_ix_{n+3-i}.$$

Note that, by Lemma 7.4, $\psi_n^{(2)}$ and $\psi_n^{(3)}$ above both belong to $\ker\phi_{\hat\sigma}$.

REMARK 7.5

The results of this section show that a Δ-basis of the type described in Lemma 7.4 exists, and therefore, $\mathcal{J} \subset \ker\phi_{\hat\sigma}$ for the associated Q-ideal $\mathcal{J}$. But we do not know if $\mathcal{J} = \ker\phi_{\hat\sigma}$.

Next, let $\{\hat\psi_n\}$ be the reduction of the balanced Δ-basis $\{\hat\beta_n\}$. The Δ-cables $\hat\psi_n$ for $n = 6e \pm 2$ $(e \geq 1)$ give us a way to implicitly calculate the ∂-cable $\hat\sigma$. Recall that $\sigma_0, \ldots, \sigma_5$ are uniquely determined and are given in Remark 4.3.

THEOREM 7.6

For $n \geq 2$, we have

$$\sigma_n = \frac{1}{2a}\sum_{i=1}^{n-1}(-1)^{i+1}\sigma_i\sigma_{n-i} \quad \text{if } n \equiv 2, 4 \pmod 6,$$

$$\sigma_n = \frac{1}{na}\sum_{i=1}^{n-1}(-1)^{i}i\sigma_i\sigma_{n-i} \quad \text{if } n \equiv 3, 5 \pmod 6,$$

$$\sigma_n = \frac{1}{n(n+1)a}\sum_{i=1}^{n-1}(-1)^{i+1}\big(3i(i-1)-n(n-2)\big)\sigma_i\sigma_{n-i} \quad \text{if } n \equiv 0 \pmod 6,$$

$$\sigma_n = \frac{1}{n(n-1)a}\sum_{i=1}^{n-1}(-1)^{i}\big((i-1)(i-2)-(n-1)(n-3)\big)i\sigma_i\sigma_{n-i}$$
$$\text{if } n \equiv 1 \pmod 6.$$

Proof

The first two equalities are equivalent to $\phi_{\hat\sigma}(\theta_n^{(0)}) = 0$ and $\phi_{\hat\sigma}(\theta_{n-1}^{(1)}) = 0$, respectively, which follow from Lemma 7.2. The last two equalities follow from Lemma 7.4 together with (15) and (16). □

To illustrate, the following relations can be used to construct $\sigma_6, \ldots, \sigma_{19}$:

$$\psi_4^{(2)} = \beta_4^{(2)} - 4\beta_6^{(0)} = 7x_0x_6 - 2x_1x_5 - x_2x_4 + x_3^2,$$
$$\psi_4^{(3)} = \beta_4^{(3)} - 4\beta_6^{(1)} = 7x_0x_7 - 2x_2x_5 + x_3x_4,$$
$$\psi_8^{(0)} = \beta_8^{(0)} = 2x_0x_8 - 2x_1x_7 + 2x_2x_6 - 2x_3x_5 + x_4^2,$$

$$
\begin{aligned}
\psi_8^{(1)} = \beta_8^{(1)} &= 9x_0x_9 - 7x_1x_8 + 5x_2x_7 - 3x_3x_6 + x_4x_5, \\
\psi_{10}^{(0)} = \beta_{10}^{(0)} &= 2x_0x_{10} - 2x_1x_9 + 2x_2x_8 - 2x_3x_7 + 2x_4x_6 - x_5^2, \\
\psi_{10}^{(1)} = \beta_{10}^{(1)} &= 11x_0x_{11} - 9x_1x_{10} + 7x_2x_9 - 5x_3x_8 + 3x_4x_7 - x_5x_6, \\
\psi_{10}^{(2)} = \beta_{10}^{(2)} - 20\beta_{12}^{(0)} &= 26x_0x_{12} - 15x_1x_{11} + 6x_2x_{10} + x_3x_9 - 6x_4x_8 \\
&\quad + 9x_5x_7 - 5x_6^2, \\
\psi_{10}^{(3)} = \beta_{10}^{(3)} - 20\beta_{12}^{(1)} &= 26x_0x_{13} - 15x_2x_{11} + 21x_3x_{10} - 20x_4x_9 + 14x_5x_8 - 5x_6x_7, \\
\psi_{14}^{(0)} = \beta_{14}^{(0)} &= 2x_0x_{14} - 2x_1x_{13} + 2x_2x_{12} - 2x_3x_{11} + 2x_4x_{10} \\
&\quad - 2x_5x_9 + 2x_6x_8 - x_7^2, \\
\psi_{14}^{(1)} = \beta_{14}^{(1)} &= 15x_0x_{15} - 13x_1x_{14} + 11x_2x_{13} - 9x_3x_{12} + 7x_4x_{11} \\
&\quad - 5x_5x_{10} + 3x_6x_9 - x_7x_8, \\
\psi_{16}^{(0)} = \beta_{16}^{(0)} &= 2x_0x_{16} - 2x_1x_{15} + 2x_2x_{14} - 2x_3x_{13} + 2x_4x_{12} - 2x_5x_{11} \\
&\quad + 2x_6x_{10} - 2x_7x_9 + x_8^2, \\
\psi_{16}^{(1)} = \beta_{16}^{(1)} &= 17x_0x_{17} - 15x_1x_{16} + 13x_2x_{15} - 11x_3x_{14} + 9x_4x_{13} \\
&\quad - 7x_5x_{12} + 5x_6x_{11} - 3x_7x_{10} + x_8x_9, \\
\psi_{16}^{(2)} = \beta_{16}^{(2)} - 48\beta_{12}^{(0)} &= 57x_0x_{18} - 40x_1x_{17} + 25x_2x_{16} - 12x_3x_{15} + x_4x_{14} \\
&\quad + 8x_5x_{13} - 25x_6x_{12} + 20x_7x_{11} - 23x_8x_{10} + 12x_9^2, \\
\psi_{16}^{(3)} = \beta_{16}^{(3)} - 48\beta_{12}^{(1)} &= 57x_0x_{19} - 40x_2x_{17} + 65x_3x_{16} - 77x_4x_{15} + 78x_5x_{14} \\
&\quad - 70x_6x_{13} + 55x_7x_{12} - 35x_8x_{11} + 12x_9x_{10}.
\end{aligned}
$$

REMARK 7.7

The reader can compare these relations with relations for the sequence w_n given in [5]. In particular, $w_0, \dots, w_{13}$ are given in [5, pp. 162, 165]. The construction used there is as follows. Given $n = 6e - 4$ ($e \geq 2$), suppose that $w_0, \dots, w_{6e-5}$ are known. Then $w_{6e-4}, \dots, w_{6e+1}$ are defined by solving certain systems of linear equations, but in the language of cables this amounts to finding $\psi_n^{(0)}, \dots, \psi_n^{(5)}$. Our current approach uses the simpler relations

$$\psi_n^{(0)}, \qquad \psi_n^{(1)}, \qquad \psi_{n+2}^{(0)}, \qquad \psi_{n+2}^{(1)}, \qquad \psi_{n+2}^{(2)}, \qquad \psi_{n+2}^{(3)}.$$

Note that if $\psi_n^{(j)} = \sum_{i=0}^n c_{(n,i)}^{(j)} x_i x_{n-i}$, then the coefficient $c_{(n,i)}^{(j)}$ is a polynomial of degree j in i. Thus, using smaller j-values has a big advantage computationally. However, the reader should note that both methods produce the same sequence σ_n, by the uniqueness established in Theorem 5.7.

8. Roberts's derivations in dimension 7

Roberts [12] constructed a family of counterexamples to Hilbert's fourteenth problem in the form of subrings $\mathcal{A}_m \subset k^{[7]}$ for integers $m \geq 2$. Although Roberts does not use the language of derivations, the maps he defines are triangular derivations. In this section, we give a description of the ring $\mathcal{A}_2$ as a cable algebra.

Let $\mathcal{B} = k[X,Y,Z,S,T,U,V] = k^{[7]}$. For $m \geq 2$, the subring $\mathcal{A}_m$ is the kernel of the derivation $\mathcal{D}_m$ of $\mathcal{B}$ defined by

$$S \to X^{m+1}, \qquad T \to Y^{m+1}, \qquad U \to Z^{m+1},$$
$$V \to (XYZ)^m, \qquad X,Y,Z \to 0.$$

Define $H_m \in \mathcal{A}_m$ by $H_m = Y^{m+1}S - X^{m+1}T$. Define an action of the cyclic group $\mathbb{Z}_3 = \langle \alpha \rangle$ on $\mathcal{B}$ by

$$\alpha(X,Y,Z,S,T,U,V) = (Z,X,Y,U,S,T,V).$$

Then α, $\mathcal{D}_m$, and the partial derivative $\partial/\partial V$ commute pairwise with each other. Therefore, α and $\partial/\partial V$ restrict to $\mathcal{A}_m$. We denote the restriction of $\partial/\partial V$ to $\mathcal{A}_m$ by δ_m.

Let $m \geq 2$ be given. In [12, Lemma 3], Roberts showed the existence of a sequence in $\mathcal{A}_m$ of the form $XV^i + (\text{terms of lower degree in } V)$, $i \geq 0$. By combining this with homogeneity conditions, he concluded that $\mathcal{A}_m$ is not finitely generated over k. Note that, by applying α, we also obtain sequences in $\mathcal{A}_m$ of the form $YV^i + (\text{terms of lower degree in } V)$ and $ZV^i + (\text{terms of lower degree in } V)$ for $i \geq 0$. The second author showed the following.

THEOREM 8.1 ([8, THEOREM 3.3])

Given $m \geq 2$, let $I_{(m,X,i)}, I_{(m,Y,i)}, I_{(m,Z,i)} \in \mathcal{A}_m$ $(i \geq 0)$ be sequences of the form

$$I_{(m,X,i)} = XV^i + (\textit{terms of lower degree in } V),$$
$$I_{(m,Y,i)} = YV^i + (\textit{terms of lower degree in } V),$$
$$I_{(m,Z,i)} = ZV^i + (\textit{terms of lower degree in } V).$$

Then

$$\mathcal{A}_m = k\big[\{H_m, \alpha H_m, \alpha^2 H_m\} \cup \big\{ I_{(m,W,i)} \,\big|\, i \geq 0, W \in \{X,Y,Z\}\big\}\big].$$

We use this to show the following result.

THEOREM 8.2

There exists an infinite δ_2-cable $\hat{P}$ in $\mathcal{A}_2$ rooted at X, and for any such $\hat{P}$ we have

$$\mathcal{A}_2 = k[H_2, \alpha H_2, \alpha^2 H_2, \hat{P}, \alpha \hat{P}, \alpha^2 \hat{P}].$$

To construct $\hat{P}$ we first study the restriction of $\mathcal{D}_2$ to a subring $\mathcal{B}'$ of $\mathcal{B}$, where $\mathcal{B}' \cong k^{[6]}$.

8.1. The derivation E in dimension 6

Let $\mathcal{R} = k[x,y,s,t,u,v] = k^{[6]}$, and define the triangular derivation E of $\mathcal{R}$ by

$$(17)\qquad v \to x^2y^2, \qquad u \to y^3t, \qquad t \to y^3s, \qquad s \to x^3, \qquad x \to 0, \qquad y \to 0.$$

Then E commutes with $\frac{\partial}{\partial v}$ and we let τ denote the restriction of $\frac{\partial}{\partial v}$ to $\ker E$.

THEOREM 8.3

There exists an infinite τ-cable $\hat{\kappa}$ rooted at x.

Proof

Let $\pi_v : \mathcal{R} \to (\ker E)_{xy}$ be the Dixmier map for E associated to the local slice v. According to [4, (6) and Lemma 2], there exists a sequence $w_n \in k[x,y,z,s,t,u]$, $n \geq 0$, with the following properties.

(i) $w_0 = 1$.
(ii) $E^{3i} w_{3m} = (x^3y^3)^{2i} w_{3(m-i)}$ $(m \geq 1, 0 \leq i \leq m)$.
(iii) $(-1)^{3m}\pi_v(xw_{3m}) \in \mathcal{R}$ $(m \geq 0)$.

Given $m \geq 0$, define $\kappa_{3m} \in \mathcal{R}$ by $\kappa_{3m} = (-1)^{3m}\pi_v(xw_{3m})$. By using (1) in Section 2.1, we see that for $m \geq 1$

$$\begin{aligned}
\frac{\partial^3}{\partial v^3}\kappa_{3m} &= \frac{\partial^2}{\partial v^2}(-1)^{3m-1}\pi_v(xEw_{3m})\frac{\partial}{\partial v}\frac{v}{x^2y^2} \\
&= \frac{\partial}{\partial v}(-1)^{3m-2}\pi_v(xE^2w_{3m})\frac{1}{x^2y^2}\frac{\partial}{\partial v}\frac{v}{x^2y^2} \\
&= (-1)^{3m-3}\pi_v(xE^3w_{3m})\frac{1}{x^4y^4}\frac{\partial}{\partial v}\frac{v}{x^2y^2} \\
&= (-1)^{3m-3}\pi_v\big(x(x^3y^3)^2w_{3(m-1)}\big)\frac{1}{x^6y^6} \\
&= (-1)^{3(m-1)}\pi_v(xw_{3(m-1)}) \\
&= \kappa_{3(m-1)}.
\end{aligned}$$

Define

$$\kappa_{3m-1} = \frac{\partial}{\partial v}\kappa_{3m} \qquad \text{and} \qquad \kappa_{3m-2} = \frac{\partial}{\partial v}\kappa_{3m-1} \quad (m \geq 1).$$

Then $\hat{\kappa} := (\kappa_n)$ is a τ-cable rooted x. □

8.2. Proof of Theorem 8.2

Given $f_1, \dots, f_n \in \mathcal{B}$, recall that the Wronskian of $f_1, \dots, f_n$ relative to $\mathcal{D}_2$ is (see [5, Section 2.6])

$$W_{\mathcal{D}_2}(f_1, \dots, f_n) = \det(\mathcal{D}_2^i f_j) \quad \text{where } 0 \leq i \leq n-1, 1 \leq j \leq n.$$

Define $F_1, F_2, F_3 \in \mathcal{B}$ by

$$F_1 = S, \qquad F_2 = \frac{1}{2}W_{\mathcal{D}_2}(S, TU), \qquad F_3 = \frac{1}{6}X^{-3}W_{\mathcal{D}_2}(S, TU, STU).$$

Then $\mathcal{D}_2$ restricts to the subring $\mathcal{B}' = k[X, YZ, F_1, F_2, F_3, V] = k^{[6]}$, where

$$\mathcal{D}_2 F_3 = (YZ)^3 F_2, \qquad \mathcal{D}_2 F_2 = (YZ)^3 F_1, \qquad \mathcal{D}_2 F_1 = X^3, \qquad \mathcal{D}_2 V = X^2 (YZ)^2.$$

Therefore, setting $x = X$, $y = YZ$, $s = F_1$, $t = F_2$, $u = F_3$, and $v = V$, we see that the restriction of $\mathcal{D}_2$ to $\mathcal{B}'$ equals E, as defined in (17) above. By Theorem 8.3 there exists a δ_2-cable $\hat{P}$ rooted at X such that $\hat{P} \subset \mathcal{B}'$. In particular, $\hat{P} = (P_i)$ has the form $P_i = \frac{1}{i!} X V^i +$ (terms of lower degree in V).

Consequently, $\alpha \hat{P}$ is a δ_2-cable $\hat{P}$ rooted at Y, and $\alpha^2 \hat{P}$ is a δ_2-cable $\hat{P}$ rooted at Z. The proof is thus completed by applying Kuroda's result (Theorem 8.1 above). □

REMARK 8.4

It seems likely that the structure of $\mathcal{A}_2$ given in Theorem 8.2 can be extended from $m = 2$ to all $m \geq 2$. To do so by the method above requires a generalization of Theorem 8.3.

9. Further comments and questions

9.1. Tanimoto's generators

Tanimoto [13] gives a set of generators for the ring A by specifying a SAGBI basis consisting of h together with homogeneous sequences λ_n, μ_n, and ν_n whose leading v-terms are av^n, Fv^n, and Gv^n, respectively. From Corollary 5.5(a) we see that A is generated as a k-algebra by h and the sequence λ_n, meaning that μ_n and ν_n are redundant. Tanimoto also computed the Hilbert series for A, which is rational even though A is not finitely generated.

9.2. Fundamental problem for cable algebras

If B is an affine k-domain and $D \in \mathrm{LND}(B)$ is nonzero, then B is a cable algebra and (B, D) is a cable pair. We ask the following.

QUESTION

Let B be an affine k-domain, and let $D \in \mathrm{LND}(B)$. If $I_\infty \neq (0)$, does B have an infinite D-cable? Equivalently, if every D-cable of B is terminal, does $I_\infty = (0)$?

Note that if every D-cable of B is terminal, then since B is affine, there exist an integer $n \geq 1$ and terminal D-cables $\hat{t}_1, \ldots, \hat{t}_n$ such that $B = k[\hat{t}_1, \ldots, \hat{t}_n]$.

9.3. Q-ideals

We would like to know which Q-ideals are prime ideals of Ω. For each even $n \geq 2$, consider the following statements regarding the fundamental Q-ideals.

(a) $\mathcal{Q}_n$ is a prime ideal of Ω.
(b) $\mathrm{tr.deg}_k \, \Omega/\mathcal{Q}_n = \frac{n}{2} + 1$.
(c) $\Omega/\mathcal{Q}_n$ is a simple cable algebra over k.

It is shown above that these are true statements for $n = 2$ and $n = 4$. Are these statements true for $n \geq 6$?

9.4. The dimension 4 case

Nagata [11] presented the first counterexamples to Hilbert's fourteenth problem. In one of these, the transcendence degree of the ring of invariants over the ground field is 4, and Nagata asked whether this could be reduced to 3. The second author [7] gave an affirmative answer to Nagata's question in the form of the kernel of a derivation of $k^{[4]}$, but this derivation is not locally nilpotent (see also [9]).

It remains an open question whether an algebraic $\mathbb{G}_a$-action on the polynomial ring $k^{[4]}$ always has a finitely generated ring of invariants. In [3] it is shown that this is the case for triangular actions, and this result was later generalized in [1] to the case of actions having rank less than 4. The next natural case to consider is the case in which T is a locally nilpotent derivation of $k^{[4]}$ of rank 4 and T restricts to a coordinate subring $B = k^{[3]}$. If $k^{[4]} = B[v]$, then the partial derivative $\partial/\partial v$ restricts to $\ker T$. It is hoped that a good understanding of cable structures of invariant rings might lead to a complete solution of the dimension 4 case.

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Freudenburg: Department of Mathematics, Western Michigan University, Kalamazoo, Michigan 49008, USA; gene.freudenburg@wmich.edu

Kuroda: Department of Mathematics and Information Sciences, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan; kuroda@tmu.ac.jp