Moduli spaces of bundles over nonprojective K3 surfaces

Arvid Perego and Matei Toma

Abstract We study moduli spaces of sheaves over nonprojective K3 surfaces. More precisely, let ω be a Kähler class on a K3 surface S, let $r \geq 2$ be an integer, and let $v = (r, \xi, a)$ be a Mukai vector on S. We show that if the moduli space M of μ_ω-stable vector bundles with associated Mukai vector v is compact, then M is an irreducible holomorphic symplectic manifold which is deformation equivalent to a Hilbert scheme of points on a K3 surface. Moreover, we show that there is a Hodge isometry between $v^\perp$ and $H^2(M, \mathbb{Z})$ and that M is projective if and only if S is projective.

1. Introduction

Moduli spaces of sheaves on projective K3 surfaces have been studied since the 1980s. Fujiki [9] considered the Hilbert scheme $\mathrm{Hilb}^2(S)$ of two points on a K3 surface S. His result was widely generalized by Beauville [4], who studied $\mathrm{Hilb}^n(S)$ for any $n \in \mathbb{N}$, showing that it is an irreducible hyper-Kähler manifold; that is, a compact Kähler manifold which is simply connected, is holomorphically symplectic, and has $h^{2,0} = 1$.

Moduli spaces of μ-stable sheaves are a generalization of Hilbert schemes of points, and they have been extensively studied when the base surface S is a projective K3 surface. Mukai [28] showed that, on the moduli space M of simple sheaves of Mukai vector $v = (r, c_1(L), a)$ (i.e., of rank r, determinant L, and second Chern character $a - r$), there is a natural holomorphic symplectic form associated to the one on S. This moduli space M is a nonseparated scheme containing as a smooth open subset the moduli space $M^\mu_v(S, H)$ of μ_H-stable sheaves (with respect to some ample line bundle H on S) of Mukai vector v; Mukai's construction thus produces a holomorphic symplectic form on $M^\mu_v(S, \omega)$.

If H is generic and r and L are prime to each other, then $M^\mu_v(S, H)$ is a projective holomorphically symplectic manifold. Moreover, it is an irreducible hyper-Kähler manifold deformation equivalent to a Hilbert scheme of points on S (see [30], [43]).

If S is a nonprojective K3 surface and ω is a Kähler class on it, one still defines the notion of μ_ω-stable sheaf and constructs the moduli space $M^\mu_v(S, \omega)$

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of μ_ω-stable sheaves of Mukai vector v. In [36] it was shown that $M_v^\mu(S,\omega)$ is a smooth complex manifold carrying a holomorphic symplectic form. If ω is generic and r is prime to $c_1(L)$, then $M_v^\mu(S,\omega)$ is even compact (see Section 2.2 for the precise notion of genericity we use for Kähler classes, called v-*genericity* in analogy to the projective case).

It is natural to ask if $M_v^\mu(S,\omega)$ is irreducible symplectic and, in this case, what its deformation class is. We first show the following.

THEOREM 1.1

Let S be a K3 surface, and let $v=(r,\xi,a)\in H^*(S,\mathbb{Z})$, where $\xi\in NS(S)$, $r>1$ prime to ξ, and $v^2\geq 0$. Suppose ω to be v-generic.

(1) *The moduli space $M_v^\mu(S,\omega)$ is a compact, connected complex manifold of dimension v^2+2 which is holomorphically symplectic and deformation equivalent to a Hilbert scheme of points on a projective K3 surface.*

(2) *On $H^2(M_v^\mu(S,\omega),\mathbb{Z})$ there is a nondegenerate quadratic form, and there is an isometry between $H^2(M_v^\mu,\mathbb{Z})$ and $v^\perp$ if $v^2>0$ (resp., $v^\perp/\mathbb{Z}v$ if $v^2=0$).*

The condition $v^2\geq 0$ implies that $M_v^\mu(S,\omega)\neq\emptyset$ (see [2], [33], [23]). As recalled above, if S is projective and $\omega=c_1(H)$ for a generic ample line bundle H, we even know that $M_v^\mu(S,\omega)$ is an irreducible symplectic manifold. To prove Theorem 1.1, we study the two remaining cases: S is projective and $\omega\notin NS(S)$; and S is nonprojective.

When S is projective and ω is not the first Chern class of an ample line bundle, we show that there is a v-generic ample line bundle H such that $M_v^\mu(S,\omega)=M_v^\mu(S,H)$. This is done by showing that the v-chamber in which ω lies intersects the ample cone and that moving the polarization inside a v-chamber does not affect the moduli space. When S is nonprojective, the strategy to prove Theorem 1.1 is to deform $M_v^\mu(S,\omega)$ along the twistor family $\mathcal{X}\longrightarrow\mathbb{P}^1$ of (S,ω): even if the sheaves in $M_v^\mu(S,\omega)$ do not necessarily deform along such a twistor family, we can still deform them as twisted sheaves.

We then provide a construction of a relative moduli space of stable twisted sheaves extending Yoshioka's construction [42] to nonprojective base manifolds, and we show that we can connect the K3 surface S to a projective K3 surface S' only by means of twistor lines, in such a way that $M_v^\mu(S,\omega)$ deforms to $M_v^\mu(S',\omega')$ for some v-generic polarization ω' on S'. Theorem 1.1 holds true even if we replace $M_v^\mu(S,\omega)$ with a moduli space of stable twisted sheaves.

The nondegenerate quadratic form on $H^2(M_v^\mu(S,\omega),\mathbb{Z})$ is defined as a quadratic form on the second complex cohomology by using the same definition of the Beauville form, the only difference being that we have to fix one holomorphic symplectic form to define it, because a priori we have $h^{2,0}\geq 1$. We then show that it is nondegenerate. The construction of the isometry with $v^\perp$ is standard and uses the same strategy as in the projective case.

As one might see from the statement on Theorem 1.1, there is only one missing property for $M_v^\mu(S,\omega)$ to be an irreducible symplectic manifold; namely, we do not know if $M^{\mu}_{v}(S,\omega)$ is Kähler. This is a long-standing problem: on the open subset $M^{\mu-lf}_{v}(S,\omega)$ of $M^{\mu}_{v}(S,\omega)$ parameterizing locally free sheaves, we have a natural Kähler metric, the Weil–Petersson metric (see [19], [20]), but at present nothing is known about how this metric could extend to a Kähler metric on the whole $M^{\mu}_{v}(S,\omega)$.

The strategy to prove Theorem 1.1 together with [13, Theorem 3.3] may be employed to obtain another proof of the existence of a Kähler metric on $M^{\mu-lf}_{v}(S,\omega)$ and of a description of a twistor family for such a hyper-Kähler metric. However, as pointed out to us by Daniel Huybrechts, this strategy does not allow one to show that $M^{\mu}_{v}(S,\omega)$ carries a Kähler metric too. Let us remark, however, that there are choices of Mukai vectors for which $M^{\mu}_{v}(S,\omega)$ coincides with $M^{\mu-lf}_{v}(S,\omega)$ and is therefore a compact irreducible hyper-Kähler manifold. Moreover, such compact moduli spaces of stable locally free sheaves may acquire any positive even complex dimension (see Proposition 4.27).

As an application of the previous result, we will show the following projectivity criterion for the moduli spaces of slope-stable sheaves on a K3 surface.

THEOREM 1.2
Let S be a K3 surface, and let $v=(r,\xi,a)\in H^{2*}(S,\mathbb{Z})$, where $\xi\in NS(S)$, $r\geq 2$, $(r,\xi)=1$, and $v^2\geq 0$. If ω is a v-generic polarization, then the moduli space $M^{\mu}_{v}(S,\omega)$ is projective if and only if S is projective.

2. Moduli spaces of stable sheaves

In the following, S will be a K3 surface, possibly nonprojective. If $\mathscr{F}$ is a coherent sheaf on S, we let the *Mukai vector* of $\mathscr{F}$ be

$$v(\mathscr{F}):=\mathrm{ch}(\mathscr{F})\cdot\sqrt{\mathrm{td}(S)}\in H^{2*}(S,\mathbb{Z}).$$

If v_i is the component of $v(\mathscr{F})$ in $H^{2i}(S,\mathbb{Z})$, we have $v_0=\mathrm{rk}(\mathscr{F})$, $v_1=c_1(\mathscr{F})$, and $v_2=\mathrm{ch}_2(\mathscr{F})+\mathrm{rk}(\mathscr{F})=\frac{1}{2}c_1^2(\mathscr{F})-c_2(\mathscr{F})+\mathrm{rk}(\mathscr{F})$, which will be viewed as an integer (i.e., we fix an isomorphism $H^4(S,\mathbb{Z})\simeq\mathbb{Z}$).

We recall that on $H^{2*}(S,\mathbb{Z})$ we have a pure weight-two Hodge structure and a lattice structure with respect to the Mukai pairing (see [15, Definitions 6.1.5, 6.1.11]): the obtained lattice will be referred to as the *Mukai lattice*, and we will write v^2 for the square of $v\in H^{2*}(S,\mathbb{Z})$ with respect to the Mukai pairing. Explicitly, $v^2=v_1^2-2v_0v_2$.

When $v_0\neq 0$ we define the *discriminant* of v, or, respectively, of $\mathscr{F}$ in the case in which $v=v(\mathscr{F})$, as

$$\Delta(v):=\frac{1}{2v_0^2}v^2+1.$$

This coincides with the definition of [2] for instance, where

$$\Delta(\mathscr{F})=\Delta\big(v(\mathscr{F})\big)=\frac{1}{\mathrm{rk}(\mathscr{F})}\Big(c_2(\mathscr{F})-\frac{\mathrm{rk}(\mathscr{F})-1}{2\,\mathrm{rk}(\mathscr{F})}c_1^2(\mathscr{F})\Big).$$

2.1. The stability condition

Let g be a Kähler metric on S, and let ω be the associated Kähler class, which will be called a *polarization* on S. If $\mathscr{F} \in \mathrm{Coh}(S)$ has positive rank, then the *slope* of $\mathscr{F}$ with respect to ω is

$$\mu_\omega(\mathscr{F}) := \frac{c_1(\mathscr{F}) \cdot \omega}{\mathrm{rk}(\mathscr{F})}.$$

DEFINITION 2.1

A torsion-free coherent sheaf $\mathscr{F}$ is μ_ω-*stable* if for every coherent subsheaf $\mathscr{E} \subseteq \mathscr{F}$ such that $0 < \mathrm{rk}(\mathscr{E}) < \mathrm{rk}(\mathscr{F})$ we have $\mu_\omega(\mathscr{E}) < \mu_\omega(\mathscr{F})$. If $\mu_\omega(\mathscr{E}) \leq \mu_\omega(\mathscr{F})$ for all such subsheaves $\mathscr{E}$, then we say that $\mathscr{F}$ is μ_ω-*semistable.*

The family of μ_ω-stable sheaves of Mukai vector v admits a moduli space $M_v^\mu(S,\omega)$. If S is projective and ω is the first Chern class of an ample line bundle H, then $M_v^\mu(S,\omega)$ is the moduli space $M_v^\mu(S,H)$ of μ_H-stable sheaves on S with Mukai vector v. We have the following proposition dealing also with the nonprojective case (see [36]).

PROPOSITION 2.2

Let S be a K3 surface, let $v \in H^{2*}(S,\mathbb{Z})$ be a Mukai vector, and let ω be a polarization on S. The moduli space $M_v^\mu(S,\omega)$ is a smooth, holomorphically symplectic manifold (possibly noncompact), and if it is not empty, then its dimension is $v^2 + 2$.

In the following we will restrict to the case of those $M_v^\mu(S,\omega)$'s which are nonempty and compact. We introduce in the next section some hypotheses on v and ω under which $M_v^\mu(S,\omega)$ is compact. We now present a condition which guarantees its nonemptiness and even the existence of a stable vector bundle with respect to any polarization.

Recall that over any nonalgebraic surface there exist nonfilterable holomorphic rank two vector bundles (see [2], [34, p. 18]). By definition they do not admit coherent subsheaves of rank one; hence, they are stable with respect to any polarization.

We now extend this type of result to arbitrary rank in the case of Kähler surfaces. Following [2] we say that a coherent sheaf on the surface S is *irreducible* if its only coherent subsheaf of lower rank is the zero sheaf. In particular, an irreducible sheaf is stable with respect to any polarization. We have the following result about the existence of locally free irreducible vector bundles.

PROPOSITION 2.3

Let S be a Kähler nonalgebraic compact complex surface, let r be a positive integer, and let $\xi \in NS(S)$. Then there exists a bound $b := b(r,\xi) \in \mathbb{Z}$ depending on r and on ξ such that for any integer $c \geq b$ there is on S an irreducible locally free sheaf $\mathscr{F}$ of rank r, $c_1(\mathscr{F}) = \xi$, and $c_2(\mathscr{F}) = c$.

Proof
For $r = 2$, a statement of this type was proved in [2] and in [34] without the Kähler assumption. The idea there was to look at the versal deformation space of a rather arbitrary coherent sheaf $\mathscr{F}$ and show that if $c_2 \gg 0$, then the deformation of $\mathscr{F}$ must contain irreducible objects. For $r > 2$ we will this time consider deformations of suitably chosen coherent sheaves and make essential use of the fact that S is Kähler. In this way, we reduce our argument to one used by Bănică and Le Potier in the case in which the algebraic dimension of S is zero (see [2, Théorème 5.3]).

We proceed by induction on r. The statement is trivial for $r = 1$ and already proven for $r = 2$. Let then $r \geq 3$, and suppose that the statement is true for rank $r-1$. Take an irreducible locally free sheaf $\mathscr{E}$ on S of rank $r-1$, $c_1(\mathscr{E}) = \xi$, and $c_2(\mathscr{E}) = c$. Consider an irreducible component B of the versal deformation space of $\mathscr{F}_0 := \mathscr{O}_S \oplus \mathscr{E}$ and the corresponding family $\mathscr{F}$ of coherent sheaves over $S \times B$.

We will check that if $c \gg 0$, then the relative Douady space $D_{(X \times B)/B}(\mathscr{F}, k)$ of flat quotients of rank k of $\mathscr{F}$ over B does not cover B for $1 \leq k \leq r-1$. Let $b : D_{(X \times B)/B}(\mathscr{F}, k) \to B$ be the natural morphism, and let $Q \subset B$ be a relatively compact subdomain of B containing the origin $0 \in B$. Fujiki [10] proved that any irreducible component of $b^{-1}(Q)$ is proper over Q. By another result of Fujiki [8], there are countably many such components.

The idea is to show by a dimension count that very general neighbors of $\mathscr{F}_0$ are not in the image of $D_{(X \times B)/B}(\mathscr{F}, k)$ for $2 \leq k \leq r-2$. We remark that if $\mathscr{F}_b$ is such a neighbor sitting in a short exact sequence

$$0 \to F' \to \mathscr{F}_b \to F'' \to 0$$

with F'' torsion-free, then F' and F'' are irreducible of different ranks; hence, $\mathrm{Hom}(F', F'') = 0 = \mathrm{Hom}(F'', F')$. This remark makes the arguments in the proof of [2, Théorème 5.3] work by replacing the corresponding inequality in [2, Lemme 5.12]. Hence, we have the statement. □

2.2. The v-genericity for Kähler forms

Let S be a K3 surface, and let $\mathscr{K}_S$ be its Kähler cone, which is an open and convex cone in $H^{1,1}(S)$. For $v = (r, \xi, a)$ with $r \geq 2$ and $\xi \in NS(S)$, we define a system of hyperplanes in $H^{1,1}(S)$, which is locally finite in $\mathscr{K}_S$ and has the property that, for any $\omega \in \mathscr{K}_S$ not lying on such hyperplanes, a torsion-free sheaf $\mathscr{F}$ on S with $v(\mathscr{F}) = v$ is μ_ω-stable if and only if it is μ_ω-semistable. Polarizations verifying this will be called *v-generic*.

2.2.1. The notion of v-genericity
To start, let S be any compact Kähler surface, fix $r, c_2 \in \mathbb{Z}$, fix $c_1 \in NS(S)$, and suppose $r > 0$. We let $\tau := (r, c_1, c_2)$, and if $\mathscr{F} \in \mathrm{Coh}(S)$ of rank r and Chern classes c_1 and c_2, we call τ the *topological type* of $\mathscr{F}$. If S is a K3 surface and $\mathscr{F} \in \mathrm{Coh}(S)$ has Mukai vector $v = (r, \xi, a)$, then its topological type is $\tau_v = (r, \xi, \xi^2/2 + r - a)$.

Note that the discriminant $\Delta(\mathscr{F})$ only depends on the topological type of $\mathscr{F}$. Hence, we can talk about the *discriminant* $\Delta(\tau)$ of τ: more precisely, if $\tau = (r, c_1, c_2)$, then

$$\Delta(\tau) = \frac{1}{r}\Big(c_2 - \frac{r-1}{2r}c_1^2\Big).$$

We set

$$W_\tau := \Big\{D \in NS(S) \,\Big|\, -\frac{r^4}{2}\Delta(\tau) \leq D^2 < 0\Big\},$$

and for every $\alpha \in H^{1,1}(S)$, we write

$$\alpha^\perp := \big\{\beta \in H^{1,1}(S) \,\big|\, \alpha \cdot \beta = 0\big\}.$$

When $\alpha \neq 0$, the set $\alpha^\perp$ is a hyperplane in $H^{1,1}(S)$. Using the same argument of [15, Lemma 4.C.2], one shows that if $\beta \in H^{1,1}(S)$, then there is an open neighborhood U of β in $H^{1,1}(S)$ such that $U \cap D^\perp \neq \emptyset$ for at most a finite number of $D \in W_\tau$. If the surface S is K3, we will use the notation W_v for W_{τ_v}.

DEFINITION 2.4

For every $D \in W_\tau$, the hyperplane $D^\perp \cap \mathscr{K}_S$ will be called a *τ-wall in the Kähler cone of S*. A connected component of $\mathscr{K}_S \setminus \bigcup_{D \in W_\tau} D^\perp$ is an open convex cone called a *τ-chamber in the Kähler cone of S*. A Kähler class in a τ-chamber of $\mathscr{K}_S$ is called a *τ-generic polarization*.

If S is a K3 surface and v is a Mukai vector, we will call a *v-wall in the Kähler cone* (resp., a *v-chamber in the Kähler cone*, *v-generic polarization*) a τ_v-wall in the Kähler cone (resp., a τ_v-chamber in the Kähler cone, τ_v-generic polarization).

Recall that the ample cone of S is $\mathrm{Amp}(S) = \mathscr{K}_S \cap NS_{\mathbb{R}}(S)$ (where $NS_{\mathbb{R}}(S) = NS(S) \otimes \mathbb{R}$). If S is a projective K3 surface and $\mathcal{C} \subseteq \mathscr{K}_S$ is a v-chamber in the Kähler cone of S, then $\mathcal{C} \cap NS_{\mathbb{R}}(S)$ is a v-chamber in the ample cone of S in the usual terminology. If H is an ample line bundle on S, then $c_1(H)$ is a v-generic polarization if and only if H is v-generic as in [15].

2.2.2. Compactness of $M_v^\mu(S,\omega)$ when ω is v-generic

Using the same proof as in the projective case (see [15, Theorem 4.C.3]), we show that v-generic polarizations enjoy the above-stated property concerning the existence of properly semistable sheaves.

LEMMA 2.5

Let ω be a Kähler class on a compact Kähler surface S, and let $\mathscr{F}$ be a μ_ω-semistable sheaf of topological type $\tau = (r, \xi, c_2)$. Suppose that there is $\mathscr{E} \subseteq \mathscr{F}$ of rank $0 < s < r$, first Chern class ζ, and such that $\mu_\omega(\mathscr{E}) = \mu_\omega(\mathscr{F})$. Then $D := r\zeta - s\xi$ is such that

$$-\frac{r^4}{2}\Delta(\tau) \leq D^2 \leq 0,$$

and $D^2 = 0$ if and only if $D = 0$.

Proof
We can suppose that $\mathscr{E}$ is saturated, so that $\mathscr{G} := \mathscr{F}/\mathscr{E}$ is torsion-free, μ_ω-semistable, and of rank $r-s$. Notice that, as $\mu_\omega(\mathscr{E}) = \mu_\omega(\mathscr{F})$, we have $D \cdot \omega = 0$. As ω is a Kähler class, from the Hodge index theorem we then have $D^2 \leq 0$, and $D^2 = 0$ if and only if $D = 0$. We then just need to show that $D^2 \geq -\frac{r^4}{2}\Delta(\tau)$.

By the definition of the discriminant, it follows that

$$\Delta(\mathscr{F}) - \frac{s}{r}\Delta(\mathscr{E}) - \frac{r-s}{r}\Delta(\mathscr{G}) = -\frac{D^2}{2s(r-s)r^2}.$$

Now, recall that the Bogomolov inequality is surely satisfied by $\mathscr{E}$ and $\mathscr{G}$, so that $\Delta(\mathscr{E}), \Delta(\mathscr{G}) \geq 0$. But this implies that

$$\begin{aligned} -D^2 &\leq 2s(r-s)r^2\Delta(\mathscr{F}) \\ &= 2s(r-s)r^2\Delta(\tau) \leq \frac{r^4}{2}\Delta(\tau), \end{aligned}$$

and we are done. □

Using the main result of [36] we then get the following result.

PROPOSITION 2.6
Let S be a K3 surface, let $r \geq 2$ be an integer, and let $\xi \in NS(S)$ be such that $(r, \xi) = 1$. Let $a \in \mathbb{Z}$, let $v := (r, \xi, a)$, and let ω be a v-generic polarization. If $M_v^\mu(S, \omega) \neq \emptyset$, then it is a smooth, compact, holomorphically symplectic manifold.

Proof
The statement follows from the main result of [36] if S is nonalgebraic. When S is projective, we will show in Section 3.1 that there exists some integer ample class H in the same v-chamber as ω. The (semi)stability with respect to ω or with respect to H will then come down to the same thing, and $M_v^\mu(S, \omega)$ will coincide with the Gieseker moduli space $M_v(S, H)$ of H-semistable sheaves, which is known to be smooth, projective, and holomorphically symplectic (see [15]). □

3. Projective K3 surfaces with nonample polarizations

In this section we prove that if S is a projective K3 surface, $v = (r, \xi, a)$ is a Mukai vector with $(r, \xi) = 1$, and ω is a v-generic polarization, then $M_v^\mu(S, \omega)$ is an irreducible holomorphically symplectic manifold, deformation equivalent to a Hilbert scheme of points on S.

3.1. Changing polarization in a chamber

We first show that changing polarization inside a chamber does not affect the moduli space. The following adaptation of [15, Lemma 4.C.5] to the case of Kähler polarizations works also on Kähler manifolds (see [11, Lemma 6.2]).

LEMMA 3.1

Let ω, ω' be two Kähler classes on a compact Kähler manifold X, and let $\mathscr{F}$ be a torsion-free sheaf on X which is μ_ω-stable but not $\mu_{\omega'}$-stable. Denote by

$$[\omega,\omega'] := \{\omega_t := t\omega' + (1-t)\omega \mid t \in [0,1]\}$$

the segment from ω to ω'. Then there is a Kähler class $\omega_t \in [\omega,\omega']$ such that $\mathscr{F}$ is properly μ_{ω_t}-semistable.

As a consequence of this, changing the polarization inside a chamber does not affect the moduli space. This is well known for v-generic ample line bundles and requires the same proof. We let $M^\mu_\tau(S,\omega)$ be the moduli space of μ_ω-stable sheaves whose topological type is τ. If S is a K3 surface, then $M^\mu_{\tau_v}(S,\omega) = M^\mu_v(S,\omega)$.

PROPOSITION 3.2

Let S be a smooth projective surface, and let $\tau = (r,\xi,c_2)$ be such that $r \geq 2$ and $\xi \in NS(S)$. Let $\mathcal{C}$ be a τ-chamber in the Kähler cone of S, and let $\omega,\omega' \in \mathcal{C}$. Then $M^\mu_\tau(S,\omega) = M^\mu_\tau(S,\omega')$.

Proof

We show that if $\mathscr{F}$ is a μ_ω-stable sheaf of topological type τ, then it is $\mu_{\omega'}$-stable as well. Indeed, suppose that $\mathscr{F}$ is not $\mu_{\omega'}$-stable. By Lemma 3.1 this implies that there is $\omega_t \in [\omega,\omega']$ such that $\mathscr{F}$ is properly μ_{ω_t}-semistable. Hence, there is $\mathscr{E} \subseteq \mathscr{F}$ of rank $0 < s < r$ and first Chern class ζ such that $\mu_{\omega_t}(\mathscr{E}) = \mu_{\omega_t}(\mathscr{F})$.

Let $D := r\zeta - s\xi$. Hence, $D \cdot \omega_t = 0$, and by Lemma 2.5, we have $D \in W_\tau \cup \{0\}$. Note that, as $\mathscr{F}$ is μ_ω-stable, we have $D \cdot \omega < 0$, so that $D \in W_\tau$. It follows that $\omega_t \notin \mathcal{C}$, which is not possible as $\mathcal{C}$ is convex. In conclusion, $\mathscr{F}$ is $\mu_{\omega'}$-stable. $\square$

3.2. Conclusion for projective K3 surfaces

We first introduce some notation. If S is a projective surface, we let $NS_\mathbb{R}(S)$ be the real Néron–Severi space of S, which is a linear subspace of $H^{1,1}(S)$. Recall that on $H^{1,1}(S)$ we have a nondegenerate intersection product whose restriction to $NS_\mathbb{R}(S)$ remains nondegenerate. Let $T_\mathbb{R}(S)$ be the orthogonal of $NS_\mathbb{R}(S)$ in $H^{1,1}(S)$, so that we have $H^{1,1}(S) = NS_\mathbb{R}(S) \oplus T_\mathbb{R}(S)$.

Finally, for every $\alpha \in H^{1,1}(S)$ we let $p_{NS} : H^{1,1}(S) \longrightarrow NS_\mathbb{R}(S)$ and $p_T : H^{1,1}(S) \longrightarrow T_\mathbb{R}(S)$ be the two projections. Moreover, for every $\alpha \in H^{1,1}(S)$ we let $\alpha_{NS} := p_{NS}(\alpha)$ and $\alpha_T := p_T(\alpha)$.

The first result we show is the following.

LEMMA 3.3

Let S be a projective surface, and let ω be a Kähler class on S.

1. *The class ω_{NS} is an ample class on S.*
2. *For every $\xi \in NS_\mathbb{R}(S)$ we have $\xi \cdot \omega = \xi \cdot \omega_{NS}$.*

Proof
Recall that $\omega = \omega_{NS} + \omega_T$. It follows that for every nonzero effective curve class C we have

$$\omega_{NS} \cdot C = \omega \cdot C - \omega_T \cdot C = \omega \cdot C > 0,$$

since ω_T is orthogonal to $NS_{\mathbb{R}}(S)$ (where C lies), and ω is a Kähler class. This implies that ω_{NS} is a nef class on S.

In particular, this means that ω_{NS} is a class in the closure of the ample cone of S. Now, recall that the projection p_{NS} is an open map; moreover, the previous part of the proof shows that the image of the Kähler cone of S under p_{NS} is contained in the nef cone of S. As the Kähler cone is open in $H^{1,1}(S)$ and the interior of the nef cone is the ample cone, it follows that the image of the Kähler cone by projection is contained in the ample cone.

The last point of the statement is simply the fact that ω_T is orthogonal to $NS_{\mathbb{R}}(S)$. □

Using the previous lemma, we can finally prove the following, which shows Theorem 1.1(1).

THEOREM 3.4
Let S be a projective K3 surface, and let $v = (r, \xi, a) \in H^{2*}(S, \mathbb{Z})$ such that $r \geq 2$, $\xi \in NS(S)$, and $(r, \xi) = 1$. If ω is v-generic and $M_v^{\mu}(S, \omega) \neq \emptyset$, then $M_v^{\mu}(S, \omega)$ is a projective irreducible hyper-Kähler manifold deformation equivalent to a Hilbert scheme of points on S.

Proof
The class ω_{NS} is ample by Lemma 3.3, and $\omega_{NS} \cdot \xi = \omega \cdot \xi$ for every $\xi \in NS_{\mathbb{R}}(S)$. It follows that for every $\mathscr{F} \in \mathrm{Coh}(S)$ we have $\mu_\omega(\mathscr{F}) = \mu_{\omega_{NS}}(\mathscr{F})$. In particular, a coherent sheaf is μ_ω-stable if and only if it is $\mu_{\omega_{NS}}$-stable, so that $M_v^{\mu}(S, \omega) = M_v^{\mu}(S, \omega_{NS})$.

Moreover, if $D \in W_v$, then $\omega_{NS} \cdot D = \omega \cdot D$. As ω is v-generic, it follows that ω_{NS} is v-generic. Let $\mathcal{C}$ be the v-chamber of the ample cone where ω_{NS} lies. As $\mathcal{C}$ is open in $\mathrm{Amp}(S)$, there is $\epsilon > 0$ such that the ball $B_\epsilon(\omega_{NS}) \subseteq \mathrm{Amp}(S)$ of ray ϵ and centered at ω_{NS} is contained in $\mathcal{C}$. Let $\omega' \in B_\epsilon(\omega_{NS}) \cap H^2(S, \mathbb{Q})$. By Proposition 3.2, we have $M_v^{\mu}(S, \omega_{NS}) = M_v^{\mu}(S, \omega')$.

As $\omega' \in H^2(S, \mathbb{Q}) \cap H^{1,1}(S)$, there are $p \in \mathbb{N}$ and $H \in Pic(S)$ such that $p\omega' = c_1(H)$. As $\omega' \in \mathcal{C}$ and $\mathcal{C}$ is a cone, we have $c_1(H) \in \mathcal{C}$. Hence, H is a v-generic ample line bundle, and $M_v^{\mu}(S, \omega') = M_v^{\mu}(S, H)$. By [30] and [43], $M_v^{\mu}(S, H)$ is an irreducible hyper-Kähler manifold deformation equivalent to a Hilbert scheme of points, and we are done. □

REMARK 3.5
A useful corollary of Lemma 3.3 is that if $\mathcal{C}$ is a v-chamber in the Kähler cone of S, then $\mathcal{C}$ intersects the ample cone (and the intersection is clearly a v-chamber

in the ample cone of S). Indeed, consider the segment $[\omega, \omega_{NS}]$. As the projection p_{NS} is a linear map, we have that $[\omega, \omega_{NS}] \cap NS_{\mathbb{R}}(S) = \{\omega_{NS}\}$ and that $p_{NS}([\omega, \omega_{NS}]) = \{\omega_{NS}\}$.

We show that $\omega_{NS} \in \mathcal{C}$ (showing that $\mathcal{C}$ intersects the ample cone by Lemma 3.3). Indeed, suppose that ω_{NS} does not lie in $\mathcal{C}$. It follows that there is $\omega' \in [\omega, \omega_{NS}]$ lying on a v-wall. This means that there is $D \in W_v$ such that $\omega' \cdot D = 0$. But as $p_{NS}(\omega') = p_{NS}(\omega) = \omega_{NS}$, it follows that $\omega \cdot D = 0$, which is not possible.

4. Moduli spaces of stable twisted sheaves

In this section we recall the notion of twisted sheaf on a complex manifold, and we introduce the notion of stability for coherent twisted sheaves. We will then construct (relative) moduli spaces of stable twisted sheaves on a K3 surface (not necessarily projective). They will be used to show that the moduli spaces $M_v^{\mu}(S,\omega)$ of μ_ω-stable sheaves with Mukai vector $v = (r, \xi, a)$ such that r and ξ are prime to each other are compact, connected, simply connected, and deformation equivalent to a Hilbert scheme of points on a projective K3 surface (whenever the polarization ω is v-generic).

4.1. Twisted sheaves and stability

We recall some basic facts about twisted sheaves on a complex manifold X. (We refer the interested reader to [6] or [25] for more details.)

There are several definitions of twisted sheaves giving equivalent categories. We use three of them: the first one is due to Căldăraru [6] and presents twisted sheaves as a twisted gluing of local coherent sheaves on X; the second one (found again in [6]) presents twisted sheaves as modules over an Azumaya algebra on X; the last one, due to Yoshioka [42], presents twisted sheaves as a full subcategory of the category of coherent sheaves on some projective bundle over X.

We begin by recalling these definitions. As our aims are moduli spaces of stable twisted sheaves, we need to introduce several notions. First, we recall the Chern character and the slope of a twisted sheaf. (For projective manifolds, this was done in [18] and [42].) We then introduce μ_ω-stability for twisted sheaves (with respect to a Kähler form ω).

4.1.1. Twisted sheaves following Căldăraru

Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an open covering of X, and let $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$. Choose a 2-cocycle $\{\alpha_{ijk}\}$, where $\alpha_{ijk} \in \mathscr{O}_X^*(U_{ijk})$, defining a class $\alpha \in H^2(X, \mathscr{O}_X^*)$. An $\{\alpha_{ijk}\}$-*twisted coherent sheaf* is a collection $\mathscr{F} = \{\mathscr{F}_i, \phi_{ij}\}$, where $\mathscr{F}_i \in \mathrm{Coh}(U_i)$ for every $i \in I$, and for every $i, j \in I$, $\phi_{ij} : \mathscr{F}_{j|U_{ij}} \longrightarrow \mathscr{F}_{i|U_{ij}}$ is an isomorphism in $\mathrm{Coh}(U_{ij})$ such that

(1) $\phi_{ii} = \mathrm{id}_{\mathscr{F}_i}$ for every $i \in I$;
(2) $\phi_{ij} = \phi_{ji}^{-1}$ for every $i, j \in I$;
(3) $\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$ for every $i, j, k \in I$.

By a *morphism of* $\{\alpha_{ijk}\}$*-twisted sheaves*

$$f:\mathscr{F}=\{\mathscr{F}_i,\phi_{ij}\}\longrightarrow\mathscr{G}=\{\mathscr{G}_i,\psi_{ij}\}$$

we mean a collection $f=\{f_i\}$ of morphisms $f_i:\mathscr{F}_i\longrightarrow\mathscr{G}_i$ of $\mathscr{O}_{U_i}$-modules such that $\psi_{ij}\circ f_j=f_i\circ\phi_{ij}$ for every $i,j\in I$.

The $\{\alpha_{ijk}\}$-twisted coherent sheaves form an abelian category $\mathrm{Coh}(X,\{\alpha_{ijk}\})$. If $\{\alpha_{ijk}\}$ and $\{\alpha'_{ijk}\}$ are two representatives of the same class $\alpha\in H^2(X,\mathscr{O}_X^*)$, then there is an equivalence between $\mathrm{Coh}(X,\{\alpha_{ijk}\})$ and $\mathrm{Coh}(X,\{\alpha'_{ijk}\})$, so that we can speak of the category $\mathrm{Coh}(X,\alpha)$ of coherent α-twisted sheaves. If $\mathscr{F}\in\mathrm{Coh}(X,\alpha)$ and $\mathscr{G}\in\mathrm{Coh}(X,\beta)$, we can define in a natural way $\mathscr{F}\otimes\mathscr{G}$ and $\mathscr{H}\mathrm{om}(\mathscr{F},\mathscr{G})$: the first one is a coherent $\alpha\beta$-twisted sheaf, while the second is a coherent $\alpha^{-1}\beta$-twisted sheaf.

We now recall an important definition: a sheaf $\mathcal{A}$ of $\mathscr{O}_X$-modules is said to be an *Azumaya algebra* if it is a sheaf of $\mathscr{O}_X$-algebras whose generic fiber is a central simple algebra. Equivalence classes of Azumaya algebras form a group $\mathrm{Br}(X)$, the *Brauer group* of X, which has an injection into $H^2(X,\mathscr{O}_X^*)$. One of the main properties we will use is the following (see [6, Theorem 1.3.5]).

PROPOSITION 4.1

Let X be a complex manifold, and let $\alpha\in\mathrm{Br}(X)$. Then there exists a locally free α-twisted sheaf on X.

For the rest of this section, we suppose $\alpha\in\mathrm{Br}(X)$ and define the twisted Chern character and twisted Mukai vector for α-twisted sheaves. More precisely, let $\mathscr{F}$ be an α-twisted coherent sheaf on X, and let E be a locally free α-twisted coherent sheaf. The *Chern character* of $\mathscr{F}$ with respect to E is

$$\mathrm{ch}_E(\mathscr{F}):=\frac{\mathrm{ch}(\mathscr{F}\otimes E^\vee)}{\sqrt{\mathrm{ch}(E\otimes E^\vee)}}.$$

The *Mukai vector* of $\mathscr{F}$ with respect to E is

$$v_E(\mathscr{F}):=\mathrm{ch}_E(\mathscr{F})\cdot\sqrt{\mathrm{td}(X)}.$$

The *slope* of a torsion-free α-twisted sheaf $\mathscr{F}$ with respect to E and to a Kähler class ω is

$$\mu_{E,\omega}(\mathscr{F}):=\frac{c_{E,1}(\mathscr{F})\cdot\omega}{\mathrm{rk}(\mathscr{F})},$$

where $c_{E,1}(\mathscr{F})$ is the component of $\mathrm{ch}_E(\mathscr{F})$ lying in $H^2(S,\mathbb{Q})$.

We collect some explicit formulas when $X=S$ is a K3 surface. Let $r:=\mathrm{rk}(\mathscr{F})$, $s:=\mathrm{rk}(E)$, $\xi:=c_1(\mathscr{F}\otimes E^\vee)$, $a:=\mathrm{ch}_2(\mathscr{F}\otimes E^\vee)$, and $b:=\mathrm{ch}_2(E\otimes E^\vee)$. Then

$$\mathrm{ch}_E(\mathscr{F})=\big(r,\xi/s,(2as-rb)/2s^2\big),$$
$$v_E(\mathscr{F})=\big(r,\xi/s,r+(2as-rb)/2s^2\big),$$

so that

$$\mu_{E,\omega}(\mathscr{F}) = \frac{\xi\cdot\omega}{rs} = \frac{c_1(\mathscr{F}\otimes E^\vee)\cdot\omega}{\operatorname{rk}(\mathscr{F})\operatorname{rk}(E)} = \mu_\omega(\mathscr{F}\otimes E^\vee) \tag{1}$$

and

$$v_E^2(\mathscr{F}) = \frac{\xi^2}{s^2} - \frac{2ra}{s} + \frac{r^2 b}{s^2} - 2r^2. \tag{2}$$

If $\alpha = 0$, then one easily sees that $\mu_{E,\omega}(\mathscr{F}) = \mu_\omega(\mathscr{F}) - \mu_\omega(E)$ and that

$$v_E^2(\mathscr{F}) = v^2(\mathscr{F}). \tag{3}$$

If $\mathscr{F}$ is a torsion-free α-twisted sheaf on S, we let

$$\operatorname{ch}_\alpha(\mathscr{F}) := \operatorname{ch}_{\mathscr{F}^{\vee\vee}}(\mathscr{F}), \qquad v_\alpha(\mathscr{F}) := v_{\mathscr{F}^{\vee\vee}}(\mathscr{F}),$$

which we call the *twisted Chern character* and *twisted Mukai vector* of $\mathscr{F}$, respectively. The *twisted slope* of $\mathscr{F}$ with respect to ω is

$$\mu_{\alpha,\omega}(\mathscr{F}) := \frac{c_{\alpha,1}(\mathscr{F})\cdot\omega}{\operatorname{rk}(\mathscr{F})},$$

where $c_{\alpha,1}(\mathscr{F})$ is the component of $\operatorname{ch}_\alpha(\mathscr{F})$ in $H^2(S,\mathbb{Q})$.

Using twisted slopes, we introduce the notion of stability for twisted sheaves. Fix $\alpha \in \operatorname{Br}(X)$, and let E be an α-twisted locally free sheaf.

DEFINITION 4.2

We say that a torsion-free $\mathscr{F} \in \operatorname{Coh}(X,\alpha)$ is *$\mu_{E,\omega}$-stable* if for every α-twisted coherent subsheaf $\mathscr{E} \subseteq \mathscr{F}$ such that $0 < \operatorname{rk}(\mathscr{E}) < \operatorname{rk}(\mathscr{F})$ we have $\mu_{E,\omega}(\mathscr{E}) < \mu_{E,\omega}(\mathscr{F})$. If $\mu_{E,\omega}(\mathscr{E}) \le \mu_{E,\omega}(\mathscr{F})$ for every such subsheaf, we say that $\mathscr{F}$ is *$\mu_{E,\omega}$-semistable.* A $\mu_{\mathscr{F}^{\vee\vee},\omega}$-(semi)stable sheaf will be called *$\mu_{\alpha,\omega}$-(semi)stable.*

To conclude this section, we show that the $\mu_{E,\omega}$-stability does not depend on E.

LEMMA 4.3

Let $\alpha \in \operatorname{Br}(S)$, $\mathscr{F} \in \operatorname{Coh}(S,\alpha)$, and $\omega \in \mathscr{K}_S$. If $E', E \in \operatorname{Coh}(S,\alpha)$ are locally free, then $\mathscr{F}$ is $\mu_{E,\omega}$-stable if and only if it is $\mu_{E',\omega}$-stable. In particular, $\mathscr{F}$ is $\mu_{E,\omega}$-stable if and only if it is $\mu_{\alpha,\omega}$-stable. If $\alpha = 0$, then the sheaf $\mathscr{F}$ is $\mu_{0,\omega}$-stable if and only if it is μ_ω-stable.

Proof

Let $\mathscr{F} \in \operatorname{Coh}(S,\alpha)$, let $\mathscr{G}$ be an α-twisted coherent subsheaf of $\mathscr{F}$, and let H be a locally free α-twisted coherent sheaf. Then

$$\begin{aligned}&\operatorname{rk}(H)\operatorname{rk}(\mathscr{F})c_1(\mathscr{G}\otimes H^\vee) - \operatorname{rk}(H)\operatorname{rk}(\mathscr{G})c_1(\mathscr{F}\otimes H^\vee)\\ &\quad = c_1(\mathscr{G}\otimes\mathscr{F}^\vee\otimes H\otimes H^\vee) = \operatorname{rk}^2(H)c_1(\mathscr{G}\otimes\mathscr{F}^\vee).\end{aligned} \tag{4}$$

Suppose now that $\mathscr{F}$ is $\mu_{E,\omega}$-stable but not $\mu_{E',\omega}$-stable. Hence, there is an α-twisted coherent subsheaf $\mathscr{G}$ of $\mathscr{F}$ of rank $0 < s < \operatorname{rk}(\mathscr{F})$ such that $\mu_{E',\omega}(\mathscr{G}) \ge \mu_{E',\omega}(\mathscr{F})$. By the $\mu_{E,\omega}$-stability of $\mathscr{F}$, we even have $\mu_{E,\omega}(\mathscr{G}) < \mu_{E,\omega}(\mathscr{F})$. Writing these two inequalities explicitly, we have

$$\omega \cdot \big(\mathrm{rk}(E')rc_1\big(\mathscr{G} \otimes (E')^\vee\big) - \mathrm{rk}(E')sc_1\big(\mathscr{F} \otimes (E')^\vee\big)\big) \geq 0, \tag{5}$$

$$\omega \cdot \big(\mathrm{rk}(E)rc_1(\mathscr{G} \otimes E^\vee) - \mathrm{rk}(E)sc_1(\mathscr{F} \otimes E^\vee)\big) < 0. \tag{6}$$

By using (4) for $H = E'$, (5) becomes $\omega \cdot c_1(\mathscr{G} \otimes \mathscr{F}^\vee) \geq 0$. By using (4) for $H = E$, (6) becomes $\omega \cdot c_1(\mathscr{G} \otimes \mathscr{F}^\vee) < 0$, giving a contradiction. □

4.1.2. Twisted sheaves as $\mathcal{A}$-modules

Again, let X be a complex manifold, and let $\mathcal{A}$ be an Azumaya algebra on X. We let $\mathrm{Coh}(X, \mathcal{A})$ be the abelian category of coherent sheaves on X having the structure of an $\mathcal{A}$-module. The following is [6, Proposition 1.3.6].

PROPOSITION 4.4

Let X be a complex manifold, let $\mathcal{A}$ be an Azumaya algebra on X, and let α be its class in $\mathrm{Br}(X)$. If E is a locally free α-twisted coherent sheaf such that $\mathscr{E}\mathrm{nd}(E) \simeq \mathcal{A}$, then we have an equivalence

$$\mathrm{Coh}(X, \alpha) \longrightarrow \mathrm{Coh}(X, \mathcal{A}), \qquad \mathscr{F} \mapsto \mathscr{F} \otimes E^\vee.$$

We now define Chern characters, Mukai vectors, and slopes for the objects of $\mathrm{Coh}(X, \mathcal{A})$, which allow us to define a notion of stability. For $\mathscr{F} \in \mathrm{Coh}(X, \mathcal{A})$ we define

$$\mathrm{ch}_{\mathcal{A}}(\mathscr{F}) := \frac{\mathrm{ch}(\mathscr{F})}{\sqrt{\mathrm{ch}(\mathcal{A})}},$$

$$v_{\mathcal{A}}(\mathscr{F}) := \mathrm{ch}_{\mathcal{A}}(\mathscr{F}) \cdot \sqrt{\mathrm{td}(X)},$$

and if ω is a Kähler class and $\mathscr{F}$ is torsion-free, we let

$$\mu_{\mathcal{A},\omega}(\mathscr{F}) := \frac{c_{\mathcal{A},1}(\mathscr{F}) \cdot \omega}{\mathrm{rk}(\mathscr{F})},$$

where $c_{\mathcal{A},1}(\mathscr{F})$ is the component of $\mathrm{ch}_{\mathcal{A}}(\mathscr{F})$ in $H^2(S, \mathbb{Q})$. We now introduce the notion of stability for $\mathcal{A}$-modules.

DEFINITION 4.5

A torsion-free $\mathscr{F} \in \mathrm{Coh}(X, \mathcal{A})$ is *$\mu_{\mathcal{A},\omega}$-stable* if, for every $\mathscr{E} \subseteq \mathscr{F}$ in $\mathrm{Coh}(X, \mathcal{A})$ such that $0 < \mathrm{rk}(\mathscr{E}) < \mathrm{rk}(\mathscr{F})$, we have $\mu_{\mathcal{A},\omega}(\mathscr{E}) < \mu_{\mathcal{A},\omega}(\mathscr{F})$. If $\mu_{\mathcal{A},\omega}(\mathscr{E}) \leq \mu_{\mathcal{A},\omega}(\mathscr{F})$ for every such subobject, we say that $\mathscr{F}$ is *$\mu_{\mathcal{A},\omega}$-semistable*.

We note that if $\mathscr{G} \in \mathrm{Coh}(X, \alpha)$ and E is a locally free α-twisted sheaf such that $\mathscr{E}\mathrm{nd}(E) \simeq \mathcal{A}$, then $\mathrm{ch}_E(\mathscr{G}) = \mathrm{ch}_{\mathcal{A}}(\mathscr{G} \otimes E^\vee)$. It follows that

$$v_E(\mathscr{G}) = v_{\mathcal{A}}(\mathscr{G} \otimes E^\vee), \qquad \mu_{E,\omega}(\mathscr{G}) = \mu_{\mathcal{A},\omega}(\mathscr{G} \otimes E^\vee),$$

so that $\mathscr{F} \in \mathrm{Coh}(X, \alpha)$ is $\mu_{E,\omega}$-stable if and only if $\mathscr{F} \otimes E^\vee$ is $\mu_{\mathcal{A},\omega}$-stable.

REMARK 4.6

We note that $\Lambda := (\mathscr{O}_X, \mathcal{A})$ is a sheaf of rings of differential operators following the definition from [32] and that $\mathrm{Coh}(X, \mathcal{A})$ is the category of Λ-modules (always in the sense of [32]). Moreover, $\mu_{\mathcal{A},\omega}$-stable $\mathcal{A}$-modules are exactly μ-stable Λ-modules (always in the sense of [32]). Even though the definitions from [32] are given only for projective manifolds, they can immediately be extended to compact complex manifolds.

4.1.3. Twisted sheaves following Yoshioka

Yoshioka [42] introduced twisted sheaves as a full subcategory of the category of coherent sheaves on a projective bundle. More precisely, let X be a complex manifold, let $\alpha \in \mathrm{Br}(X)$, and let E be a locally free α-twisted sheaf. On an open cover $\mathscr{U} = \{U_i\}_{i\in I}$ we represent E by $\{E_i, \phi_{ij}\}_{i,j\in I}$. Let $\mathbb{P}_i := \mathbb{P}(E_i)$, together with the map $\pi_i : \mathbb{P}_i \longrightarrow U_i$. The twisted gluing data ϕ_{ij} turn to gluing data φ_{ij} of the $\mathbb{P}_i$'s and of the π_i's, giving a projective bundle $\pi : \mathbb{P} \longrightarrow X$ (whose class in $\mathrm{Br}(X)$ is α).

As shown in [42, Lemma 1.1], we have $\mathrm{Ext}^1(T_{\mathbb{P}/X}, \mathscr{O}_{\mathbb{P}}) = \mathbb{C}$. Hence, up to scalars, there is a unique nontrivial extension

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}} \longrightarrow G \longrightarrow T_{\mathbb{P}/X} \longrightarrow 0.$$

The vector bundle G can be described in another way. Fix a tautological line bundle $\mathscr{O}(\lambda_i)$ over $\mathbb{P}_i$, so that the twisted gluing data ϕ_{ij} give isomorphisms $\psi_{ij} : \mathscr{O}(\lambda_i) \longrightarrow \mathscr{O}(\lambda_j)$, and $L := \{\mathscr{O}(\lambda_i), \psi_{ij}\}$ is an $\pi^*(\alpha^{-1})$-twisted line bundle on $\mathbb{P}$. Then the vector bundles $\pi_i^* E_i(\lambda_i)$ glue together to give a locally free sheaf which is isomorphic to G.

DEFINITION 4.7

A coherent sheaf $\mathscr{F}$ on $\mathbb{P}$ is called $\mathbb{P}$-*sheaf* if the canonical morphism $\pi^*\pi_*(G^\vee \otimes \mathscr{F}) \longrightarrow G^\vee \otimes \mathscr{F}$ is an isomorphism. We let $\mathrm{Coh}(\mathbb{P}, X)$ be the full subcategory of $\mathrm{Coh}(\mathbb{P})$ given by $\mathbb{P}$-sheaves.

Lemma 1.5 of [42] shows that $\mathscr{F} \in \mathrm{Coh}(\mathbb{P}, X)$ if and only if $\mathscr{F}_{|\mathbb{P}_i} \simeq \pi^*\mathscr{E}_{|U_i} \otimes \mathscr{O}(\lambda_i)$ for some $\mathscr{E} \in \mathrm{Coh}(U_i)$. Using this, one shows the following result.

PROPOSITION 4.8

Let X be a complex manifold, and let $\pi : \mathbb{P} \longrightarrow X$ be a projective bundle whose class in $\mathrm{Br}(X)$ is α. Then there is an equivalence of categories

$$P : \mathrm{Coh}(\mathbb{P}, X) \longrightarrow \mathrm{Coh}(X, \alpha), \qquad P(\mathscr{F}) := \pi_*(\mathscr{F} \otimes L^\vee).$$

Following Yoshioka, we have a definition of the Chern character, Mukai vector, and slope of a $\mathbb{P}$-sheaf $\mathscr{F}$. More precisely, we have

$$\mathrm{ch}_{\mathbb{P}}(\mathscr{F}) := \frac{\mathrm{ch}(\pi_*(G^\vee \otimes \mathscr{F}))}{\sqrt{\mathrm{ch}(\pi_*(G^\vee \otimes G))}},$$

so that

$$v_{\mathbb{P}}(\mathscr{F}) = \mathrm{ch}_{\mathbb{P}}(\mathscr{F}) \cdot \sqrt{\mathrm{td}(S)}, \qquad \mu_{\mathbb{P},\omega}(\mathscr{F}) := \frac{c_{\mathbb{P},1}(\mathscr{F}) \cdot \omega}{\mathrm{rk}(\mathscr{F})},$$

where $c_{\mathbb{P},1}(\mathscr{F})$ is the component of $\mathrm{ch}_{\mathbb{P}}(\mathscr{F})$ in $H^2(S, \mathbb{Q})$. We now introduce the notion of stability for $\mathbb{P}$-sheaves.

DEFINITION 4.9

We say that a torsion-free $\mathscr{F} \in \mathrm{Coh}(\mathbb{P}, X)$ is $\mu_{\mathbb{P},\omega}$-*stable* if, for every subobject $\mathscr{E}$ of $\mathscr{F}$ in $\mathrm{Coh}(\mathbb{P}, X)$ such that $0 < \mathrm{rk}(\mathscr{E}) < \mathrm{rk}(\mathscr{F})$, we have $\mu_{\mathbb{P},\omega}(\mathscr{E}) < \mu_{\mathbb{P},\omega}(\mathscr{F})$. If $\mu_{\mathbb{P},\omega}(\mathscr{E}) \leq \mu_{\mathbb{P},\omega}(\mathscr{F})$ for every such subobject, we say that $\mathscr{F}$ is $\mu_{\mathbb{P},\omega}$-*semistable*.

If $\mathbb{P} = \mathbb{P}(E)$ for some locally free α-twisted sheaf E, the equivalence P gives

$$\mathrm{ch}_{\mathbb{P}}(\mathscr{F}) = \mathrm{ch}_E\big(P(\mathscr{F})\big), \qquad v_{\mathbb{P}}(\mathscr{F}) = v_E\big(P(\mathscr{F})\big),$$
$$\mu_{\mathbb{P},\omega}(\mathscr{F}) = \mu_{E,\omega}\big(P(\mathscr{F})\big).$$

It follows that $\mathscr{F} \in \mathrm{Coh}(\mathbb{P}, X)$ is $\mu_{\mathbb{P},\omega}$-stable if and only if $P(\mathscr{F})$ is $\mu_{E,\omega}$-stable.

If $\mathscr{F}$ is a $\mu_{\mathbb{P},\omega}$-stable $\mathbb{P}$-sheaf, as $\mathrm{Coh}(\mathbb{P}, S)$ is a full subcategory of $\mathrm{Coh}(\mathbb{P})$ and as the functor P is an equivalence, we have that

$$\mathrm{Ext}^1_{\mathrm{Coh}(\mathbb{P},S)}(\mathscr{F}, \mathscr{F}) \simeq \mathrm{Ext}^1_{\mathrm{Coh}(S,\alpha)}\big(P(\mathscr{F}), P(\mathscr{F})\big)$$

and

$$\mathrm{Ext}^2_{\mathrm{Coh}(\mathbb{P},S)}(\mathscr{F}, \mathscr{F}) \simeq \mathrm{Ext}^2_{\mathrm{Coh}(S,\alpha)}\big(P(\mathscr{F}), P(\mathscr{F})\big).$$

4.1.4. Chern classes following Huybrechts and Stellari

If we consider twisted sheaves following Căldăraru, there is another possible definition of their Chern classes and characters, introduced by Huybrechts and Stellari [17], which we recall here.

Consider a complex manifold X and $\alpha \in H^2(X, \mathcal{O}_X^*)$, and fix a Čech 2-cocycle $\{\alpha_{ijk}\}$ representing α on an open covering $\{U_i\}_{i \in I}$ of X. Moreover, choose a Čech 2-cocyle $\{B_{ijk}\}$, where $B_{ijk} \in \Gamma(U_{ijk}, \mathbb{Q})$, such that $\alpha_{ijk} = \exp(B_{ijk})$ (viewed as local sections of $\mathbb{R}/\mathbb{Z} = U(1) \subseteq \mathcal{O}_X^*$).

As the sheaf $\mathcal{C}_X^\infty$ of C^∞-functions on X is acyclic, up to supposing the covering $\{U_i\}_{i \in I}$ is sufficiently fine, there are $a_{ij} \in \Gamma(U_{ij}, C^\infty)$ such that

$$B_{ijk} = -a_{ij} + a_{ik} - a_{jk}.$$

Now, let us consider an α-twisted sheaf given by $\mathscr{F} = \{\mathscr{F}_i, \phi_{ij}\}$, and let

$$\psi_{ij} := \phi_{ij} \cdot \exp(a_{ij}),$$

which is clearly an isomorphism between the restrictions of $\mathscr{F}_i$ and $\mathscr{F}_j$ to U_{ij}. It is, moreover, easy to show that

$$\psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = \mathrm{id}.$$

Hence, the sheaf $\mathscr{F}_B = \{\mathscr{F}_i, \psi_{ij}\}_{i,j\in I}$ is an untwisted sheaf. We then let

$$\mathrm{ch}^B(\mathscr{F}) := \mathrm{ch}(\mathscr{F}_B).$$

The definition given in this way depends only on B.

The relation between ch^B and the previous Chern characters is explained in [18] and goes as follows, supposing that $\alpha \in \mathrm{Br}(X)$. Let E be a locally free α-twisted sheaf, and let

$$B_E := \frac{c_1^B(E)}{\mathrm{rk}(E)},$$

where $c_1^B(E)$ is the degree two part of $\mathrm{ch}^B(E)$. Then we have

$$\mathrm{ch}^B(\mathscr{F}) = \mathrm{ch}_E(\mathscr{F}) \cdot \exp(B_E).$$

4.2. Genericity for polarizations

We now extend the notion of genericity for polarization to the twisted case. As we did in Section 2.2, we first introduce a notion of discriminant for twisted sheaves, which depends on the choice of a locally free $E \in \mathrm{Coh}(S,\alpha)$.

4.2.1. Discriminant of a twisted sheaf

If $\mathscr{F}$ is an α-twisted coherent sheaf, we call the *discriminant* of $\mathscr{F}$ with respect to E the number

$$\Delta_E(\mathscr{F}) := \frac{1}{2\,\mathrm{rk}^2(\mathscr{F})} v_E^2(\mathscr{F}) + 1.$$

If $\alpha = 0$, this is just $\Delta(\mathscr{F})$ by (3). More generally, the discriminant does not depend on E, as shown in the following result.

LEMMA 4.10

Let $\alpha \in \mathrm{Br}(S)$ and $\mathscr{F} \in \mathrm{Coh}(S,\alpha)$. If $E_1, E_2 \in \mathrm{Coh}(S,\alpha)$ are locally free, then $\Delta_{E_1}(\mathscr{F}) = \Delta_{E_2}(\mathscr{F})$.

Proof

Let $E \in \mathrm{Coh}(S,\alpha)$ be locally free of rank s, and set $r := \mathrm{rk}(\mathscr{F})$, $\xi := c_1(\mathscr{F} \otimes E^\vee)$, $a := \mathrm{ch}_2(\mathscr{F} \otimes E^\vee)$, and $b := \mathrm{ch}_2(E \otimes E^\vee)$. By (2) we have

$$\Delta_E(\mathscr{F}) = \frac{1}{2r^2}\Big(\frac{\xi^2}{s^2} - \frac{2ra}{s} + \frac{r^2 b}{s^2} - 2r^2\Big) + 1.$$

An easy computation shows that

$$\begin{aligned}\frac{\xi^2}{s} - \frac{2ra}{s} + \frac{r^2 b}{s^2} &= -\frac{\mathrm{ch}_2(\mathscr{F} \otimes \mathscr{F}^\vee \otimes E \otimes E^\vee)}{s^2} + \frac{r^2\,\mathrm{ch}_2(E \otimes E^\vee)}{2s^2} \\ &= -\,\mathrm{ch}_2(\mathscr{F} \otimes \mathscr{F}^\vee),\end{aligned}$$

so that

$$\Delta_E(\mathscr{F}) = \frac{1}{2r^2}\big(-\,\mathrm{ch}_2(\mathscr{F} \otimes \mathscr{F}^\vee) - 2r^2\big) + 1, \tag{7}$$

which does not depend on E, implying the statement. □

For $v \in H^{2*}(S,\mathbb{Q})$ and $\alpha \in \mathrm{Br}(S)$, we let

$$\Delta_\alpha(v) := \Delta_{\mathscr{F}^{\vee\vee}}(\mathscr{F}),$$

where $\mathscr{F} \in \mathrm{Coh}(S,\alpha)$ is torsion-free and $v_\alpha(\mathscr{F}) = v$. By Lemma 4.10 this is well defined, and if $\alpha = 0$, then $\Delta_0(v) = \Delta(v)$. We now prove a generalization to twisted sheaves of the Bogomolov inequality.

PROPOSITION 4.11

Let $\alpha \in \mathrm{Br}(S)$, let $\mathscr{F} \in \mathrm{Coh}(S,\alpha)$, and let ω be a Kähler class on S. If $\mathscr{F}$ is $\mu_{\alpha,\omega}$-semistable, then $\Delta_\alpha(\mathscr{F}) \geq 0$.

Proof

It is easy to see that $\mathscr{F}$ is $\mu_{\alpha,\omega}$-semistable if and only if $\mathscr{F}^\vee$ is $\mu_{\alpha^{-1},\omega}$-semistable. In particular, this implies that $\mathscr{F}$ is $\mu_{\alpha,\omega}$-semistable if and only if $\mathscr{F}^{\vee\vee}$ is $\mu_{\alpha,\omega}$-semistable.

Now, note that $\mathscr{F}^{\vee\vee} \otimes \mathscr{F}^\vee = (\mathscr{F} \otimes \mathscr{F}^\vee)^{\vee\vee}$. Hence, if l is the length of the singular locus of $\mathscr{F} \otimes \mathscr{F}^\vee$, it follows that

$$\mathrm{ch}_2(\mathscr{F}^{\vee\vee} \otimes \mathscr{F}^\vee) = \mathrm{ch}_2(\mathscr{F} \otimes \mathscr{F}^\vee) + l.$$

By (2), it then follows that

$$v_\alpha^2(\mathscr{F}^{\vee\vee}) = v_\alpha^2(\mathscr{F}) - 2l \leq v_\alpha^2(\mathscr{F}).$$

As $\mathrm{rk}(\mathscr{F}) = \mathrm{rk}(\mathscr{F}^{\vee\vee})$ it follows that $\Delta_\alpha(\mathscr{F}) \geq \Delta_\alpha(\mathscr{F}^{\vee\vee})$. Hence, we just need to show the statement for $\mathscr{F}^{\vee\vee}$.

Now let $F := \mathscr{F}^{\vee\vee}$, which is locally free and $\mu_{\alpha,\omega}$-semistable. By the Kobayashi–Hitchin correspondence for twisted sheaves as proved by Wang [41], the sheaf $\mathscr{E}\mathrm{nd}(F) = F \otimes F^\vee$ is μ_ω-polystable, so that $\Delta(F \otimes F^\vee) \geq 0$ by the Bogomolov inequality.

Choose now a locally free $E \in \mathrm{Coh}(S,\alpha)$ of rank s, and let $b := \mathrm{ch}_2(E \otimes E^\vee)$. By Lemma 4.10 we have $\Delta(F \otimes F^\vee) = \Delta_{E\otimes E^\vee}(F \otimes F^\vee)$, so that $\Delta_{E\otimes E^\vee}(F \otimes F^\vee) \geq 0$. If $\xi = c_1(F \otimes E^\vee)$ and $a = \mathrm{ch}_2(F \otimes E^\vee)$, it follows from (2) that

$$v_{E\otimes E^\vee}^2(F \otimes F^\vee) = \frac{2r^2\xi^2}{s^2} - \frac{4r^3 a}{s} + \frac{2r^2 b}{s^2} - 2r^4 = 2r^2 v_E^2(F) + 2r^4.$$

Hence,

$$0 \leq \Delta_{E\otimes E^\vee}(F \otimes F^\vee) = \frac{1}{2r^4} v_{E\otimes E^\vee}^2(F \otimes F^\vee) + 1 = 2\Delta_E(F).$$

Hence, $\Delta_\alpha(F) = \Delta_E(F) \geq 0$, and we are done. □

4.2.2. Walls and chambers

Now, let

$$W_{\alpha,v} := \left\{ D \in NS(S) \;\middle|\; -\frac{r^4}{2}\Delta_\alpha(v) \leq D^2 < 0 \right\}.$$

If $\alpha = 0$, then we have $W_{0,v} = W_v$.

DEFINITION 4.12

For $D \in W_{\alpha,v}$, we call the hyperplane $D^{\perp}$ an (α,v)-*wall*. A connected component of $\mathscr{K}_S \setminus \bigcup_{D\in W_{\alpha,v}} D^{\perp}$ is called an (α,v)-*chamber*. A polarization $\omega \in \mathscr{K}_S$ is (α,v)-*generic* if it lies in an (α,v)-chamber.

A polarization ω is $(0,v)$-generic if and only if it is v-generic. We are now ready to prove one of the main results of this section about changing polarization inside a chamber. The argument is the same as that for untwisted sheaves, here adapted to the twisted case.

PROPOSITION 4.13

Let $\alpha \in \mathrm{Br}(S)$, let $v \in H^{2*}(S,\mathbb{Q})$, and let ω, ω' be two (α,v)-generic polarizations lying in the same (α,v)-chamber. If $\mathscr{F} \in \mathrm{Coh}(S,\alpha)$ is a torsion-free sheaf such that $v_\alpha(\mathscr{F}) = v$, then $\mathscr{F}$ is $\mu_{\alpha,\omega}$-stable if and only if it is $\mu_{\alpha,\omega'}$-stable.

Proof

The proof is divided into two steps.

Step 1. Choose an α-twisted locally free sheaf E, and let $r := \mathrm{rk}(\mathscr{F})$, $\xi := c_1(\mathscr{F} \otimes E^{\vee})$, $a := \mathrm{ch}_2(\mathscr{F} \otimes E^{\vee})$, $s := \mathrm{rk}(E)$, and $b := \mathrm{ch}_2(E \otimes E^{\vee})$. Let ω be any polarization, and suppose that $\mathscr{F}$ is properly $\mu_{\alpha,\omega}$-semistable. Hence, there is an α-twisted subsheaf $\mathscr{E} \subseteq \mathscr{F}$ such that $\mu_{E,\omega}(\mathscr{E}) = \mu_{E,\omega}(\mathscr{F})$. We let $r' := \mathrm{rk}(\mathscr{E})$, $\xi' := c_1(\mathscr{E} \otimes E^{\vee})$, and $a' := \mathrm{ch}_2(\mathscr{E} \otimes E^{\vee})$, where $0 < r' < r$ and $\xi' \in NS(S) \otimes \mathbb{Q}$. Moreover, let

$$D := r\frac{\xi'}{s} - r'\frac{\xi}{s},$$

so that $D \cdot \omega = 0$. Hence, $D^2 \leq 0$, as ω is a Kähler form.

Now, let $\mathscr{G} := \mathscr{F}/\mathscr{E}$, and we suppose without loss of generality that $\mathscr{E}$ is saturated and that $\mathscr{E}$ and $\mathscr{G}$ are $\mu_{\alpha,\omega}$-semistable. Moreover, let $r'' := \mathrm{rk}(\mathscr{G})$, $\xi'' := c_1(\mathscr{G} \otimes E^{\vee})$, and $a'' := \mathrm{ch}_2(\mathscr{G} \otimes E^{\vee})$, so that $r'' = r - r'$, $\xi'' = \xi - \xi'$, and $a'' = a - a'$. Finally, let

$$K := \frac{v_\alpha^2(\mathscr{F})}{r} - \frac{v_\alpha^2(\mathscr{E})}{r'} - \frac{v_\alpha^2(\mathscr{G})}{r''}.$$

We note that as $\mathscr{E}$ and $\mathscr{G}$ are $\mu_{\alpha,\omega}$-semistable, by Proposition 4.11 we have $\Delta_\alpha(\mathscr{E}), \Delta_\alpha(\mathscr{G}) \geq 0$, meaning $v_\alpha^2(\mathscr{E}) \geq -2(r')^2$ and $v_\alpha^2(\mathscr{G}) \geq -2(r'')^2$. Hence, we get

$$K \leq \frac{v_\alpha^2(\mathscr{F})}{r} + 2r. \tag{8}$$

On the other hand, by (2) we have

$$K = \frac{\xi^2}{rs^2} - \frac{(\xi')^2}{r's^2} - \frac{(\xi'')^2}{r''s^2} = -\frac{r^2(\xi')^2 + (r')^2\xi^2 - 2rr'\xi\xi'}{s^2rr'r''}.$$

By the definition of D we have

$$D^2 = \frac{r^2(\xi')^2 + (r')^2\xi^2 - 2rr'\xi\xi'}{s^2},$$

so that (8) implies

$$D^2 = -rr'r''K \geq -r'(r-r')v_\alpha^2(\mathscr{F}) - 2r^2r'(r-r').$$

But as $r'(r-r') \leq r^2/4$, we finally get

$$D^2 \geq -\frac{r^2}{4}v_\alpha^2(\mathscr{F}) - \frac{r^4}{2} = -\frac{r^4}{2}\Delta_\alpha(\mathscr{F}) = -\frac{r^4}{2}\Delta_\alpha(v).$$

In conclusion, $D \in W_{\alpha,v} \cup \{0\}$.

Step 2. Suppose that $\mathscr{F}$ is $\mu_{\alpha,\omega}$-stable but not $\mu_{\alpha,\omega'}$-stable. Let

$$[\omega,\omega'] := \{\omega_t := t\omega' + (1-t)\omega \mid t \in [0,1]\}$$

be the segment from ω to ω', and let B_α be the family of subsheaves of $\mathscr{F}$ whose slope with respect to E and ω' is bounded from below.

If $\mathscr{E} \in B_\alpha$, then $\mathscr{E} \otimes E^\vee$ is a subsheaf of $\mathscr{F} \otimes E^\vee$, and $\mu_{E,\omega}(\mathscr{E}) = \mu_\omega(\mathscr{E} \otimes E^\vee)$. This implies that $\mathscr{E} \otimes E^\vee$ is in the family B of subsheaves of $\mathscr{F} \otimes E^\vee$ whose slope with respect to ω is bounded from below. As the family B is bounded, it follows that the family B_α is bounded. Using the same argument as in the proof of Lemma 3.1, one can then conclude that there is $t \in]0,1]$ such that $\mathscr{F}$ is properly μ_{α,ω_t}-semistable.

Hence, there is a subsheaf $\mathscr{E}$ of $\mathscr{F}$ of rank $0 < s < r$ such that $\mu_{E,\omega_t}(\mathscr{E}) = \mu_{E,\omega_t}(\mathscr{F})$. If $D = rc_{E,1}(\mathscr{E}) - sc_{E,1}(\mathscr{F})$, it follows that $D \cdot \omega_t = 0$. As $D \cdot \omega \neq 0$, we have $D \neq 0$. Hence, $D^2 < 0$. But as $\mathscr{F}$ is μ_{E,ω_t}-semistable, Step 1 implies that $D \in W_{\alpha,v}$, which is not possible as ω_t is in the same (α,v)-chamber as ω and ω'. In conclusion, the sheaf $\mathscr{F}$ has to be $\mu_{E,\omega'}$-stable, and we are done. □

4.3. Moduli space of stable twisted sheaves

We now introduce (relative) moduli spaces of stable twisted sheaves. On projective manifolds these were constructed by Yoshioka [42] (viewing twisted sheaves as $\mathbb{P}$-sheaves and using a geometric invariant theory construction) and by Lieblich [25] (viewing twisted sheaves as sheaves on some $\mathscr{O}^*$-gerbe).

Here we first provide a relative moduli space of simple twisted sheaves by viewing them as simple $\mathbb{P}$-sheaves. The relative moduli space of stable sheaves will then be an open subset of it.

4.3.1. The relative moduli space of simple twisted sheaves

Consider a smooth and proper morphism $\pi : \mathscr{X} \longrightarrow T$ such that for every $t \in T$ the fiber X_t over t is a K3 surface. We assume for simplicity that T is a complex manifold, although the constructions work over complex spaces as well.

Suppose, moreover, that we are given a complex manifold $\mathscr{P}$ together with a morphism $f : \mathscr{P} \longrightarrow \mathscr{X}$ of T-complex spaces such that, for every $t \in T$, the morphism $f_t : P_t \longrightarrow X_t$ is a projective bundle, where $P_t = f^{-1}(X_t)$. For every $t \in T$ the projective bundle $P_t \longrightarrow X_t$ defines a class α_t in the Brauer group $\mathrm{Br}(X_t)$.

Now, let $f' := \pi \circ f$, so that we get a map $f' : \mathscr{P} \longrightarrow T$. By [22, Theorem 6.4], there is a complex space $\mathscr{M}(\mathscr{P}/T)$ together with a holomorphic surjective map

$$\phi : \mathscr{M}(\mathscr{P}/T) \longrightarrow T$$

which is a relative moduli space of simple coherent sheaves on $\mathscr{P}$. For every $t \in T$ the fiber $\mathscr{M}_t$ of ϕ over t is the moduli space of simple coherent sheaves on P_t.

Now, $\mathscr{F} \in \mathrm{Coh}(P_t)$ is simple if and only if $\mathrm{End}(\mathscr{F}) \simeq \mathbb{C}$. As $\mathrm{Coh}(P_t, X_t)$ is a full subcategory of $\mathrm{Coh}(P_t)$, a P_t-sheaf $\mathscr{F}$ is simple in $\mathrm{Coh}(P_t, X_t)$ if and only if it is simple in $\mathrm{Coh}(P_t)$. Hence, simple P_t-sheaves form a subset $\mathscr{M}^s(\mathscr{P}/T)$ of $\mathscr{M}(\mathscr{P}/T)$.

As the condition defining $\mathbb{P}$-sheaves is open (see [42, Lemma 1.5]), it follows that $\mathscr{M}^s(\mathscr{P}/T)$ is open in $\mathscr{M}(\mathscr{P}/T)$. Hence, it is a complex space together with a holomorphic map $\psi : \mathscr{M}^s(\mathscr{P}/T) \longrightarrow T$. This is the relative moduli space of simple $\mathscr{P}$-sheaves on $\mathscr{X}$.

The relative projective bundle $f : \mathscr{P} \longrightarrow \mathscr{X}$ corresponds to the existence of a relative Azumaya algebra $\mathcal{A}$ on $\mathscr{X}$. For every $t \in T$, we have $P_t = \mathbb{P}(E_t)$ for some locally free α_t-twisted sheaf on X_t, and we let $\mathcal{A}_t := E_t \otimes E_t^\vee$. The previous equivalence of categories of twisted sheaves then shows that $\mathscr{M}^s(\mathscr{P}/T)$ is the relative moduli space of simple $\mathcal{A}$-modules on $\mathscr{X}$ or, equivalently, the relative moduli space of simple twisted sheaves on $\mathscr{X}$.

4.3.2. The relative moduli space of stable twisted sheaves

We now produce out of $\psi : \mathscr{M}^s(\mathscr{P}/T) \longrightarrow T$ the relative moduli space of stable twisted sheaves. Choose $v = (v_0, v_1, v_2) \in H^{2*}(S, \mathbb{Q})$ such that $v_1 \in NS(S_t)$ for every $t \in T$ and $v_0 \geq 2$. We let $\mathscr{M}^s_v(\mathscr{P}/T)$ be the component of $\mathscr{M}^s(\mathscr{P}/T)$ parameterizing simple $\mathbb{P}$-sheaves of Mukai vector v, and we write $\psi_v : \mathscr{M}^s_v(\mathscr{P}/T) \longrightarrow T$ for $\psi_{|\mathscr{M}^s_v(\mathscr{P}/T)}$.

In order to define the moduli space of stable twisted sheaves of Mukai vector v, we need a section $\widetilde{\omega} \in R^2\pi_*\mathbb{C}$ such that $\omega_t := \widetilde{\omega}_{|X_t}$ is a Kähler class for every $t \in T$, which is used to define stability on every fiber. As stable twisted sheaves are simple, we let $\mathscr{M}^\mu_v(\mathscr{P}/T, \widetilde{\omega})$ be the subset of $\mathscr{M}^s_v(\mathscr{P}/T)$ whose fiber over $t \in T$ is given by the simple P_t-sheaves which are μ_{P_t,ω_t}-stable and whose Mukai vector is v. We then have a natural map (of sets)

$$p : \mathscr{M}^\mu_v(\mathscr{P}/T, \widetilde{\omega}) \longrightarrow T.$$

The main result of this section is the following.

PROPOSITION 4.14

Let $\pi : \mathscr{X} \longrightarrow T$, $f : \mathscr{P} \longrightarrow \mathscr{X}$, v, and $\widetilde{\omega}$ be as before. Then $\mathscr{M}^\mu_v(\mathscr{P}/T, \widetilde{\omega})$ is an open subset of $\mathscr{M}^s_v(\mathscr{P}/T)$. Hence, it is a complex manifold, and the map p is holomorphic.

Proof

By Remark 4.6, the openness can be proved as in [32, Lemma 3.7]. Indeed, if $\mathscr{F} \in \mathrm{Coh}(\mathbb{P}, S)$ and $F := P(\mathscr{F})^{\vee\vee}$, then $\mathscr{F}$ is $\mu_{\mathbb{P},\omega}$-stable if and only if $P(\mathscr{F}) \otimes$

$P(F)^{\vee}$ is $\mu_{\mathcal{A},\omega}$-stable in $\mathrm{Coh}(S,\mathcal{A})$, where $\mathcal{A} = P(F) \otimes P(F)^{\vee}$. Moreover, the openness of stability in the analytic case may be proved in the usual way, by using boundedness results which are contained in [37] and [38]. The separatedness follows from [22, Proposition 6.6] since the parameterized sheaves are stable. □

Standard deformation arguments following [5] allow us to show that if $p : \mathscr{M} := \mathscr{M}_v^{\mu}(\mathscr{P}/\mathscr{X}, \widetilde{\omega}) \longrightarrow T$ is the relative moduli space of twisted stable sheaves, then for every $t \in T$ and for every $\mathscr{F} \in p^{-1}(t) = \mathscr{M}_t$ we have

$$T_{[\mathscr{F}]}\mathscr{M}_t \simeq \mathrm{Ext}^1_{\mathrm{Coh}(X_t,\alpha_t)}\big(P(\mathscr{F}), P(\mathscr{F})\big),$$

that the obstruction for the existence of deformations of $\mathscr{F}$ lives in

$$\mathrm{Ext}^2_{\mathrm{Coh}(X_t,\alpha_t)}\big(P(\mathscr{F}), P(\mathscr{F})\big),$$

and that we have an exact sequence

$$\begin{aligned} \mathrm{Ext}^1_{\mathrm{Coh}(X_t,\alpha_t)}\big(P(\mathscr{F}), P(\mathscr{F})\big) &\longrightarrow T_{[\mathscr{F}]}\mathscr{M} \\ &\longrightarrow T_tT \longrightarrow \mathrm{Ext}^2_{\mathrm{Coh}(X_t,\alpha_t)}\big(P(\mathscr{F}), P(\mathscr{F})\big)_0. \end{aligned} \tag{9}$$

It follows, from this exact sequence and by the previous discussion, that the morphism $p : \mathscr{M} \longrightarrow T$ is smooth.

If T is reduced to a point, then X is just a K3 surface S and $P \longrightarrow S$ is a projective bundle whose class in $\mathrm{Br}(S)$ is α. The *moduli space of $\mu_{\alpha,\omega}$-stable α-twisted sheaves of twisted Mukai vector v on S* will then be denoted $M^{\mu}_{\alpha,v}(S,\omega)$.

REMARK 4.15

Suppose that $\alpha = 0$, and let

$$\gamma := \frac{\mathrm{ch}(\mathscr{F}^{\vee})}{\sqrt{\mathrm{ch}(\mathscr{F} \otimes \mathscr{F}^{\vee})}}$$

for $\mathscr{F} \in M^{\mu}_{0,v}(S,\omega)$. Then $v_0(\mathscr{F}) = v$ if and only if $v(\mathscr{F}) = v/\gamma$, so that $M^{\mu}_{0,v}(S,\omega) \simeq M^{\mu}_{v/\gamma}(S,\omega)$. We even notice that ω is $(0,v)$-generic if and only if it is v/γ-generic.

4.3.3. Moduli spaces of stable twisted sheaves over projective K3 surfaces

If the base K3 surface S is projective, from [42] we have some more information about the moduli spaces of stable twisted sheaves. We make use of the following notation: let $\alpha \in \mathrm{Br}(S)$, and let $\mathscr{F}$ be a torsion-free α-twisted sheaf whose twisted Mukai vector is $w = (r, 0, a)$.

We let F be a locally free α-twisted sheaf and ξ be a representative of the class of $\mathbb{P}(E)$ in $H^2(S,\mathbb{Z})$. We let $e^{\xi/r} := (1, \xi/r, \xi^2/2r^2)$ and $w_{\xi} := e^{\xi/r} \cdot w$, so that

$$w_{\xi} = (r, \xi, a + \xi^2/2r).$$

It is worth noting that there is a topological vector bundle E_ξ on S such that $v(E_\xi) = w_\xi$. As shown in [42], we have $w_\xi \in H^2(S,\mathbb{Z})$ (while in general we have $w \in H^2(S,\mathbb{Q})$).

REMARK 4.16
If $\alpha = 0$ and $\mathscr{F}$ is a μ_ω-stable sheaf whose Mukai vector is $v = (r,\xi,a)$, write $a = c + r$, where $c = \mathrm{ch}_2(\mathscr{F})$. The 0-twisted Mukai vector of $\mathscr{F}$ is then $w = (r, 0, r + a'/2r)$, where $a' = \mathrm{ch}_2(\mathscr{F} \otimes \mathscr{F}^{\vee\vee})$. We note that $a' = 2rc - \xi^2$. Hence, $w = (r, 0, r + c - \xi^2/2r)$. A representative of the class of $\mathbb{P}(E)$ in this case can be chosen to be ξ itself. Hence, we have

$$w_\xi = e^{\xi/r} w = (r, \xi, r + c) = v.$$

The following is [42, Theorem 3.16].

THEOREM 4.17
Let S be a projective K3 surface, let $w = (r,\zeta,b) \in H^2(S,\mathbb{Q})$, and let $\alpha \in \mathrm{Br}(S)$. Choose a representative ξ of α in $H^2(S,\mathbb{Z})$, and suppose that w_ξ is primitive. Moreover, let H be an (α,w)-generic ample line bundle on S. Then the moduli space $M^s_{\alpha,w}(S,H)$ is an irreducible symplectic manifold which is deformation equivalent to a Hilbert scheme of points on S.

We have the following result, which is the twisted version of Theorem 3.4.

PROPOSITION 4.18
Let S be a projective K3 surface, let $w = (r,\zeta,b)$ be a Mukai vector, and let $\alpha \in \mathrm{Br}(S)$. Choose ξ to be a representative of α in $H^2(S,\mathbb{Z})$, and suppose that r and ξ are prime to each other. If ω is an (α,w)-generic polarization, then $M^\mu_{\alpha,w}(S,\omega)$ is an irreducible symplectic manifold which is deformation equivalent to a Hilbert scheme of points on S.

Proof
By Lemma 3.3 and Proposition 4.13 and following the proof of Theorem 3.4, we see that there is an ample line bundle H such that $M^\mu_{\alpha,w}(S,\omega) = M^\mu_{\alpha,w}(S,H)$. This last moduli space is an irreducible symplectic manifold which is deformation equivalent to a Hilbert scheme of points on S by Theorem 4.17. □

4.3.4. Quasi-universal families
We conclude this section with the following result about the existence of a quasi-universal family (see [1] for the absolute untwisted case).

PROPOSITION 4.19
Let $\pi : \mathscr{X} \longrightarrow T$, $f : \mathscr{P} \longrightarrow \mathscr{X}$, $v = (v_0, v_1, v_2)$, and $\widetilde{\omega}$ be as before. Let $\mathcal{A}$ be a relative Azumaya algebra corresponding to $\mathscr{P}$, and for every $t \in T$ let $\alpha_t \in \mathrm{Br}(X_t)$ be the class of $\mathcal{A}_t$. Suppose that there is a locally free $\mathcal{A}$-module $\mathcal{V}$ verifying the

following two properties for every $t \in T$:

(1) *the restriction $\mathcal{V}_t$ of $\mathcal{V}$ to X_t is μ_{α_t,ω_t}-stable;*
(2) *the twisted Mukai vector of $\mathcal{V}_t$ is (v_0, v_1, w_2), where $w_2 < v_2$.*

Then there is a quasi-universal family on $\mathscr{M}^{\mu}_{v}(\mathscr{P}/T, \widetilde{\omega}) \times_T \mathscr{X}$.

Proof
Let $\mathscr{M} := \mathscr{M}^{\mu}_{\widetilde{v}}(\mathscr{P}/T, \widetilde{\omega})$. As for stable coherent sheaves, there is an open covering $\mathscr{U} = \{U_i\}_{i\in I}$ of $\mathscr{M}$ given by analytic subsets endowed with universal $\mathcal{A}$-modules $\mathscr{F}_i$.

Let $p_i : U_i \times_T \mathscr{X} \to U_i$ and $q_i : U_i \times_T \mathscr{X} \to \mathscr{X}$ denote the two projections. We put $\mathscr{E}_i := \mathscr{F}_i \otimes_{q_i^*\mathcal{A}} q_i^*\mathcal{V}^\vee$. By the choice of $\mathcal{V}$ we have $R^0 p_{i,*}\mathscr{E}_i = 0 = R^2 p_{i,*}\mathscr{E}_i$, and $W_i := R^1 p_{i,*}\mathscr{E}_i$ is a nontrivial locally free $\mathscr{O}_{U_i}$-module whose rank is independent of i. It is easy to check now that the $\mathcal{A}$-modules $\mathscr{F}_i \otimes_{\mathscr{O}} p_i^* W_i^\vee$ glue together to give the desired quasi-universal family. □

4.4. Deformation of stable twisted sheaves along twistor lines

In this section we describe and generalize a construction used by several authors in the case of stable locally free sheaves of slope zero (see [35], [39], [40], [27]).

Let (S, I, ω) be a polarized K3 surface, and let $\pi : Z(S) \longrightarrow \mathbb{P}^1$ be its twistor family. We suppose that the fiber over 0 is $S_0 = (S, I)$, and we write $S_t = (S, I_t)$ for the fiber over t. Here $I = I_0$ and I_t denote the complex structures on S. With this convention we have $S_\infty = (S, I_\infty) = (S, -I)$. Recall that the choice of ω on (S, I) is equivalent to the choice of a Riemannian metric g which is compatible with I and whose associated Kähler class is ω. Along the twistor line the metric g remains compatible with I_t, the associated class ω_t is Kähler, and we get a section $\widetilde{\omega}$ of $R^2\pi_*\mathbb{C}$ which is ω_t on S_t. Slope stability on S_t will be considered with respect to ω_t.

Before we introduce deformations of sheaves along twistor lines, we make an observation about $(1,1)$-forms on the twistor space of S. Recall that, as a differentiable manifold, $Z(S)$ is the product $S \times \mathbb{P}^1$, which is endowed with a complex structure in the following way (see [13]). Cover $\mathbb{P}^1$ by two charts (each isomorphic to $\mathbb{C}$), and take ζ as the complex coordinate function on one of them and ζ^{-1} as that on the other. Further, let I, J, K be the complex structures on S which make it into a hyper-Kähler manifold. Denoting by $I_{\mathbb{P}^1}$ the complex structure on $\mathbb{P}^1$, put the following complex structure to act on the tangent space $T_S \times T_{\mathbb{P}^1}$ of $S \times \mathbb{P}^1$:

$$\mathfrak{I} := \Big(\frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}} I + \frac{\zeta+\bar{\zeta}}{1+\zeta\bar{\zeta}} J + i\frac{\zeta-\bar{\zeta}}{1+\zeta\bar{\zeta}} K, I_{\mathbb{P}^1}\Big).$$

With respect to this complex structure the projection $q : S \times \mathbb{P}^1 \to S$ is not holomorphic but only C^∞.

LEMMA 4.20
Let ψ be a $(1,1)$-form on (S, I, ω). Its pullback $q^*\psi$ is a $(1,1)$-form on $Z(S)$ if and only if ψ is anti-self-dual on (S, I, ω).

Proof
Let $\Psi := q^*\psi$. It is a 2-form on $Z(S)$, so it is of type $(1,1)$ if and only if $\Psi(\mathfrak{I}v, \mathfrak{I}w) = \Psi(v,w)$ for any pair (v,w) of real tangent vectors at a point of $Z(S)$. As $\mathfrak{I}$ preserves the horizontal and the vertical directions on $Z(S) = S \times \mathbb{P}^1$ and as $\Psi(v,w) = 0$ if one of the tangent vectors v or w is horizontal, it suffices to check that $\Psi(\mathfrak{I}v, \mathfrak{I}w) = \Psi(v,w)$ only on vertical vectors, meaning that the restrictions of Ψ to the fibers of $\pi : Z(S) \longrightarrow \mathbb{P}^1$ are of type $(1,1)$.

Suppose that ψ is anti-self-dual. This property only depends on g and on the orientation of S. As g is compatible with each complex structure I_t, it follows that the restriction of Ψ to each fiber of π is then anti-self-dual. In particular, it is of type $(1,1)$. Hence, Ψ is also of type $(1,1)$ on $Z(S)$.

Conversely, if ψ is not anti-self-dual, then it decomposes as $\psi = \psi^{\mathrm{SD}} + \psi^{\mathrm{ASD}}$, where the self-dual part is of the form $\psi^{\mathrm{SD}} = f\omega_I$ for some nonzero function f. But ω_I is not of type $(1,1)$ with respect to J so neither will be Ψ. □

We now turn to deformations of sheaves along the twistor line.

4.4.1. Deformation of a locally free polystable sheaf with trivial slope
Let E_0 be a polystable vector bundle on S_0 whose slope is zero, and denote by E^∞ the C^∞-vector bundle underlying E_0. The Kobayashi–Hitchin correspondence provides E^∞ with an anti-self-dual connection. By Lemma 4.20, the curvature of the connection is of $(1,1)$-type on each S_t. We therefore obtain holomorphic structures E_t on E^∞ over each S_t, induced by the structure E_0 in a canonical way. In fact, we even get a holomorphic structure on q^*E^∞. Denote by $\tilde{E}$ the corresponding sheaf of holomorphic sections over $Z(S)$. As E_t is holomorphic and carries an anti-self-dual connection, it is polystable for every $t \in \mathbb{P}^1$. It is easy to see that if E_0 is stable, then E_t is stable for every $t \in \mathbb{P}^1$.

4.4.2. Deformation of an Azumaya algebra
Let now $\mathcal{A}_0$ be an Azumaya algebra on S_0, and let α_0 be its class in $\mathrm{Br}(S_0)$. Choose a locally free α_0-twisted sheaf E_0 such that $\mathcal{A}_0 \simeq \mathscr{E}\mathrm{nd}(E_0)$. We will suppose that E_0 is μ_{α_0,ω_0}-stable.

The Kobayashi–Hitchin correspondence for twisted sheaves established by Wang [41] shows that $\mathcal{A}_0$ is μ_{ω_0}-polystable. Note that $\mu_{\omega_0}(\mathcal{A}_0) = 0$. Hence, by Section 4.4.1 the vector bundle $\mathcal{A} := q^*\mathcal{A}_0$ carries a holomorphic structure, and for every $t \in \mathbb{P}^1$ its restriction $\mathcal{A}_t$ to the fiber S_t is a μ_{ω_t}-polystable vector bundle with trivial slope. We need to show that $\mathcal{A}_t$ is an Azumaya algebra.

To do so, we argue as in the proof of [27, Lemma 6.5(3)]. The Azumaya algebra structure on $\mathcal{A}_0$ is given by a holomorphic map $m_0 : \mathcal{A}_0 \otimes \mathcal{A}_0 \longrightarrow \mathcal{A}_0$ verifying some identities among holomorphic sections. This means that m_0 is a holomorphic section of the vector bundle $\mathscr{H}\mathrm{om}(\mathcal{A}_0 \otimes \mathcal{A}_0, \mathcal{A}_0)$. But this is μ_{ω_0}-polystable as $\mathcal{A}_0$ is; hence, it carries an anti-self-dual connection, and m_0 is parallel with respect to it.

As a consequence, m_0 defines a parallel section of $\mathscr{H}om(\mathcal{A}_t \otimes \mathcal{A}_t, \mathcal{A}_t)$ and, hence, a holomorphic map $m_t : \mathcal{A}_t \otimes \mathcal{A}_t \longrightarrow \mathcal{A}_t$. Hence, $\mathcal{A}_t$ is an $\mathscr{O}_{S_t}$-algebra. As the same identities among sections which are verified on $\mathcal{A}_0$ are verified even on $\mathcal{A}_t$, it follows that $\mathcal{A}_t$ is an Azumaya algebra.[1]

4.4.3. Deformation of a stable twisted vector bundle

Let $\alpha_0 \in \mathrm{Br}(S_0)$, and let F_0 be an α_0-twisted locally free sheaf which is μ_{α_0,ω_0}-stable. Choose an α_0-twisted locally free sheaf E_0 which is μ_{α_0,ω_0}-stable in such a way that $c_{E_0,1}(\mathscr{F}_0) = 0$.

We let $G_0 := F_0 \otimes E_0^\vee$ and $\mathcal{A}_0 := E_0 \otimes E_0^\vee$. Then $\mathcal{A}_0$ is an Azumaya algebra, and as we saw in Section 4.4.2, it is a polystable sheaf. Moreover, G_0 is a locally free sheaf of trivial slope, and it has the structure of an $\mathcal{A}_0$-module. The Kobayashi–Hitchin correspondence for twisted sheaves in [41] shows that G_0 is a polystable sheaf.

Following Section 4.4.2, $q^*\mathcal{A}_0$ is a holomorphic vector bundle, and for every $t \in \mathbb{P}^1$ its restriction $\mathcal{A}_t$ to S_t is a polystable sheaf having the structure of an Azumaya algebra. We let α_t be its class in $\mathrm{Br}(S_t)$.

By Section 4.4.2 the polystable sheaf G_0 gives rise, for every $t \in \mathbb{P}^1$, to a polystable sheaf G_t with trivial slope. The same argument used in Section 4.4.2 to show that $\mathcal{A}_t$ is an Azumaya algebra, applied this time to $m_t : \mathcal{A}_t \otimes G_t \longrightarrow G_t$, shows that the sheaf G_t has the structure of an $\mathcal{A}_t$-module.

As G_t is an $\mathcal{A}_t$-module, it corresponds to an α_t-twisted locally free sheaf F_t on S_t. In particular, E_0 gives rise to an α_t-twisted locally free sheaf E_t on S_t such that $\mathscr{E}nd(E_t) \simeq \mathcal{A}_t$ and $F_t \otimes E_t^\vee \simeq G_t$.

LEMMA 4.21

The sheaves F_t and E_t are μ_{α_t,ω_t}-stable.

Proof

We show that E_t is μ_{α_t,ω_t}-stable. The proof for F_t is similar. Suppose that E_t is not μ_{α_t,ω_t}-stable, and let $\mathscr{E}_t \subseteq E_t$ in $\mathrm{Coh}(S_t, \alpha_t)$ with $\mu_{E_t,\omega_t}(\mathscr{E}_t) \geq \mu_{E_t,\omega_t}(E_t)$. We suppose that $\mathscr{E}_t$ is μ_{α_t,ω_t}-stable.

We let $\mathcal{H}_t := \mathscr{E}_t \otimes E_t^\vee$, which is an $\mathcal{A}_t$-module, and we have $\mathcal{H}_t \subseteq \mathcal{A}_t$. The inequality $\mu_{E_t,\omega_t}(\mathscr{E}_t) \geq \mu_{E_t,\omega_t}(E_t)$ gives $\mu_{\mathcal{A}_t,\omega_t}(\mathcal{H}_t) \geq \mu_{\mathcal{A}_t,\omega_t}(\mathcal{A}_t)$, so that

[1] If E_0 is an untwisted sheaf, we can give a more direct proof. The multiplication of two holomorphic sections ϕ_1, ϕ_2 of $\mathcal{A}_t$ remains holomorphic. (Hence, $\mathcal{A}_t$ is a sheaf of algebras on S_t.) This is a consequence of the formula $\hat{D}(\phi_1 \circ \phi_2) = \hat{D}\phi_1 \circ \phi_2 + \phi_1 \circ \hat{D}\phi_2$, where $\hat{D}$ is the connection induced by D on $\mathcal{A}_0$.

By [6, Theorem 1.1.6], we just need to show that $\mathcal{A}_t$ is locally of the form $\mathscr{E}nd(F)$ for some locally free sheaf F of $\mathscr{O}_{S_t}$-modules. To do so, consider the self-dual part R_{SD} of the curvature R of D. We have that $R_{\mathrm{SD}} = c \cdot \mathrm{Id} \cdot \omega_0$ for a suitable constant c. By solving the equation $dd^c\phi = -\frac{c}{r}\omega_0$ on an open subset U, we find a holomorphic Hermitian line bundle (L, h) on U whose curvature is $-\frac{c}{r}\omega_0$. Hence, $F^\infty := E_0 \otimes L$ is a rank r vector bundle on U with a Hermite–Einstein connection, and $\mathcal{A}^\infty \cong \mathscr{E}nd^\infty(F^\infty)$ as anti-self-dual vector bundles. Hence, on F^∞ we have a holomorphic structure F_t compatible with the corresponding I_t, and $\mathcal{A}_t \cong \mathscr{E}nd(F_t)$.

$\mu_{\omega_t}(\mathcal{H}_t) \geq \mu_{\omega_t}(\mathcal{A}_t) = 0$. As $\mathcal{A}_t$ is μ_{ω_t}-polystable, this implies that $\mu_{\omega_t}(\mathcal{H}_t) = 0$ and that it is a direct summand of $\mathcal{A}_t$. In particular, it is μ_{ω_t}-polystable.

By using the same argument given before, the sheaf $\mathcal{H}_t$ gives rise to a μ_{ω_0}-polystable sheaf $\mathcal{H}_0$ on S_0, which is contained in $\mathcal{A}_0$ and has the structure of an $\mathcal{A}_0$-module, and $\mu_{\omega_0}(\mathcal{H}_0) = \mu_{\omega_0}(\mathcal{A}_0) = 0$. The equivalence between $\mathrm{Coh}(S_0, \alpha_0)$ and $\mathrm{Coh}(S_0, \mathcal{A}_0)$ given by tensoring with $E_0^\vee$ then produces a subsheaf $\mathscr{E}_0$ of E_0 such that $\mu_{E_0,\omega_0}(\mathscr{E}_0) = \mu_{E_0,\omega_0}(E_0)$. But this is not possible as E_0 is μ_{α_0,ω_0}-stable. In conclusion, the sheaf E_t is μ_{α_t,ω_t}-stable. □

4.5. Relative moduli space of twisted sheaves on twistor lines

In this section we show that the relative moduli space of stable twisted sheaves gives us a way to deform the moduli spaces $M^\mu_{\alpha,w}(S,\omega)$ to an irreducible symplectic manifold (which is moreover deformation equivalent to a Hilbert scheme of points on a projective K3 surface).

We let S be a K3 surface, $w = (r, 0, a) \in H^{2*}(S, \mathbb{Z})$ with $r \geq 2$, $\alpha \in \mathrm{Br}(S)$, and ω be an (α, w)-generic polarization. The Kähler class ω corresponds to the choice of a Riemannian metric g which is compatible with the complex structure I of S and whose associated Kähler class is ω. Let $\pi : Z(S) \longrightarrow \mathbb{P}^1$ be the twistor family of g. We denote by S_t the fiber of π over t, which corresponds to a complex structure I_t on S associated with t. The metric g is compatible with I_t, the associated class ω_t is Kähler, and w is a Mukai vector on S_t for every $t \in \mathbb{P}^1$.

Choose now a $\mu_{\alpha,\omega}$-stable α-twisted sheaf $\mathscr{E}$ on S of rank r, and let $E := \mathscr{E}^{\vee\vee}$. This is a $\mu_{\alpha,\omega}$-stable α-twisted vector bundle of rank r, and we let $\mathcal{A}_0 := \mathscr{E}nd(E)$ be the corresponding Azumaya algebra. We suppose that $v_E(\mathscr{E}) = w$. By Section 4.4.2, there is a holomorphic vector bundle $\mathcal{A}$ on $Z(S)$ whose restriction $\mathcal{A}_t$ to S_t is an Azumaya algebra on S_t for every $t \in \mathbb{P}^1$. We let $\alpha_t \in \mathrm{Br}(S_t)$ be its class and $\mathcal{A}_t \simeq \mathscr{E}nd(E_t)$, where E_t is the deformation of E along the twistor line (see Section 4.4.3).

By Section 4.3.2 there is then a relative moduli space of stable twisted sheaves $p : \mathscr{M} \longrightarrow \mathbb{P}^1$ such that for every $t \in \mathbb{P}^1$ the fiber over t is the moduli space $M^\mu_{\alpha_t,w}(S_t, \omega_t)$ of μ_{α_t,ω_t}-stable α_t-twisted sheaves whose twisted Mukai vector with respect to E_t is w.

REMARK 4.22

On $\mathscr{M} \times_{\mathbb{P}^1} Z(S)$ we have a quasi-universal family. For $\mathscr{F} \in M^\mu_{\alpha,w}(S,\omega)$, let $F := \mathscr{F}^{\vee\vee}$ and $\mathcal{V}_0 := F \otimes E^\vee$. We let $\mathcal{V}$ in Proposition 4.19 be $\mathcal{V} := q^*\mathcal{V}_0$.

We first prove some geometric properties of the relative moduli space $p : \mathscr{M} \longrightarrow \mathbb{P}^1$.

PROPOSITION 4.23

Let S be a K3 surface, let $w = (r, 0, a) \in H^{2*}(S, \mathbb{Z})$ with $r \geq 2$, $\alpha \in \mathrm{Br}(S)$, and let ω be an (α, w)-generic polarization. The relative moduli space $p : \mathscr{M} \longrightarrow \mathbb{P}^1$ of stable twisted sheaves verifies the following properties:

(1) *the morphism p is submersive;*

(2) *if T_p^* denotes the relative cotangent bundle of p, then there is a holomorphic global section of $\wedge^2 T_p^* \otimes \mathscr{O}_{\mathbb{P}^1}(2)$ whose restriction to any fiber is a holomorphic symplectic form.*

Proof
We divide the proof into several parts.

Step 1: Submersivity. As every $\mathscr{E} \in \mathscr{M}_t$ is simple and the canonical bundle of a K3 surface is trivial, we have $\mathrm{Ext}^2(\mathscr{E},\mathscr{E})_0 = 0$. The exact sequence (9) implies then that the map p is submersive, so that condition (1) of the statement is proved.

Step 2: Section through locally free sheaves. Let $t_0 \in \mathbb{P}^1$, and choose $F \in M^{\mu}_{\alpha_{t_0},w}(S_{t_0},\omega_{t_0})$ a locally free sheaf. As we saw in Section 4.4.3, the sheaf F gives rise to a sheaf $F_t \in M^{\mu}_{\alpha_t,w}(S_t,\omega_t)$ for every $t \in \mathbb{P}^1$. This produces a section

$$s_F : \mathbb{P}^1 \longrightarrow \mathscr{M}, \qquad s_F(t) := F_t$$

of p, which is holomorphic. If we let E_t be the α_t-twisted μ_{α_t,ω_t}-stable sheaf such that $\mathcal{A}_t = \mathscr{E}\mathrm{nd}(E_t)$ (an Azumaya algebra on S_t whose class in $\mathrm{Br}(S_t)$ is α_t), and $G_t := F_t \otimes E_t^{\vee}$, we let $\mathcal{G} := q^* G_t$, which is a holomorphic vector bundle on $Z(S)$. The restriction of the relative tangent bundle T_p of p to the section s is

$$s^* T_p \simeq R^1 \pi_* \mathscr{E}\mathrm{nd}(\mathcal{G}).$$

Step 3: Relative symplectic form. We prove that condition (2) is fulfilled. We first note that for every $t \in \mathbb{P}^1$ the restriction $T_{p|t}$ of T_p to $\mathscr{M}_t$ is the tangent bundle of $M^{\mu}_{\alpha_t,w}(S_t,\omega_t)$, and similarly, the restriction $(T_p^*)_{|t}$ of T_p^* to $\mathscr{M}_t$ is the cotangent bundle of $M^{\mu}_{\alpha_t,w}(S_t,\omega_t)$. As on $M^{\mu}_{\alpha_t,w}(S_t,\omega_t)$ we have a holomorphic symplectic form (if S_t is projective, this is done in [42]; the proof in the general case is similar), so we get an isomorphism $T_{p|t} \simeq (T_p^*)_t$.

This implies the existence of a line bundle $\mathscr{O}_{\mathbb{P}^1}(d)$ for some $d \in \mathbb{Z}$ together with an isomorphism $T_p \longrightarrow T_p^* \otimes p^* \mathscr{O}_{\mathbb{P}^1}(d)$. We then just need to show that $d = 2$. To do so, consider a locally free sheaf $F \in \mathscr{M}_0$. As seen in Step 2 we have a holomorphic section $s : \mathbb{P}^1 \longrightarrow \mathscr{M}$ of p, and

$$s^* T_p \simeq R^1 p_* \mathscr{E}\mathrm{nd}(\mathcal{G}),$$

where $\mathcal{G} = q^*(F \otimes E_0^{\vee})$. By the relative Serre duality we get

$$R^1 p_* \mathscr{E}\mathrm{nd}(\mathcal{G}) \simeq \left(R^1 p_* \mathscr{E}\mathrm{nd}(\mathcal{G})^* \otimes K_\pi\right)^*,$$

where K_π is the relative canonical bundle of $\pi : Z(S) \longrightarrow \mathbb{P}^1$.

Now, as $\mathcal{G}$ is locally free, we have $\mathscr{E}\mathrm{nd}(\mathcal{G}) \simeq \mathscr{E}\mathrm{nd}(\mathcal{G})^*$. Moreover, $K_\pi \simeq \mathscr{O}_{\mathbb{P}^1}(-2)$ (see [13]). Hence,

$$R^1 p_* \mathscr{E}\mathrm{nd}(\mathcal{G}) \simeq R^1 p_* \mathscr{E}\mathrm{nd}(\mathcal{G})^* \otimes \mathscr{O}_{\mathbb{P}^1}(2).$$

In conclusion,

$$s^* T_p \simeq s^* T_p^* \otimes \mathscr{O}_{\mathbb{P}^1}(2).$$

As $s^*T_p \simeq s^*T_p^* \otimes \mathscr{O}_{\mathbb{P}^1}(d)$, it follows that $d = 2$. This shows that condition (2) is fulfilled. □

We now prove some geometric properties of the moduli spaces of stable twisted sheaves we are considering. In particular, we show that they are all compact and connected.

PROPOSITION 4.24

Let S be a K3 surface, let $w = (r, 0, a) \in H^{2*}(S, \mathbb{Z})$ with $r \geq 2$, $\alpha \in \mathrm{Br}(S)$, and let ω be an (α, w)-generic polarization. Moreover, let ξ be a representative of α in $H^2(S, \mathbb{Z})$ which is prime to r. The moduli space $M^{\mu}_{\alpha,w}(S, \omega)$ is a compact, connected manifold.

Proof

The compactness of $M^{\mu}_{\alpha,w}(S, \omega)$ is well known when S is projective, and a proof in the nonprojective and nontwisted case has been given in [36]. This proof uses in an essential way the comparison map from the moduli space of stable sheaves to the corresponding Donaldson–Uhlenbeck compactification of the moduli space of anti-self-dual connections in a Hermitian vector bundle on S. These arguments may be extended to the twisted case. We refer the reader to [36] and [41] for the ingredients.

To show that $M^{\mu}_{\alpha,w}(S, \omega)$ is connected, we will follow the strategy used by Mukai and by Kaledin, Lehn, and Sorger to prove the analogous result when S is projective, ω is the first Chern class of an ample line bundle, and the sheaves are untwisted (see the proof of [21, Theorem 4.1]).

We first suppose that the moduli space $M^{\mu}_{\alpha,w}(S, \omega)$ is not connected, and we choose a connected component Y. Moreover, we fix a sheaf $F \in Y$ and a sheaf $G \in M^{\mu}_{\alpha,w}(S, \omega) \setminus Y$.

Let $p : Y \times S \longrightarrow Y$ and $q : Y \times S \longrightarrow S$ be the two projections, and consider a $p^*\beta \cdot q^*\alpha$-twisted universal family $\mathscr{F}$ on $Y \times S$. We then define two complexes

$$K^{\bullet}_F := \mathscr{E}\mathrm{xt}^{\bullet}_p(q^*F, \mathscr{F}), \qquad K^{\bullet}_G := \mathscr{E}\mathrm{xt}^{\bullet}_p(q^*G, \mathscr{F})$$

of β-twisted sheaves on Y.

As the sheaves F and G have the same topological invariants (since their Mukai vectors are equal), by the Grothendieck–Riemann–Roch theorem and letting $d := \dim(Y)$, we have $c^B_d(K^{\bullet}_F) = c^B_d(K^{\bullet}_G)$, where c^B_d is the component of degree $2d$ of c^B (for some B-field giving the twist β). We now compute more explicitly these twisted Chern classes, and we start from $K^{\bullet}_G$. We note that if $E \in Y$, then E is a stable twisted sheaf having the same slope as G, but which is not isomorphic to G. It follows that

$$\mathrm{Ext}^0(G, F) = \mathrm{Ext}^2(G, F) = 0.$$

As

$$\mathscr{E}\mathrm{xt}^j_p(q^*G, \mathscr{F})_E \simeq \mathrm{Ext}^j(G, E),$$

it follows that

$$\mathscr{E}\mathrm{xt}^j_p(q^*G,\mathscr{F})=0$$

if $j=0,2$, and that $\mathscr{E}\mathrm{xt}^1_p(q^*G,\mathscr{F})$ is a locally free β-twisted sheaf whose rank is $d-2$.

As a consequence we have

$$c_d^B(K_G^\bullet)=-c_d^B\left(\mathscr{E}\mathrm{xt}^1_p(q^*G,\mathscr{F})\right)=0,$$

as $\mathscr{E}\mathrm{xt}^1_p(q^*G,\mathscr{F})$ is a locally free β-twisted vector bundle of rank $d-2<d$. Recall that c^B of $\mathscr{E}\mathrm{xt}^1_p(q^*G,\mathscr{F})$ is defined as the Chern class of some untwisted vector bundle of the same rank. Hence, as this rank is smaller than the dimension of Y, the dth B-twisted Chern class is trivial.

We now need to compute $c_d^B(K_F^\bullet)$. To do so, we first recall that by [3] there is locally on Y a complex

$$A^\bullet=\cdots\xrightarrow{a_{-1}}A^0\xrightarrow{a_0}A^1\xrightarrow{a_1}A^2\longrightarrow 0$$

of free sheaves such that for every $\sigma:Y'\longrightarrow Y$ and for every $j\in\mathbb{Z}$ we have

$$\mathscr{E}\mathrm{xt}^j_{p'}\left(\sigma^*(q')^*F,\sigma^*\mathscr{F}\right)\simeq\mathcal{H}^j(\sigma^*A^\bullet),$$

where $p':Y'\times S\longrightarrow Y'$ and $q':Y'\times S\longrightarrow S$ are the two projections, and where $\mathcal{H}^j$ denotes the jth cohomology of the complex.

Let us now cover Y with open subsets U_i so that F is contained in only one of them, and let us moreover suppose that the previous complex $A^\bullet$ exists over U_i. If $E\in U_i$ and E is not F, then $\mathcal{H}^j(A^\bullet)_E=0$. Hence, the rank of each map a_i of the complex $A^\bullet$ is constant on $Y\setminus\{F\}$. But we have

$$\mathcal{H}^0(A^\bullet)_F\simeq\mathcal{H}^0(A^\bullet)_F\simeq\mathbb{C}.$$

Hence, the rank of a_0 and a_1 at F drops by 1, while the rank of a_i is constant on Y for $i\leq -1$. The same proof as that of [21, Lemma 4.3] shows that the degeneracy locus of a_0 and a_1 is the reduced point F, while a_i does not degenerate if $i\leq -1$.

Let us now consider the blowup $\sigma:Z\longrightarrow Y$ of Y at F with reduced structure, and let D be the exceptional divisor on Z. Consider the complex

$$\sigma^*A^\bullet=\cdots\xrightarrow{\sigma^*a_{-1}}\sigma^*A^0\xrightarrow{\sigma^*a_0}\sigma^*A^1\xrightarrow{\sigma^*a_1}\sigma^*A^2\longrightarrow 0.$$

The degeneracy locus of σ^*a_0 and σ^*a_1 is then the exceptional divisor D with reduced structure, while the σ^*a_i's do not degenerate on Z for $i\leq -1$.

The maps σ^*a_0 and σ^*a_1 hence factor through

$$(A')^0\xrightarrow{a_0'}\sigma^*A^1\xrightarrow{a_1'}(A')^2,$$

where $\sigma^*A^0\subseteq(A')^0$, $(A')^2\subseteq\sigma^*A^2$, and the sheaves

$$M:=(A')^0/\sigma^*A^0,\qquad L:=\sigma^*A^2/(A')^2$$

are supported on D. As in the proof of [21, Theorem 4.1, Step 4], the sheaves L and M are characterized by canonical isomorphisms

$$L \otimes \mathcal{O}_D \simeq \mathrm{Ext}^2(F,F) \otimes \mathcal{O}_D, \qquad \mathrm{Tor}_1^{\mathcal{O}_D}(M, \mathcal{O}_D) \simeq \mathrm{Hom}(F,F) \otimes \mathcal{O}_D.$$

Here the computation is done in a neighborhood of the divisor D.

As in [21], it follows from this that

$$\mathscr{E}\mathrm{xt}^0_{p'}(\sigma^* q^* F, \sigma^* \mathscr{F}) \simeq \mathcal{O}_D(D), \qquad \mathscr{E}\mathrm{xt}^2_{p'}(\sigma^* q^* F, \sigma^* \mathscr{F}) \simeq \mathcal{O}_D,$$

viewed as $\sigma^*\beta$-twisted sheaves, and that $\mathscr{E}\mathrm{xt}^1_{p'}(\sigma^* q^* F, \sigma^* \mathscr{F})$ is a locally free $\sigma^*\beta$-twisted sheaf of rank $d-2$. It follows that

$$c_d^B(\sigma^* K_F^\bullet) = D^d = -1.$$

But note that

$$c_d^B(\sigma^* K_F^\bullet) = \sigma^* c_d^B(K_F^\bullet) = \sigma^* c^B(K_G^\bullet) = 0,$$

giving a contradiction. In conclusion, the moduli space $M^\mu_{\alpha,w}(S,\omega)$ has to be connected. □

We can now prove the following result, which is the main result of this section and which concludes the proof of Theorem 1.1(1).

PROPOSITION 4.25

Let S be a K3 surface, let $w = (r, 0, a) \in H^{2*}(S,\mathbb{Z})$ with $r \geq 2$, $\alpha \in \mathrm{Br}(S)$, and let ω be an (α, w)-generic polarization. Moreover, let ξ be a representative of α in $H^2(S,\mathbb{Z})$ which is prime to r. Consider the relative moduli space of stable twisted sheaves $p : \mathscr{M} \longrightarrow \mathbb{P}^1$ along the twistor family of (S,ω).

(1) *There is a $\overline{t} \in \mathbb{P}^1$ such that $\mathscr{M}_{\overline{t}}$ is an irreducible symplectic manifold which is deformation equivalent to a Hilbert scheme of points on a projective K3 surface S.*

(2) *The moduli space $M^\mu_{\alpha,w}(S,\omega)$ is a compact, connected complex manifold which is simply connected and carries a holomorphic symplectic form.*

Proof

We let $\pi : Z(S) \longrightarrow \mathbb{P}^1$ be the twistor family of (S,ω). By [16, Lemma 2.1] there is a $\overline{t}$ such that $S_{\overline{t}}$ is a projective K3 surface. The polarization $\omega_{\overline{t}}$ is $(\alpha_{\overline{t}}, w)$-generic, and $w_\xi = v(E_\xi)$ for some topological vector bundle E_ξ. Such a topological vector bundle remains constant along $\mathbb{P}^1$. Hence, $w_\xi = (r, \xi, b)$, where r and ξ are prime to each other. It follows from Proposition 4.18 that $M^\mu_{\alpha_{\overline{t}},w}(S_{\overline{t}}, \omega_{\overline{t}})$ is an irreducible symplectic manifold which is deformation equivalent to a Hilbert scheme of points on $S_{\overline{t}}$.

By Proposition 4.24, all the fibers are compact, connected manifolds, and by Proposition 4.23(a) the morphism p is submersive. By [7, the proposition in Section 1], it follows that p is a smooth and proper morphism. Hence, it is a deformation of $M^\mu_{\alpha,w}(S,\omega)$, and we are done. □

4.6. Moduli spaces of locally free sheaves

The previous results can be largely improved if we suppose something more on $M_v^\mu(S,\omega)$, namely, that it parameterizes only locally free sheaves. However, this case has already been considered by differential geometers. We therefore only state the following result and refer the reader to [19] and [20] for the proof.

PROPOSITION 4.26

Let S be a K3 surface, let $v=(r,\xi,a)$ be a Mukai vector such that r and ξ are prime to each other, and let ω be a v-generic polarization. Then the open part $\mathscr{M}^{lf}$ of the relative moduli space $p:\mathscr{M}\longrightarrow\mathbb{P}^1$ along the twistor family of (S,ω), parameterizing locally free sheaves, is the twistor family of the moduli space $M_v^{\mu-lf}(S,\omega)$ of ω-stable locally free sheaves with Mukai vector v on S.

If moreover $v^2=0$, then a standard argument shows that every sheaf in $M_v^\mu(S,\omega)$ is locally free (see [15, Remark 6.1.9]), and thus, the previous proposition applies to $M_v^\mu(S,\omega)$, which is moreover compact. The next proposition shows that compact moduli spaces of stable locally free sheaves as above may attain any even complex dimension.

PROPOSITION 4.27

Let r be a positive integer, let $d\in[0,2r-2]$ be an even integer, and let g be an integer such that $g\leq-(r^2-1)(r-1)$ and g is congruent to $\frac{d}{2}$ modulo r. Then there exists a K3 surface X with $NS(X)$ generated by one element ξ such that $\xi^2=2g-2$, and there exist torsion-free coherent sheaves E on X of rank r, $c_1(E)=\xi$, and such that $2r^2\Delta(E)-2(r^2-1)=d$. Moreover, all such sheaves are locally free and irreducible. In particular, they are stable with respect to any polarization on X, and their moduli space is a compact irreducible holomorphic symplectic manifold of dimension d.

Proof

The existence of K3 surfaces X with cyclic Néron–Severi groups was proved in [24], whereas the existence of torsion-free sheaves E with the above invariants follows from [23, Theorem 2.7]. We will check that such sheaves are irreducible and locally free. Suppose that $0\to E_1\to E\to E_2\to 0$ is an exact sequence with E_i coherent sheaves without torsion on X of ranks r_i and with $c_1(E_i)=\xi_i$ $(i=1,2)$. Then $\xi_1+\xi_2=\xi$, $r_1+r_2=r$, and we directly compute

$$\Delta(E)=\frac{1}{2r}\Big(\frac{\xi^2}{r}-\frac{\xi_1^2}{r_1}-\frac{\xi_2^2}{r_2}\Big)+\frac{r_1}{r}\Delta(E_1)+\frac{r_2}{r}\Delta(E_2).$$

Since $g\leq 0$, X is nonalgebraic. Hence, $\Delta(E_i)\geq 0$ and thus

$$\Delta(E)\geq\frac{1}{2r}\Big(\frac{\xi^2}{r}-\frac{\xi_1^2}{r_1}-\frac{\xi_2^2}{r_2}\Big)=-\frac{1}{2r_1r_2}\Big(\frac{r_2\xi}{r}-\xi_2\Big)^2$$
$$\geq-\frac{\xi^2}{2r^2(r-1)}=\frac{1-g}{(r-1)r^2}>\frac{r^2-1}{r^2}=1-\frac{1}{r^2}.$$

But this implies that $d > 2r^2$, which contradicts our choice of d. Hence, E is irreducible. If E were not locally free, then an easy computation would imply that the discriminant of its double dual would be negative: a contradiction to the nonalgebraicity of X. □

5. The second integral cohomology

We now study the second integral cohomology of $M_v^\mu(S,\omega)$. We will show that it carries a nondegenerate quadratic form of signature $(3,20)$ and that we have an isometry between $H^2(M_v^\mu,\mathbb{Z})$ and $v^\perp$. If $M_v^\mu(S,\omega)$ is Kähler, it is even a Hodge isometry. As a consequence, we will show that the moduli space is projective if and only if S is projective.

5.1. The quadratic form

Throughout this section we will let $X := M_v^\mu(S,\omega)$ for a choice of a K3 surface S, a Mukai vector $v = (r,\xi,a)$ with r and ξ prime to each other, and a v-generic polarization ω. We let $2n$ be its complex dimension. We start by defining a quadratic form on $H^2(X,\mathbb{C})$ for every holomorphic symplectic form σ on X, by using the same formula as that of the Beauville form of an irreducible symplectic manifold. For every $\alpha \in H^2(X,\mathbb{C})$, we let

$$\begin{aligned} q_\sigma(\alpha) := \frac{n}{2}\int_X \alpha^2 \wedge \sigma^{n-1} \wedge \overline{\sigma}^{n-1} \int_X \sigma^n \wedge \overline{\sigma}^n \\ + (1-n)\int_X \alpha \wedge \sigma^n \wedge \overline{\sigma}^{n-1} \int_X \alpha \wedge \sigma^{n-1} \wedge \overline{\sigma}^n. \end{aligned}$$

Note that the symplectic form is always supposed to be closed, so the above definition does not depend on representatives. Note also that $q_\sigma(\sigma+\overline{\sigma}) = (\int_X \sigma^n \wedge \overline{\sigma}^n)^2 \neq 0$, so q_σ is nontrivial.

Recall next the definition of the topological quadratic form

$$\tilde{q}_X(\alpha) := c_n \int_X \alpha^2 \sqrt{\mathrm{td}(X)},$$

where c_n is a constant depending only on n chosen so that the form becomes integral on $H^2(X,\mathbb{Z})$ (see [12, Part III, Definition 26.19]). It is known that q_σ and $\tilde{q}_X$ are proportional when X is moreover supposed to be Kähler.

We finally define $\tilde{H}^{2,0} := \mathrm{Im}(\{\tau \in H^0(\Omega^2) \mid \mathrm{d}\tau = 0\} \to H^2(X,\mathbb{C}))$ and $\tilde{h}^{2,0}(X) := \dim \tilde{H}^{2,0}(X)$. We first prove the following.

PROPOSITION 5.1

Let $p: X \to C$ be a proper submersion of relative dimension $2n$ over a connected curve C such that there exists a point $0 \in C$ with $X_0 := p^{-1}(0)$ irreducible holomorphic symplectic. Suppose, moreover, that there exists a relative nondegenerate symplectic form $\sigma \in H^0(X,\Omega^2_{X/C} \otimes p^*L)$ with values in a line bundle L over C. Let $q_t := q_{\sigma_t}$ be the quadratic form defined by σ on $H^2(X_t,\mathbb{C})$ for each $t \in C$.

Then for all $t \in C$ the quadratic form q_t is a positive multiple of $\tilde{q}_{X_0}$. In particular, q_t is nondegenerate of signature $(3, b_2(X) - 3)$ and $\tilde{h}^{2,0}(X_t) = 1$.

Proof
We may suppose that L is the trivial line bundle on C. Indeed, for the general case it will suffice to take trivializations of L over Zariski-open subsets of C containing 0.

Fix some $\alpha \in H^2(X_t, \mathbb{C})$, and define for $t_1, t_2 \in C$

$$q_{t_1,t_2}(\alpha) := \frac{n}{2} \int_X \alpha^2 \wedge \sigma_{t_1}^{n-1} \wedge \overline{\sigma}_{t_2}^{n-1} \int_X \sigma_{t_1}^n \wedge \overline{\sigma}_{t_2}^n + (1-n) \int_X \alpha \wedge \sigma_{t_1}^n \wedge \overline{\sigma}_{t_2}^{n-1} \int_X \alpha \wedge \sigma_{t_1}^{n-1} \wedge \overline{\sigma}_{t_2}^n.$$

(Note again that the above formula does not depend on representatives since the symplectic forms $\sigma_{t_1}, \sigma_{t_2}$ are closed.) This defines a complex function on $C \times C$ which is holomorphic in t_1 and antiholomorphic in t_2. It becomes holomorphic on $C \times C^-$, where C^- denotes the curve C with the opposite complex structure. Over an analytical open neighborhood U of 0 in C, all fibers X_t are Kähler. Hence, for $t \in U$ the quadratic form q_t is proportional to $\tilde{q}$. Take now $\alpha, \alpha' \in H^2(X_0, \mathbb{C})$ such that $q_0(\alpha) \neq 0$. Then the meromorphic function

$$(t_1, t_2) \mapsto \frac{q_{t_1,t_2}(\alpha')}{q_{t_1,t_2}(\alpha)}$$

on $C \times C^-$ is constant on the diagonal $\Delta_U \subset U \times U^- \subset C \times C^-$. But Δ_U is Zariski-dense in $C \times C^-$. To see this, consider the system of local holomorphic curves C_t on $C \times C^-$ given as images of the maps $z \mapsto (t+z, t+\bar{z})$. Each curve C_t passes through the reference point $(t,t) \in \Delta_U$ but its intersection with Δ_U is a piece of a real line. Hence, by the principle of isolated zeros, any holomorphic function vanishing locally on Δ_U will also vanish on the curves C_t and thus also vanish on the three-dimensional real submanifold of $C \times C^-$ they cover. Therefore, the function $(t_1, t_2) \mapsto \frac{q_{t_1,t_2}(\alpha')}{q_{t_1,t_2}(\alpha)}$ is constant on $C \times C^-$. From this it follows that q_t is proportional to $\tilde{q}$ for any $t \in C$.

It remains to check that $\tilde{h}^{2,0}(X_t) = 1$ for all $t \in C$. For this we will show that the kernel K of the linear map

$$\left\{\tau \in H^0(X_t, \Omega^2) \,\middle|\, \mathrm{d}\tau = 0\right\} \to H^0(X_t, K_{X_t}), \quad \tau \mapsto \tau \wedge \sigma^{n-1},$$

consists of d-exact forms only. Let b_t be the bilinear form associated to q_t. Then for any $\tau \in K$ and $\alpha \in H^2(X_t, \mathbb{C})$ we have

$$b_t(\tau, \alpha) = \frac{n}{2} \int_X \tau \wedge \alpha \wedge \sigma_t^{n-1} \wedge \overline{\sigma}_t^{n-1} \int_X \sigma_t^n \wedge \overline{\sigma}_t^n + \frac{1-n}{2} \int_X \tau \wedge \sigma_t^n \wedge \overline{\sigma}_t^{n-1} \int_X \alpha \wedge \sigma_t^{n-1} \wedge \overline{\sigma}_t^n + \frac{1-n}{2} \int_X \alpha \wedge \sigma_t^n \wedge \overline{\sigma}_t^{n-1} \int_X \tau \wedge \sigma_t^{n-1} \wedge \overline{\sigma}_t^n = 0,$$

and our assertion follows since q_t is nondegenerate. □

5.2. Isometry with $v^{\perp}$

We now show that there is an isometry between $H^2(M_v^{\mu}(S,\omega),\mathbb{Z})$ and $v^{\perp}$ if $v^2>0$, and with $v^{\perp}/\mathbb{Z}\cdot v$ if $v^2=0$. We introduce some notation. If $v\in H^{2*}(S,\mathbb{Z})$, then we let $v^{\perp}$ be the orthogonal of v with respect to the Mukai pairing. If $v=(r,\xi,a)$ and $\xi\in NS(S)$, then the pure weight-two Hodge structure on $H^{2*}(S,\mathbb{Z})$ induces a pure weight-two Hodge structure on $v^{\perp}$: namely, a class $\alpha=(\alpha_0,\alpha_1,\alpha_2)\in v^{\perp}$ is of $(1,1)$-type if and only if $\alpha_1\in NS(S)$.

If $\alpha=(\alpha_0,\alpha_1,\alpha_2)\in H^{2*}(S,\mathbb{Q})$, then we write $\alpha^{\vee}:=(\alpha_0,-\alpha_1,\alpha_2)$. If $\alpha=\mathrm{ch}(F)$ for some locally free sheaf F, then $\alpha^{\vee}=\mathrm{ch}(F^{\vee})$. It is immediate to see that if $\alpha,\beta\in H^{2*}(S,\mathbb{Q})$, then $(\alpha\cdot\beta)^{\vee}=\alpha^{\vee}\cdot\beta^{\vee}$. In particular, this implies that $(\beta/\alpha)^{\vee}=\beta^{\vee}/\alpha^{\vee}$ and $(\sqrt{\alpha})^{\vee}=\sqrt{\alpha^{\vee}}$ whenever these expressions make sense.

We now introduce a morphism associating to any class in $v^{\perp}$ a rational cohomology class on the moduli space of stable (twisted) sheaves. The construction is inspired from the similar morphism which is used in the projective case (see [30], [43], [26], [31]). Let $\alpha\in\mathrm{Br}(S)$, let $w\in H^{2*}(S,\mathbb{Q})$ be a Mukai vector, and let ω be a w-generic polarization. Suppose moreover that $M_{\alpha,w}^{\mu}(S,\omega)$ is compact, and let $p:M_{\alpha,w}^{\mu}(S,\omega)\times S\longrightarrow M_{\alpha,w}^{\mu}(S,\omega)$ and $q:M_{\alpha,w}^{\mu}(S,\omega)\times S\longrightarrow S$ be the projections.

Choosing a quasi-universal family $\mathscr{E}$ on $M_{\alpha,w}^{\mu}(S,\omega)\times S$ of similitude ρ (which exists by Remark 4.22), we define a morphism

$$\lambda_{S,\alpha,w}:w^{\perp}\longrightarrow H^2\big(M_{\alpha,w}^{\mu}(S,\omega),\mathbb{Q}\big)$$

by letting

$$\lambda_{S,\alpha,w}(\beta):=\frac{1}{\rho}\big[p_*\big(q^*\big(\beta^{\vee}\cdot\sqrt{\mathrm{td}(S)}\big)\cdot\mathrm{ch}(\mathscr{E})\big)\big]_1,$$

where $[\cdot]_1$ is the part lying in $H^2(M_{\alpha,w}^{\mu}(S,\omega),\mathbb{Q})$. As $\beta\in w^{\perp}$, the class $\lambda_{S,\alpha,w}(\beta)$ does not depend on the chosen quasi-universal family. If $\alpha=0$, we simply write $\lambda_{S,w}$ for $\lambda_{S,0,w}$.

We now show the following, which is a generalization of known results in the projective case (see [29], [30], [43]).

PROPOSITION 5.2

Let S be a K3 surface, and let $v=(r,\xi,a)\in H^{2*}(S,\mathbb{Z})$, where $r\geq 2$, $\xi\in NS(S)$, $(r,\xi)=1$, and $v^2\geq 0$. Moreover, let ω be a v-generic polarization. Then the image of $\lambda_{S,v}$ is contained in $H^2(M_v^{\mu}(S,\omega),\mathbb{Z})$, and

(1) *if $v^2=0$, then $\lambda_{S,v}$ defines an isometry*

$$\overline{\lambda}_{S,v}:v^{\perp}/\mathbb{Z}\cdot v\longrightarrow H^2\big(M_v^{\mu}(S,\omega),\mathbb{Z}\big);$$

(2) *if $v^2>0$, then $\lambda_{S,v}$ is an isometry.*

Proof

If $v^2>0$, we just need to show the following properties:

(a) the image of $\lambda_{S,v}$ is contained in $H^2(M_v^{\mu}(S,\omega),\mathbb{Z})$;

(b) the morphism $\lambda_{S,v}$ is bijective;
(c) the morphism $\lambda_{S,v}$ is an isometry.

Let $\mathscr{E}$ be a quasi-universal family of similitude ρ on $M_v^\mu(S,\omega)\times S$, and fix a locally free μ_ω-stable vector bundle F of Mukai vector v. Let $w := v_F(F) = (r, 0, a - \xi^2/2r)$, and let

$$f : M_v^\mu(S,\omega) \longrightarrow M_{0,w}^\mu(S,\omega), \qquad f(\mathscr{F}) := \mathscr{F} \otimes F^\vee,$$

which is an isomorphism (see Remark 4.15).

We let $q : M_{0,w}^\mu(S,\omega)\times S \longrightarrow S$ be the projection, and we let

$$\mathscr{E}' := (f \times \mathrm{id}_S)_* \mathscr{E} \otimes q^* F^\vee,$$

which is a quasi-universal family of similitude ρ on $M_{0,w}^\mu(S,\omega)\times S$. Moreover, as f is an isomorphism, the morphism

$$f_* : H^2\big(M_v^\mu(S,\omega),\mathbb{Z}\big) \longrightarrow H^2\big(M_{0,w}^\mu(S,\omega),\mathbb{Z}\big)$$

is easily checked to be an isometry.

Now, we let

$$h : H^{2*}(S,\mathbb{Z}) \longrightarrow H^{2*}(S,\mathbb{Q}), \qquad h(\beta) := \frac{\beta\cdot \mathrm{ch}(F^\vee)}{\sqrt{\mathrm{ch}(F\otimes F^\vee)}}.$$

We let $(\cdot,\cdot)_S$ be the Mukai pairing on S, and we let $[\cdot]_2$ be the part lying in $H^4(S,\mathbb{Q})$. If $\beta \in v^\perp$, we have

$$\begin{aligned}\big(h(\beta), w\big)_S &= -\Big[\frac{\beta^\vee\cdot \mathrm{ch}(F)}{\sqrt{\mathrm{ch}(F\otimes F^\vee)}}\cdot v_F(F)\Big]_2\\ &= -\big[\beta^\vee\cdot \mathrm{ch}(F)\cdot\sqrt{\mathrm{td}(S)}\big]_2 = (\beta, v)_S = 0,\end{aligned}$$

so that

$$h : v^\perp \longrightarrow w^\perp.$$

The same argument shows that it is an isometry. We even have $f_*(\lambda_{S,v}(\beta)) = \lambda_{S,w}(h(\beta))$. Indeed,

$$\begin{aligned}f_*\big(\lambda_{S,v}(\beta)\big) &= \frac{1}{\rho}\big[f_* p_*\big(q^*\big(\beta^\vee\sqrt{\mathrm{td}(S)}\big)\,\mathrm{ch}(\mathscr{E})\big)\big]_1\\ &= \frac{1}{\rho}\big[p_*\big((f\times\mathrm{id}_S)_* q^*\big(\beta^\vee\sqrt{\mathrm{td}(S)}\big)\,\mathrm{ch}(\mathscr{E}')\big)\big]_1\\ &= \frac{1}{\rho}\big[p_*\big(q^*\big(h(\beta)^\vee\sqrt{\mathrm{td}(S)}\big)\,\mathrm{ch}(\mathscr{E}')\big)\big]_1 = \lambda_{S,w}\big(h(\beta)\big).\end{aligned}$$

In conclusion, we see that $\lambda_{S,v}$ verifies properties (a), (b), and (c) above if and only if $\lambda_{S,w}$ verifies them.

Now, consider the twistor line of (S,ω), and let $p : \mathscr{M} \longrightarrow \mathbb{P}^1$ be the associated relative moduli space. As we can define $\lambda_{S,v}$ in a relative way using relative quasi-universal families (which exist by Remark 4.22), properties (a), (b), and (c) above are verified on a fiber if and only if they are verified all along the twistor line. It follows that $\lambda_{S,w}$ verifies (a), (b), and (c) if and only if λ_{S_t,w_t} verifies them for some $t \in \mathbb{P}^1$.

As we saw before, there is t such that S_t is projective, and in this case λ_{S_t,w_t} is an isometry by [42]. Hence, we are done. If $v^2 = 0$, the proof is similar: the only difference is about the fact that $\mathbb{Z} \cdot v$ is the kernel of $\lambda_{S,v}$, which holds in the general case as it holds over a projective K3 surface (see [29]). □

An immediate corollary of the previous proposition is the following result.

COROLLARY 5.3
Let S be a K3 surface, $v = (r, \xi, a) \in H^{2*}(S, \mathbb{Z})$ where $\xi \in NS(S)$, $r \geq 2$, $(r, \xi) = 1$, and $v^2 \geq 0$. If ω is a v-generic polarization and $M^\mu_v(S, \omega)$ is Kähler, then the morphism λ_v is a Hodge isometry.

Theorem 1.2 can now be seen as a corollary of the previous results.

COROLLARY 5.4
Let S be a K3 surface, and let $v = (r, \xi, a) \in H^{2*}(S, \mathbb{Z})$, where $\xi \in NS(S)$, $r \geq 2$, $(r, \xi) = 1$, and $v^2 \geq 0$. If ω is a v-generic polarization, then $M^\mu_v(S, \omega)$ is projective if and only if S is projective.

Proof
First, note that if S is projective, then $M^\mu_v(S, \omega)$ is projective by Theorem 3.4. Suppose now that S is not projective; we want to prove that $M^\mu_v(S, \omega)$ is not projective as well. Suppose that $M^\mu_v(S, \omega)$ is projective: in particular, this implies that it is Kähler. Hence, by Theorem 1.1(1) it follows that it is an irreducible symplectic manifold.

Recall that an irreducible symplectic manifold X is projective if and only if there is a line bundle L on X such that $q(L) > 0$, where q is the Beauville form of X (see [14]). Hence, there is a line bundle L on $M^\mu_v(S, \omega)$ such that $q(L) > 0$, where q is the Beauville form on $M^\mu_v(S, \omega)$, which coincides with the nondegenerate quadratic form we defined in the previous section.

Moreover, by Corollary 5.3, as $M^\mu_v(S, \omega)$ is Kähler we have that λ_v is a Hodge isometry. There is then $\alpha \in v^\perp$ of type $(1,1)$ (with respect to the Hodge structure on $v^\perp$) such that $\lambda_v(\alpha) = c_1(L)$, and $(\alpha, \alpha)_S > 0$.

Let us now describe $v^\perp \otimes \mathbb{Q}$. First, an element $(0, \zeta, b) \in \widetilde{H}(S, \mathbb{Q})$ is in $v^\perp \otimes \mathbb{Q}$ if and only if $b = \zeta \cdot \xi$. As $(0, \zeta, \zeta \cdot \xi) = e^{\xi/r} \cdot (0, \zeta, 0)$, we have

$$e^{\xi/r} \cdot H^2(S, \mathbb{Q}) \subseteq v^\perp.$$

It is easy to see that $e^{\xi/r} \cdot (2r^2, 0, v^2) \in v^\perp \otimes \mathbb{Q}$. Hence,

$$e^{\xi/r} \cdot \mathbb{Q}(2r^2, 0, v^2) \subseteq v^\perp \otimes \mathbb{Q}.$$

This implies that

$$v^\perp \otimes \mathbb{Q} = e^{\xi/r} \cdot \big(H^2(S, \mathbb{Q}) \oplus \mathbb{Q}(2r^2, 0, v^2)\big),$$

so that the $(1,1)$-part $(v^\perp)^{1,1}$ of $v^\perp \otimes \mathbb{Q}$ is

$$(v^\perp)^{1,1} = e^{\xi/r} \cdot \big(NS_{\mathbb{Q}}(S) \oplus \mathbb{Q}(2r^2, 0, v^2)\big),$$

where $NS_{\mathbb{Q}}(S) := NS(S) \otimes \mathbb{Q}$.

The direct sum is orthogonal with respect to the Mukai pairing, and it is easy to see that

$$\big(e^{\xi/r}(2r^2, 0, v^2)\big)^2 = -4r^2v^2 \leq 0,$$

as $v^2 \geq 0$. Moreover, as S is nonprojective, the lattice $e^{\xi/r}NS_{\mathbb{Q}}(S)$ is negative semidefinite. It follows that $(v^\perp)^{1,1}$ is negative semidefinite. Hence, for every $\alpha \in (v^\perp)^{1,1}$ we have $(\alpha, \alpha)_S \leq 0$, which is not possible. In conclusion, if S is not projective, then the moduli space cannot be projective, and we are done. □

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Perego: Institut Élie Cartan, UMR 7502, Université de Lorraine, Vandoeuvre-lès-Nancy, France; Arvid.Perego@univ-lorraine.fr

Toma: Institut Élie Cartan, UMR 7502, Université de Lorraine, Vandoeuvre-lès-Nancy, France; Matei.Toma@univ-lorraine.fr